Convergence of Multiple Fourier Series

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I. Fouries Series

NOTATION.
We write $\mathbb{T}$ for the unit circle in the complex plane and identify it with $[0, 2\pi)$ with addition modulo 2π.

Functions defined on $\mathbb{T}$ will be thought of as 2π-periodic functions on $\mathbb{R}$. For a function in $L^1(\mathbb{T})$, its Fourier coefficients are given by:

$$\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-int} dt,$$

and the Fourier Series of f is written as:

$$\sum_{n \in \mathbb{Z}} \hat{f}(n) e^{int}$$

A fundamental question in Harmonic Analysis is about the Convergence of the Fourier series.

Consider the partial sums of the Fourier Series of an integrable function f on the torus $\mathbb{T}$:

$$S_N f(t) = \sum_{-N}^{N} \hat{f}(j) e^{ijt}$$

These are the symmetric partial sums. Observe that for each N, S_N defines a multiplier operator, given by

$$\widehat{S_N f}(k) = \chi_{[-N,N]}(k) \hat{f}(k)$$

.

In this talk we will ask whether such partial sums converge in the norm of the spaces $L^p(\mathbb{T})$, $1 \leq p < \infty$, and also in higher dimensions, i.e for the spaces $L^p(\mathbb{T}^d)$.

In one dimension the answer is No, for $p = 1$ and for $p = \infty$, and Yes for all $1 < p < \infty$. For higher dimensions, there is another story!

It turns out that by a simple transference argument, we can pass to the corresponding problem in the setting of the real line $\mathbb{R}$

Given a function in $L^1(\mathbb{R})$, its Fourier Transform is given by:

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-i\xi x} dx$$

and the inversion formula:

$$f(x) \sim \int_{\mathbb{R}} \hat{f}(\xi) e^{i\xi x} d\xi$$

We will talk about the convergence of the partial integrals:

$$S_N f(x) = \int_{-N}^{N} \hat{f}(\xi) e^{i\xi x} d\xi$$

Consider the multiplier given by $\phi(\xi) = -isgn(\xi)$ which defines the Hilbert transform given by $\widehat{(Hf)}(\xi) = -isgn(\xi)\hat{f}(\xi)$.

It is well known (F. and M. Riesz Theorem) that this operator is bounded on $L^p(\mathbb{R}), 1 < p < \infty$.

We see easily that the partial integral operator is given by

$$S_N = M_N H M_{-N} - M_{-N} H M_N$$

where M_N is the modulation operator $M_N f(t) = e^{iNt} f(t)$.

It folllows that the operators $\{S_N\}$ are uniformly bounded. We conclude that for $1 < p < \infty$, norm convergence of the Fourier series holds.

Multiple Fourier Series

Consider now integrable functions on the d-torus $\mathbb{T}^d$. For such functions, the Fourier Series is a multiple Fourier series:

$$f \sim \sum_{\mathbf{k} \in \mathbb{Z}^d} \hat{f}(\mathbf{k}) e^{i\mathbf{k}.\mathbf{t}}$$

where $\mathbf{k} = (k_1, k_2, ..., k_d)$ and $\mathbf{t} = (t_1, t_2, ..., t_d)$.

Now partial sums can be defined in several ways. Two natural ways are by taking partial sums over squares, and over spheres.

We write

$$D_N f(\mathbf{t}) = \sum_{|\mathbf{k}| \leq N} \hat{f}(\mathbf{k}) e^{i\mathbf{k}.\mathbf{t}}$$

where $|\mathbf{k}| = max(|k_1|, |k_2|, ..., |k_d|)$, and

$$S_R = \sum_{\|\mathbf{k}\| \leq R} \hat{f}(\mathbf{k}) e^{i\mathbf{k}.\mathbf{t}}$$

where $\|\mathbf{k}\| = (k_1^2 + k_2^2 + ... + k_d^2)^{1/2}$, respectively.
Once again, by a transference we can pass to multiplier operators on $L^p(\mathbb{R}^d)$.

Square summability

As before, in order to prove that the square sums of the Fourier series converge in norm, we need to prove the uniform boundedness of the multiplier operators D_N, given by

$$(D_N f)\hat{}(\xi) = \chi_{Q_N}(\xi)\hat{f}(\xi),$$

where $Q_N = \prod_1^N [-N, N]$.

