A Journey with Dynamical Properties in Dynamical Systems

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- The growth of crystals,
- The stock market,
- The formation of traffic jams,
- The behaviour of the decimal digits of the square root of 2;

and so on.

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- Once an initial world-state is chosen, applying the rule again and again, the dynamics determines the world-state at all future times.

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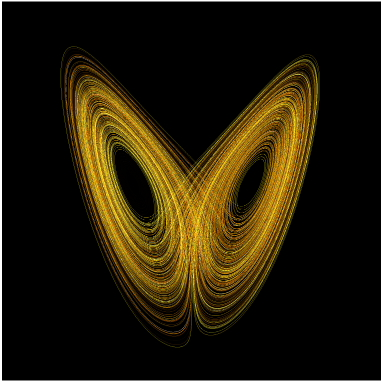
Lorenz attractor is a set of chaotic solutions of the Lorenz dynamical system of ordinary differential equations which when plotted resembles a butterfly or figure eight.



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- The name of the subject, “DYNAMICAL SYSTEMS”, came from the title of classical book: G.D. Birkhoff, Dynamical Systems. Amer. Math. Soc. Colloq. Publ. 9. American Mathematical Society, New York (1927), 295 pp.
- Many areas of biology, physics, economics and applied mathematics involve a detailed analysis of dynamical systems based on the particular laws governing their change.

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- The work of Poincaré was a great influence to the present state of the subject since it led to a change in the motivation from the quantitative to the qualitative and geometrical study of such mechanical systems and more general systems of nonlinear differential equations.
- This change was a key step for the development of the modern theory of dynamical systems during the 20^{th} century.

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- An *orbit* of a point $x \in X$, denoted by $O_f(x)$, is the set $\{x, f(x), f^2(x), \dots\}$, if f is continuous and is the set $\{\dots, f^{-2}(x), f^{-1}(x), x, f(x), f^2(x), \dots\}$, if f is a homeomorphism.

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- A property of continuous maps which is preserved under topological conjugacy is called a *dynamical property*. For example expansivity, shadowing, transitivity, mixing, minimality, chaos etc.
- We discuss various dynamical properties of maps on topological spaces.

Expansive Homeomorphisms

Expansivity (W.R. Utz, *Unstable homeomorphisms*, Proc. Amer. Math. Soc., 1950)

Let (X, d) be a metric space. A homeomorphism $f : X \to X$ is called *expansive* provided there exists a real number $\delta > 0$ such that whenever $x, y \in X$ with $x \neq y$ then there exists an integer n (depending on x, y) satisfying $d(f^n(x), f^n(y)) > \delta$; δ is called an *expansive constant* for f.

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- Expansiveness is independent of the choice of metric for compact metric spaces but not for non-compact metric spaces.

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- Linear maps on $\mathbb{R}^n$ with no eigen values of modulus 1.
- Left/right shift operator on the subspace $X = \left\{\frac{1}{n}, 1 - \frac{1}{n} : n \in \mathbb{N}\right\}$ of the real line.

Spaces admitting expansive homeomorphisms

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C. Mouron, Topology Proceedings, 2003

There exists a 2-dimensional planar continuum that admits an expansive homeomorphism and separates the plane.

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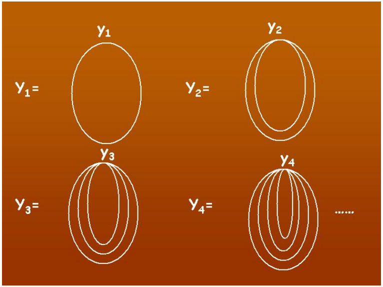
Mane, Trans. Amer. Math. Soc., 1979

Let f be an expansive homeomorphism on a compact metric space X then the topological dimension of X is finite.



Consider the subspace Y_n, $n = 1, 2, ...$ as shown in figure, of $\mathbb{R}^2$. Observe that in each Y_n all the points intersect at the point y_n. Hence removal of the point y_n will disconnect Y_n into n disjoint arcs. Let Y be the union of all these Y_n. As each y_n has unique property, every homeomorphism on Y will fix y_n. Since such points are infinite in number, there exists no expansive homeomorphism on Y.

