Fitted operator finite difference scheme for a class of singularly perturbed differential difference turning point problems exhibiting interior layer

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Outline

Singularly perturbed problems

Singularly perturbed problem

Singular perturbation problem as a differential equation is a differential equation whose highest-order derivative is multiplied by a small positive parameter ε. The perturbation parameter ε may take arbitrary values in the interval $(0, 1]$.

- Setting the value of the parameter ε equals to 0, reduces the order of the given differential equation and it fails to satisfy one or more boundary conditions.
- The solution of a singularly perturbed problem, unlike a regular problem have boundary and/or interior layers, that is, narrow sub-domains specified by the parameter ε on which the solution vary by a finite value while the derivatives of the solution in these subdomains grow without bound as epsilon tends to zero.

Singularly perturbed problems

- So, typically there are thin transition layers where the behavior of the solution varies rapidly or jumps abruptly, while away from the layers the solution behaves regularly and varies slowly.

Boundary value problems for singularly perturbed differential equations models many practical phenomena in biology and physics. They model convection-diffusion processes in applied mathematics which arise in diverse areas including

- Fluid mechanics, fluid dynamics.
- Chemical reactor theory, nuclear engineering.
- Aero dynamics, combustion, plasma dynamics.
- Magneto hydrodynamics, rarefied gas dynamics.
- Control theory, oceanography and other domains of the great world of fluid motion.

Singularly perturbed problems

A few notable examples are

- Boundary layer problems
- WKB problems
- Modeling of steady and unsteady viscous flow problems with high Reynolds numbers.
- Convective heat transport problems with large Peclet numbers
- Magneto-hydrodynamics duct problems at high Hartman numbers
- Drift- diffusion equation of semiconductor modeling, etc.

Failure of classical numerical methods

- Standard discretization methods for solving singular perturbation problems are unstable and fail to give accurate results when the perturbation parameter ε is small.
- In particular, methods based on centered differences or upwinded differences on uniform meshes yield error bounds, in the maximum norm, which depend on an inverse power of singular perturbation parameter ε.
- The violation of the monotonicity property of a boundary value problem when grid approximations are constructed even for the simplest problems can lead to both large errors and nonphysical results.
- Therefore, it is important to develop numerical methods for these problems, whose accuracy does not depend upon the perturbation parameter ε.

Methods for solving singular perturbation problems

There are two different approaches which are used for construction of special numerical methods:

- Fitted operator methods which comprise specially designed finite difference operator on standard meshes and therefore have an advantage in simplicity because meshes used are uniform.
- Fitted mesh methods which use standard classical finite difference schemes on on specially designed meshes.

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Singularly perturbed Differential difference equations [SPDDE]

Singularly perturbed differential-difference equation is an differential equation in which the highest order derivative is multiplied by a small parameter and involve atleast one delay/advance term.

Singularly perturbed Differential difference equations

- Differential-difference equations are of two types: Neutral and Retarded delay differential equations.
- Singularly perturbed differential-difference equations has arosen in many fields, such as in the study of variational problems in control theory where problems are complicated by the effect of time delays in signal transmission[E. L. Els'gol'ts,12, AMS, Providence, RI, 1964].
- In the study of bistable devices [M. W. Derstine, H. M. Gibbs, D. L. Kaplan, Physics Review A, 26 (1982), pp. 3720-3722].
- Evolutionary biology [M. Wazewska-Czyzewska, A. Lasota, Mat. Stos. 6 (1976), pp. 25-40].
- Description of human pupil light reflex [A. Longtin, J. Milton, Math. Biosc., 90 (1988) , pp. 183-199].
- A variety of models of physiological processes or diseases .