By a dilation argument, we see easily that $\|D_N\|_{op}$ are independent of N. Hence we need only look at one operator, for $N = 1$, i.e. the operator D_1.

We will restrict ourselves to $d = 2$.
Now, the multiplier D_1 is a composition of four multipliers corresponding to the indicator functions of four half-planes in $\mathbb{R}^2$. Specifically, let

$$\begin{aligned} E_1 &= \{x \in \mathbb{R}^2 : (x-(1,0)).(-1,0) \geq 0\} \\ E_2 &= \{x \in \mathbb{R}^2 : (x-(-1,0)).(1,0) \geq 0\} \\ E_3 &= \{x \in \mathbb{R}^2 : (x-(0,1)).(0,-1) \geq 0\} \\ E_4 &= \{x \in \mathbb{R}^2 : (x-(0,-1)).(0,10) \geq 0\} \end{aligned}$$

Then

$$D_1 = T_{\chi_{E_1}} \circ T_{\chi_{E_2}} \circ T_{\chi_{E_3}} \circ T_{\chi_{E_4}}$$

Therefore to prove the boundedness of D_1, it is enough to prove:

Theorem (Half Plane Multiplier)

Let $x_0 \in \mathbb{R}^2$, and let $v \in \mathbb{R}^2$ be a unit vector. Let

$$E_{x_0,v} = \{x \in \mathbb{R}^2 : (x - x_0).v \geq 0\}.$$

Then the operator defined for $f \in \mathcal{S}$, as

$$f \longrightarrow S_{x_0,v}f = (\chi_{E_{x_0,v}}\hat{f})\check{}$$

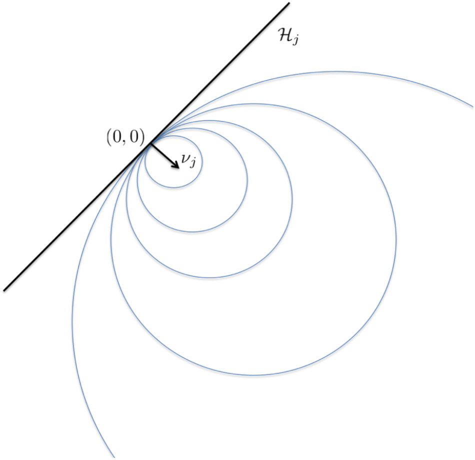
is bounded on $L^p(\mathbb{R}^2), 1 < p < \infty$, i.e. there exists a constant $C_p > 0$ such that

$$\|S_{x_0,v}f\| \leq C_p\|f\|_p$$

This theorem is easy to prove simply by using the one dimensional result.

Clearly, we can also get Polygonal summability, by taking regular polynomials of any number of sides. As the number of sides increase, so will the norm of the operator, so for spherical summation, the limiting process of approximating a disc by such polynomials does not converge.

To overcome this difficulty, Yves Meyer found an interesting approximation to the half-plane multiplier via large discs. The idea of the proof of Meyer's lemma is to use a nice approximation of a half plane by dilated discs:



Yves Meyer (unpublished) proved the following vector-valued inequality for a sequence of half-plane operators:

Lemma (Meyer)

Suppose the ball multiplier T_B given by $(T_B f)\hat{} = (\chi_B \hat{f})^\vee$ is bounded on L^p. Let $\nu_1, \nu_2, ..., \nu_j, ...$ be a sequence of unit vectors in $\mathbb{R}^2$ and let $\mathcal{H}_j$ be the corresponding half-planes $\{x \in \mathbb{R}^2 : x.\nu_j \geq 0\}$, and $S_{\mathcal{H}_j}$ the corresponding multiplier operators, then

$$\|(\sum_j |S_{\mathcal{H}_j}(f_j)|^2)^{1/2}\|_p \leq C_p \|(\sum_j |(f_j)|^2)^{1/2}\|_p.$$

Remark. For a single bounded operator T, a vector-valued inequality of the form

$$\|(\sum_j |T(f_j)|^2)^{1/2}\|_p \leq C_p \|(\sum_j |(f_j)|^2)^{1/2}\|_p.$$

is well-known. In Meyer's theorem, there is a sequence of operators also. The point is that each of these operators is the limit of a sequence of operators composed of suitable dilations and translations of a single operator, S_1, the ball multiplier.

Then Charles Fefferman constructed a counterexample by bringing in the construction of the Kakeya-Besicovitch set set. So we need to go back half a century... to a seemingly unrelated theme...