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- Notion of 'expansiveness' is defined and studied in various other settings. For instance, on uniform spaces, topological groups, topological vector spaces, Banach spaces, differentiable manifolds, etc.
- In 1993 Kato (*Continuum–wise expansive homeomorphisms*, Canad. J. Math. 45, 1993) has generalized this concept by defining Continuum-wise expansive homeomorphisms, which have been studied in detail having very good applications in continuum theory and related areas.

Topological Transitivity

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- So, it basically requires orbit of each open subset to be dense in the phase space.

Definition : Topological transitivity (TT)

A dynamical system (X, f) is said to be *topologically transitive* if for every pair (U, V) of nonempty open subsets of X, there exists a positive integer n such that $f^n(U) \cap V \neq \emptyset$.

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Characterization of topological transitivity

In a dynamical system (X, f), where X is a compact metric space and f is onto, topological transitivity (TT) is equivalent to the existence of a dense orbit in X, i.e., there is a point $x \in X$ such that orbit of x is dense in X (DO).

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Take $X = \{1/n : n \in \mathbb{N}\} \cup \{0\}$ endowed with the usual metric and $f : X \to X$ defined by $f(0) = 0$ and $f(1/n) = 1/(n+1)$, $n \in \mathbb{N}$. Then (X, f) satisfies (DO) but not (TT).

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Example

Take $I = [0, 1]$ and the standard tent map $g : I \to I$

$$g(x) = \begin{cases} 2x & \text{if } 0 \leqslant x < 1/2, \\ 2 - 2x & \text{if } 1/2 \leqslant x \leqslant 1. \end{cases}$$

Let X be the set of all periodic points of g and $f = g|_X$. Then X is infinite and every orbit is finite in X thus the system (X, f) does not satisfy the condition (DO) but the condition (TT) is fulfilled.

Remark

In a dynamical system if f^n ($f \circ f \circ \cdots \circ f$ (n-times)) is topologically transitive for some $n \in \mathbb{N}$ then trivially f is topologically transitive. The following example shows that the converse is not true.

Remark

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Example

Take $X = [0, 2]$ and $f : X \to X$ defined by

$$f(x) = \begin{cases} 2x + 1 & \text{if } 0 \leqslant x \leqslant 1/2, \\ -2x + 3 & \text{if } 1/2 \leqslant x \leqslant 1, \\ -x + 2 & \text{if } 1 \leqslant x \leqslant 2. \end{cases}$$

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Then f is topologically transitive but f^2 is not. Also $f \times f$ is not transitive.

- This motivates the concept of total transitivity in which every iterate of f becomes topologically transitive.

Total Transitivity and Mixing

Definition : Total transitivity

A dynamical system (X, f) is said to be *totally transitive* if f^n, $n \in \mathbb{N}$, are topologically transitive.

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Definition : Strongly mixing

A dynamical system (X, f) is said to be *strongly mixing* (or just *mixing*) if for every pair (U, V) of nonempty open subsets of X there exists $n_0 \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$, for all $n \geqslant n_0$.

- If $(X \times X, f \times f)$ is topologically transitive then so is (X, f). Otherway we have already seen that the transitivity of (X, f) need not imply the transitivity of $(X \times X, f \times f)$. This motivates a new dynamical property, namely weakly mixing.

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A weakly mixing dynamical system (X, f) is totally transitive.

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So in general we have the following implications:

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Consider the rotation on circle given by $f(\theta) = \theta + \lambda$, where λ is an irrational number. Then (S^1, f) is totally transitive but not weakly mixing.

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Dynamical systems which are weakly mixing but not mixing have been constructed in Toeplitz flows.

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A totally transitive dynamical system (X, f), where X is compact with an open interval J (i.e. J is homeomorphic to $(0, 1)$) having dense set of periodic points is strongly mixing.