Work done till date

- Study of singularly perturbed differential-difference equation was started by Lange and Miura in 1985. In a series of paper they gave an asymptotic approach to study such type of boundary value problems but, they discussed mainly the non-turning point problems.
- Kadalbajoo and Sharma initiated the numerical analysis of such class of problems and in a series of paper since 2000 gave many robust numerical methods for solving such type of problems.
- Many more researches such as Ramesh, Devendra, Patidar, Ramos, etc. over a period of time gave various ε-uniform numerical schemes to solve such type of problems but, all of the study above was limited to the case when the convection coefficient does not changes sign inside the domain.
- Here we are considering the case when the convection coefficient vanishes,i.e, Turning points are present inside the domain for such class of problems.

Problem formulation

$$\varepsilon y'' + a(x)y'(x) - b(x)y(x) + c(x)y(x-\delta) = f(x), \quad x \in (-1,1) = \Omega, \tag{1}$$

$$\begin{aligned} y(x) &= \phi(x), \quad -1-\delta \leq x \leq -1, \\ y(1) &= \gamma, \end{aligned} \tag{2}$$

where $\bar{\Omega} = [-1,1],\ 0 < \varepsilon \ll 1$ and

$$a(0) = 0, \quad a'(0) > 0 \tag{3}$$

$$b(x) \geq b_0 > 0, \quad \forall\, x \in [-1,1] \tag{4}$$

$$\text{and} \quad b(x) - c(x) \geq K > 0, \quad c(x) \geq 0, \quad \forall\, x \in [-1,1]. \tag{5}$$

To ensure that there is no other turning point in the region $[-1,1]$ we impose the restriction

$$|a'(x)| \geq \frac{|a'(0)|}{2}, \quad \forall\, x \in [-1,1]. \tag{6}$$

Parameter β is defined as

$$\beta = b(0)/a'(0). \tag{7}$$

The domain $\bar{\Omega}$ is divided into three subintervals, i.e.,

$$\bar{\Omega} = D_1 \cup D_2 \cup D_3 \quad D_1 = [-1, -\mu],$$

$$D_2 = (-\mu, \mu),\ D_3 = [\mu, 1],\ 0 < \mu \leq 1/2,\ D_0 = [-1-\delta, -1].$$

The differential operator L_ε corresponding to the boundary value problem (1)- (2) is defined by

$$L_\varepsilon y(x) = F(x) \tag{8}$$

$$L_\varepsilon y(x) = \begin{cases} \varepsilon y''(x) + a(x)y'(x) - b(x)y(x), & -1 < x \leq -1+\delta \\ \varepsilon y''(x) + a(x)y'(x) - b(x)y(x) + c(x)y(x-\delta), & -1+\delta < x < 1. \end{cases} \tag{9}$$

$$F(x) = \begin{cases} f(x) - c(x)\phi(x-\delta), & -1 < x \leq -1+\delta \\ f(x), & -1+\delta < x < 1. \end{cases} \quad (10)$$

Lemma

(Comparison Principle): Let $\Psi(x) \in C^2(\bar{\Omega})$ be a smooth function satisfying $\Psi(-1) \geq 0,\ \Psi(1) \geq 0$ and $L_\varepsilon \Psi(x) \leq 0$ for all $x \in \Omega$. Then $\Psi(x) \geq 0$ for all $x \in \bar{\Omega}$.

Lemma

The solution $y(x)$ of the boundary value problem (1), (2) is bounded and satisfy the following estimate

$$||y||_0 \leq ||f||_0 k^{-1} + C_1(||\phi||_{0,D_0} + |\gamma|)$$

where $k = min_{x \in \bar{\Omega}}\{(b(x) - c(x)),\ b(x)\},\ C_1 \geq 1$ is a positive constant.