Kakeya Needle Problem; Besicovotch Set

In 1917, a problem was posed by the Japanese Mathematician, Soichi Kakeya:

In the class of figures in which a segment of length 1 can be turned around through 360 degrees, remaining always within the figure, which one has the smallest area?

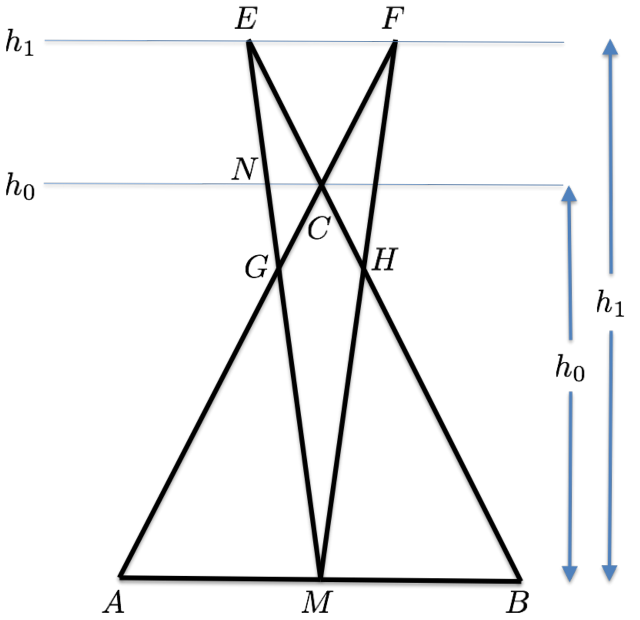
Kakeya's problem, *of course,* did not reach Russia, where, in 1920, Abram Samoilovitch Besicovitch posed a twin problem:

What is the minimum planar measure of a measurable set which is the union of segments of all directions, each of length 1.

In 1928 Besicovitch gave an unexpected and brilliant construction ...

We will consider a construction which is a modification of the construction due to Besicovitch. This modified construction is appropriate for Fefferman's construction.

Begin with an equilateral triangle ABC (with small area), and let it "sprout":


E
F
h_1
h_0
N
C
G
H
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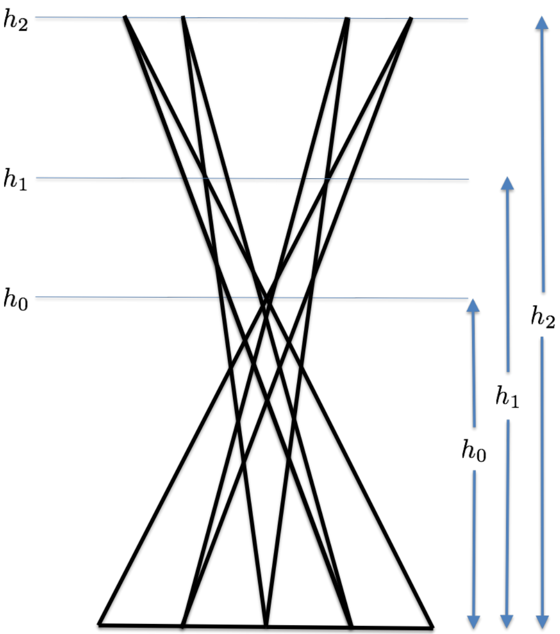
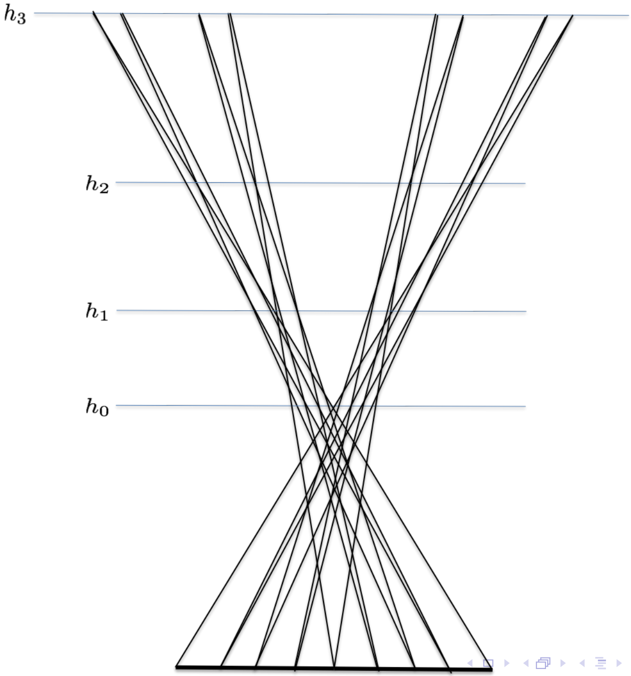
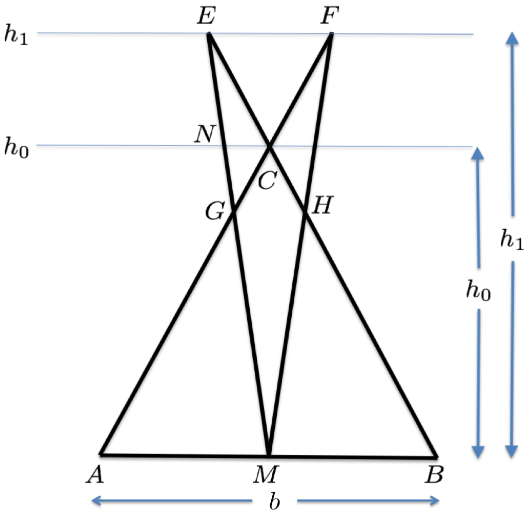
M
B
h_1
h_0