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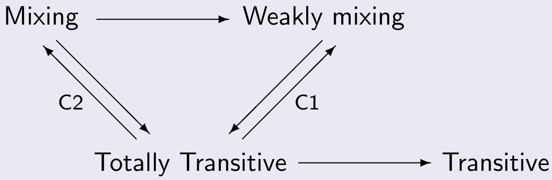

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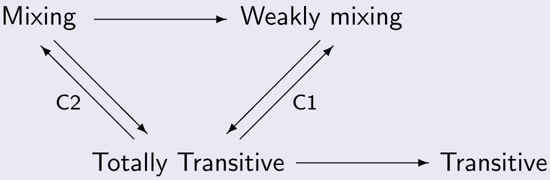

- By C1 and C2 we mean:
  - C1: For any metric space X, $f : X \to X$ continuous with $\overline{Per(f)} = X$.
  - C2: For any compact metric space with an open interval, $f : X \to X$ continuous with $\overline{Per(f)} = X$.

Pseudo Orbit Tracing Property

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Definition : Pseudo orbit tracing property (or shadowing property)

A sequence of points $\{x_i : i \in \mathbb{Z}\}$ of a metric space X is called a δ-*pseudo orbit* of a homeomorphism f if $d(f(x_i), x_{i+1}) < \delta$ for $i \in \mathbb{Z}$. Given $\epsilon > 0$ a δ-pseudo orbit $\{x_i\}$ is said to be ϵ-*traced* by a point $x \in X$ if $d(f^i(x), x_i) < \epsilon$ for every $i \in \mathbb{Z}$.
We say f has the *pseudo orbit tracing property* (abbrev. $POTP$) if for every $\epsilon > 0$ there is $\delta > 0$ such that every δ-pseudo orbit of f can be ϵ-traced by some point of X.

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- For compact spaces this property is independent of the compatible metrics used.

Theorem

Let f be a linear map of the Euclidean space $\mathbb{R}^n$ and d be the Euclidean metric for $\mathbb{R}^n$. Then f has $POTP$ under d iff f is hyperbolic, i.e. it has no eigenvalues of modulus 1.

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Let X be a compact metric space and $X^{\mathbb{Z}} = \prod_{-\infty}^{\infty} X_i$, where each X_i is a copy of X. Then the shift map $\sigma : X^{\mathbb{Z}} \to X^{\mathbb{Z}}$ has $POTP$.

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Theorem (Walters)

Let S be a closed subset $Y_k^{\mathbb{Z}}$ and $\sigma : S \to S$ be the subshift. Then $\sigma : S \to S$ has $POTP$ iff σ is subshift of finite type.

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- Notion of shadowing property is defined and studied in various other settings. For instance, on uniform spaces, topological groups, topological vector spaces, differentiable manifolds, etc.
- Recently, many researches have studied relation between shadowing property/ variants of shadowing with other dynamical properties.

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Suppose X is a compact metric space. If f is an expansive homeomorphism which has shadowing property then f is topologically stable in the class of homeomorphisms of X.

- The following result gives an important property of topologically Anosov maps (i.e. expansive and having $POTP$).

Walter, 1978

Suppose X is a compact metric space. If f is an expansive homeomorphism which has shadowing property then f is topologically stable in the class of homeomorphisms of X.

Definition : Nonwandering point

A point $x \in X$ is called a *nonwandering point* of f if for every open neighborhood U of x, there exist $n > 0$ such that $f^n(U) \cap U \neq \emptyset$. $\Omega(f)$ denotes the set of all nonwandering points of f.