Next, we let each of the sprouted triangles sprout again and again, and so on...


h_2
h_1
h_0
h_0
h_1
h_2

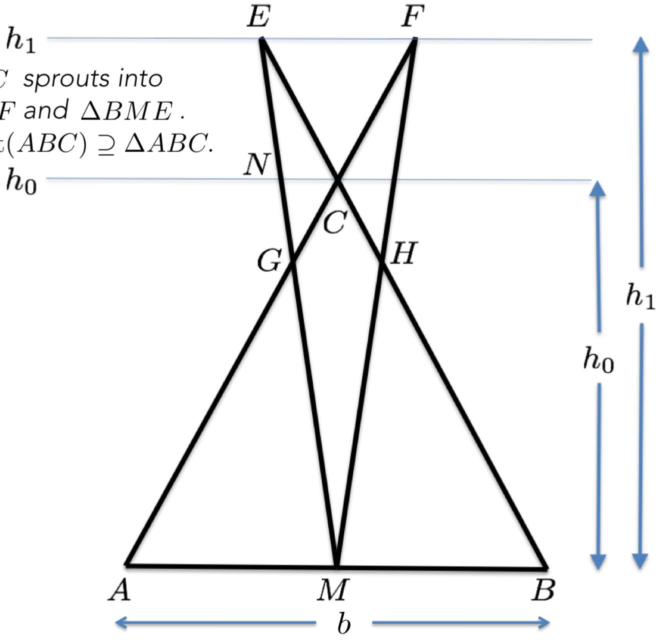

h_3
h_2
h_1
h_0

We first compute the area added at each step:

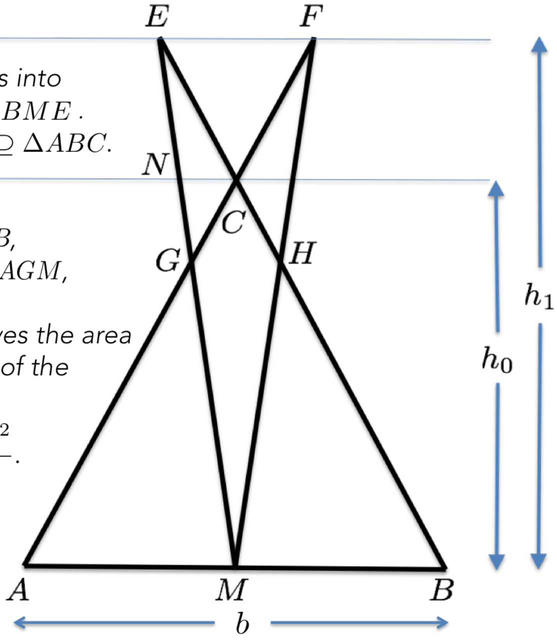


ΔABC sprouts into ΔAMF and ΔBME.

$\text{Sprout}(ABC) \supseteq \Delta ABC$.



ΔABC *sprouts into* ΔAMF *and* ΔBME.
$\text{Sprout}(ABC) \supseteq \Delta ABC.$



$\Delta ENC \sim \Delta EMB$, *and* $\Delta NGC \sim \Delta AGM$, *hence a simple computation gives the area of the two arms of the sprout as*

$$\frac{b}{2}\frac{(h_1 - h_0)^2}{2h_1 - h_0}.$$

We compute the area of the sprouted set after k steps of construction.

Given an $\epsilon > 0$, we start with base of the initial triangle, $b = \epsilon$, and height $h_0 = \epsilon$. At the jth stage of sprouting, let $h_j = \epsilon(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{j+1})$.

At step 1, the triangle has two sprouts Λ_1 and Λ_2, and at the kth step, there are 2^k sprouts, namely $\{\Lambda_{r_1,r_2,\ldots,r_k} : r_j = 1 \text{ or } 2\}$. Let

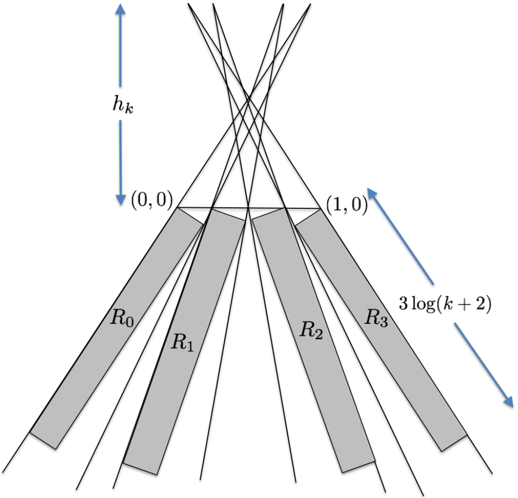
$$E_\epsilon(k) = \cup \Lambda_{r_1,r_2,\ldots,r_k}$$

Then

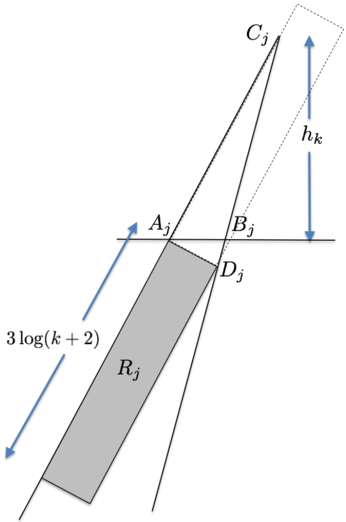
$$
\begin{aligned}
|E_\epsilon(k)| &= |\Delta ABC| + \text{areas of arms of all sprouts} \\
&= \frac{1}{2}\epsilon^2 + \sum_{j=1}^{k} 2^j \frac{2^{-(j-1)}\epsilon}{2} \frac{(h_j - h_{j-1})^2}{2h_j - h_{j-1}} \\
&\leq \frac{1}{2}\epsilon^2 + \epsilon^2 \sum_{j=1}^{k} \frac{1}{(j+1)^2} \\
&\leq C\epsilon^2.
\end{aligned}
$$

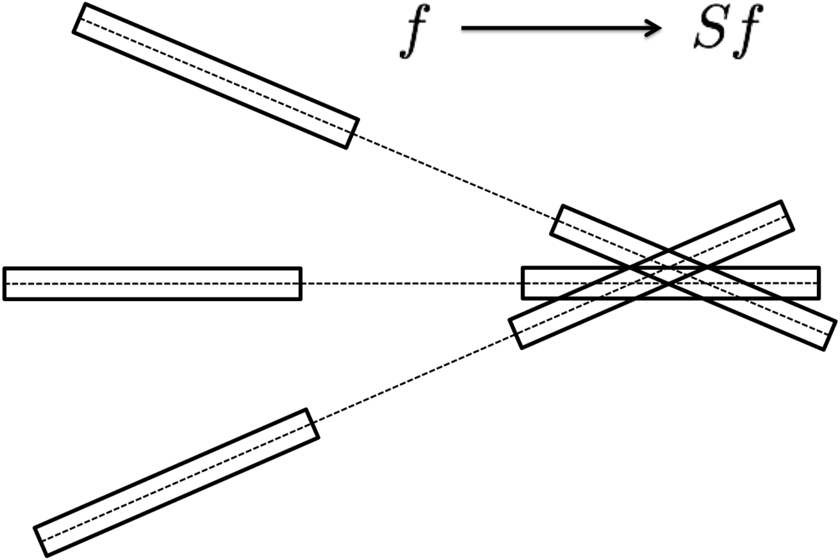
Hence, we have a Kakeya set of measure as small as we like!