Remarkable Applications of above Dynamical Properties

Spectral decomposition theorem due to Smale:

Let f be an expansive homeomorphism which has shadowing property. Then the following properties hold: $\Omega(f)$ contains a finite sequence $B_i (1 \leq i \leq l)$ of f–invariant closed subsets such that

- $\Omega(f) = \bigcup_{i=1}^{l} B_i$ (disjoint union),
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Decomposition theorem due to Bowen:

Let f be an expansive homeomorphism which has shadowing property. Then for a basic set B there exists $a > 0$ and a finite sequence $C_i(0 \leq i \leq a-1)$ of closed subsets such that

- $C_i \cap C_j = \phi(i \neq j), f(C_i) = C_{i+1}$ and $f^a(C_i) = C_i$,
- $B = \bigcup_{i=1}^{a-l} C_i$,
- $f^a_{C_i} : C_i \to C_i$ is topologically mixing.

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Minimality $\implies$ Topological transitivity

Every minimal dynamical system is topologically transitive.

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- There are different types of chaos defined and studied in the literature. We shall discuss here first accepted definition of chaos given by Robert Devaney.

Definition : Sensitive dependence to initial conditions

Let (X, f) be a dynamical system where X is a metric space with metric d and $f : X \to X$ be continuous. Then (X, f) has *sensitive dependence to initial conditions* (SDIC) if there exists an $\epsilon > 0$ such that for any $x \in X$ and neighborhood N_x about x, there exists a $y \in N_x$ and $n \in \mathbb{N}$ satisfying $d(f^n(x), f^n(y)) > \epsilon$. The number ϵ is also called the *sensitivity constant for the system*.

Definition : Devaney chaotic

Let (X, f) be a dynamical system where X is a metric space and $f : X \to X$ is continuous. Then f is said to be *Devaney chaotic* if f is transitive, has dense set of periodic points and has SDIC.

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Devaney chaotic on unit interval

On the unit interval, f is Devaney chaotic iff f is transitive.

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The following result was proved by Banks et al.

Theorem

Let (X, f) be a dynamical system, where X is a compact metric space and $f : X \to X$ is continuous. If f is transitive and has dense set of periodic points then f has sensitive dependence on initial conditions.

Devaney chaotic on unit interval

On the unit interval, f is Devaney chaotic iff f is transitive.

Devaney chaotic on unit circle

On the unit circle, f mixing $\implies$ f Devaney chaotic $\implies$ f transitive.

Dynamical Properties on G-spaces

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- Analyzing definitions of dynamical properties on metric/topological spaces, we have extended the notions of expansivity, minimality, mixing, transitivity, shadowing, chaos etc for homeomorphisms/continuous maps on G-spaces not studied earlier.
- We have obtained interesting results regarding existence/non-existence, extensions, projecting, lifting, characterizations, topological stability and decomposition theorems for homeomorphisms/continuous maps on G-spaces and on general topological spaces.

Definition : Non-autonomous systems

Let X be a topological space and $f_n : X \to X$ be continuous for each $n \in \mathbb{N}$ and let $f_{1,\infty}$ denote the sequence $(f_1, f_2, \ldots, f_n, \ldots)$. The pair $(X, f_{1,\infty})$ is said to be *non-autonomous system*.

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- Define $f_1^n(x) = f_n \circ f_{n-1} \circ \cdots \circ f_2 \circ f_1$, $n \in \mathbb{N}$ and $f_1^0 = id_X$, the identity on X. In particular, when $f_{1,\infty}$ is a constant sequence $(f, f, \ldots, f, \ldots)$ then the pair $(X, f_{1,\infty})$ is just the classical dynamical system (autonomous dynamical system) (X, f).

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- Beginning with Kolyada and Snoha work in 1996, in the recent past lots of studies have been done regarding dynamical properties in nonautonomous discrete dynamical systems. We have defined and studied expansiveness, shadowing, topological stability, chain recurrence, non-wandering points, decomposition theorems in nonautonomous discrete dynamical systems given by a sequence of continuous maps/homeomorphisms on metric spaces.

Some Recent References

- Authors have defined and studied n-expansive homeomorphisms. [C. Morales, *A generalization of expansivity*, Discrete and Continuous Dynamical Systems, 2012]

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  [Jorge Groisman, Jos Vieitez, *On transitive expansive homeomorphisms of the plane*, Topology and its Applications, 2014]
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Thank you

THANK YOU