Fefferman took the Kakeya-Besicovitch construction and looked at:



He considers long disjoint rectangles R_j, oriented along the longer sides, so that the adjacent equal rectangles R'_J overlap in the "Kakeya sense".





$f \longrightarrow Sf$

An elementary computation of the Hilbert transform:

Exercise. Let $J = [-a, a]$ and $J' = \{x : a \leq |x| \leq 3a\}$, and suppose $\phi(\xi) = \chi_{[0,\infty)}(\xi)$, then

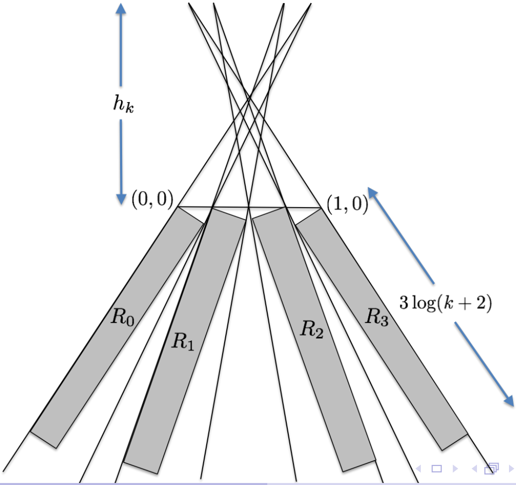
$$|T_\phi(\chi_J)| \geq \frac{1}{10}\chi_{J'}$$

As a consequence for each rectangle R we get,

$$|S_{\mathcal{H}}(\chi_R)| \geq \frac{1}{10}\chi_{R'}$$

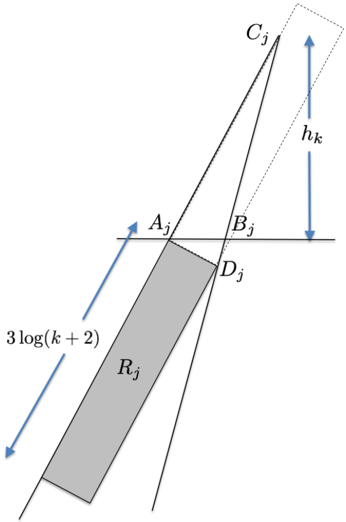
Let $\delta > 0$ be given, We construct upto a level k, so that $k + 2 > e^{1/\delta}$. Then

$$|E(k)| \leq \delta \sum_j |R_j|$$



Next, an easy computation shows that

$$|R_j' \cap E(k)| \geq \frac{1}{12}|R_j|$$



Let ν_j be a unit vector parallel to the long side of R_j, and let $\mathcal{H}_j$ be the corresponding half-plane. Recall Meyer's inequality:

$$\|(\sum_j |S_{\mathcal{H}_j}(f_j)|^2)^{1/2}\|_p \leq C_p \|(\sum_j |(f_j)|^2)^{1/2}\|_p.$$

On the one hand,

$$\begin{aligned} \int_{E(k)} \sum_j |S_{\mathcal{H}_j}(\chi_{R_j})|^2 dx &\geq \sum_j \int_{E(k)} (\frac{1}{10}\chi_{R_j'}(x))^2 dx \\ &= \frac{1}{100} \sum |E(k) \cap R_j'| \\ &\geq \frac{1}{1200} \sum_j |R_j| \end{aligned}$$

and on the other hand, using Hölder's inequality,

$$\begin{aligned}\int_{E(k)} \sum_j |S_{\mathcal{H}_j}(\chi_{R_j})|^2 dx &\leq |E(k)|^{(p-2)/2} \|(\sum_j (|S_{\mathcal{H}_j}(\chi_{R_j})|^2)^{1/2}\|_p^2 \\ &\leq C_p^2 |E(k)|^{(p-2)/2} \|(\sum_j |\chi_{R_j}(x)|^2)^{1/2}\|_p^2 \\ &= C_p^2 |E(k)|^{(p-2)/2} (\sum_j |R_j|)^{2/p} \\ &\leq C_p^2 \delta^{(p-2)/2} \sum_j |R_j|\end{aligned}$$

But this means that

$$\frac{1}{1200}\sum_j |R_j| \leq C_p^2 \delta^{(p-2)/2} \sum_j |R_j|$$

We may let $p > 2$, since the L^p and $L^{p'}$ multipliers are the same for, $1/p + 1/p' = 1$

We get a contadiction by taking δ small.

Bochner-Riesz Means

Since there is a sharp discontinuity in the Disc multiplier, we now consider family of multiplier operators, called the Bochner-Riesz Means, which attempt to smooth out this sharp discontinuitly. For $\lambda > 0$, define an oprator T_λ given by

$$\widehat{T_\lambda f}(\xi) = (1 - |\xi|^2)_+^\lambda$$

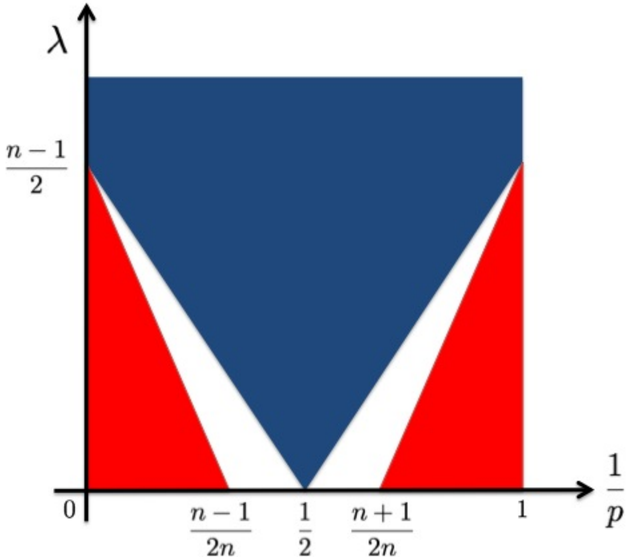
For $\lambda = 0$, we obtain the Disc Multiplier, and for $\lambda > 1$, the multipliers arise from convolution by integrable functions, hence are bounded operators.

So we restrict to $0 < \lambda < 1$.

Known Results, Bochner Riesz Conjecture

- If $\lambda > \frac{n-1}{2}$, then T_λ is bounded on all L^p.
- If $\left|\frac{1}{p} - \frac{1}{2}\right| \geq \frac{2\lambda+1}{2n}$, then T_λ is unbounded for all p.
- T_λ is bounded on L^p for $\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{\lambda}{n-1}$.
- If $\lambda > \frac{n-1}{2(n+1)}$, then T_λ is bounded on L^p for $\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{2\lambda+1}{2n}$.

Bochner-Riesz Conjecture



Kakeya Conjecture

We know that a Kakeya set can have arbitrarily small area. But in some sense the set is quite ’substantial. So perhaps its measure is not the correct parameter to look at its size. A more appropriate measure of its size could be its Hausdorff or Minkowski dimension.

Hausdorff Dimension

Let $S \subset \mathbb{R}$ and $d \in [0, \infty)$, the d-dimensional Hausdorff measure of S is defined by

$$C_H^d(S) := \inf\left\{\sum_i r_i^d : S \subset \cup_i B(r_i)\right\}.$$

The Hausdorff dimension of S is defined by

$$H(S) := \inf\{d \geq 0 : C_H^d(S) = 0\}.$$

and then we also have

$$H(S) := \sup\{d \geq 0 : C_H^d(S) = \infty\}.$$

Minkowski or Box Dimension

Suppose that $N(\epsilon)$ is the number of boxes of side length ϵ required to cover a set S. Then the box-counting dimension is defined as:

$$\dim_{\mathrm{box}}(S) := \lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log(1/\varepsilon)}.$$

In general

$$H(S) \leq \dim_{\mathrm{box}}(S) \leq n$$

For planar sets, it was proved that a Kakeya set must have full Hausdorff dimension, i.e. 2.

For $n > 2$, the following conjecture is still open: **The Hausdorff dimension of a Kakeya set in $\mathbb{R}^n$ is n.**

Bochner-Riesz Conjecture *implies* Kakeya Conjecture. and conversely, the solution of

Kakeya Conjecture will imply progress in the Bochner-Riesz conjecture.

[T. Tao, N. Katz. I. Laba, J. Bourgain, T. Wolff...]

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  (Charles Fefferman (Fields Medallist 1978) was just 23 when he wrote this paper.)

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