Theorem

Let $a(x),\ b(x),\ c(x),\ f(x) \in C^j(\bar{\Omega}),\ j > 0,\ ||a(x)||_0 = \sigma,\ |a(x)| \geq \eta > 0,\ x \in D_1 \cup D_3$. *Then there exist a positive constant* C *such that if* $a(x) < 0$ *on* D_3 *then the solution* $y(x)$ *of the problem* (1)-(2) *satisfies*

$$|y^{(k)}(x)| \leq C\left(1 + \varepsilon^{-k} \mathit{exp}\left(\frac{-\eta(1-x)}{\varepsilon}\right)\right), \quad \text{for } k = 1, \ldots, j+1,\ x \in D_3$$

whereas if $a(x) > 0$ *on* D_1 *then*

$$|y^{(k)}(x)| \leq C\left(1 + \varepsilon^{-k} \mathit{exp}\left(\frac{-\eta(x+1)}{\varepsilon}\right)\right), \quad \text{for } k = 1, \ldots, j+1,\ x \in D_1.$$

Remark:1 For $a(x),\ b(x),\ c(x),\ f(x) \in C^m(\bar{\Omega}),\ m$ a positive integer and $S_2(m) = \{||a||_m,\ ||b||_m, ||c||_m\ ||f||_m,\ |a(x)|,\ 1-\mu,\ y(-1),\ y(1), y(-\mu),\ y(\mu),\ m\}$, there exist a constant C depending only on $S_2(m)$ such that

$$|y^{(k)}(x)| \le C, \quad \text{for } k = 1, \dots, m+1 \quad x \in D_1 \cup D_3.$$

Theorem

Consider $a(x),\ b(x),\ c(x),\ f(x) \in C^m(\bar{\Omega})$ *and assume conditions* (3)-(6). *Let* $\beta_l < 1 < \beta_s,\ \beta_l < \beta < \beta_s$ *and* $0 < \beta < 1$. *Then there exist a constant C depending only on* $S(m) = \{||a||_2,\ ||b||_1,\ ||c||_1,\ ||f||_1,\ b_0,\ \beta_l,\ \beta_s,\ ||\phi|_{0,D_0},\ |\gamma|,\ ||a||_m,\ ||b||_m,\ ||c||_m, ||f||_m,\ m\}$ *such that the solution* $y(x)$ *of* (1)-(2) *satisfies*

$$|y^{(k)}(x)| \le C(|x| + \varepsilon^{1/2})^{\beta-k}, \quad k = 1, \dots, m+1,\ x \in (-1, 1).$$

Numerical discretization

Let Y_i^N be any given function defined on the computational grid.

$$D^+Y_i = \frac{(Y_{i+1} - Y_i)}{h}, \; D^-Y_i = \frac{(Y_i - Y_{i-1})}{h},$$

$$D^+D^-Y_i = \frac{(Y_{i+1} - 2Y_i + Y_{i-1})}{h^2}.$$

The discrete problem corresponding to (1)-(2) is defined as

$$L^N Y_i = F_i \tag{11}$$

$$\begin{aligned} Y_0 &= \phi_0 \\ Y_N &= \gamma \end{aligned} \tag{12}$$

$$L^N Y_i = \begin{cases} \varepsilon\rho_i D^+D^-Y_i + a_i D^-Y_i - b_i Y_i, & i \\ \varepsilon\rho_i D^+D^-Y_i + a_i D^-Y_i - b_i Y_i + \dfrac{c_i}{h}\Big[(s_i - x_{i-m-1})Y_{i-m} + (x_{i-m} - s_i)Y_{i-m-1}\Big], & \\ \varepsilon\rho_i D^+D^-Y_i + a_i D^+Y_i - b_i Y_i + \dfrac{c_i}{h}\Big[(s_i - x_{i-m-1})Y_{i-m} + (x_{i-m} - s_i)Y_{i-m-1}\Big], & i \end{cases} \tag{13}$$

$$F_i = \begin{cases} f_i - c_i\phi_{i-m}, & i = 1, \ldots, m \\ f_i, & i = m+1, \ldots, N-1 \end{cases} \tag{14}$$

$\rho_i = \dfrac{a_i h}{2\varepsilon}\coth\left(\dfrac{a_i h}{2\varepsilon}\right)$, x_{N_1} is the turning point.

Lemma

(Discrete Minimum Principle): Suppose $\Psi_0 \geq 0$ and $\Psi_N \geq 0$. Then $L^N\Psi_i \leq 0$ for all $i = 1, 2, \ldots, N-1$ implies that $\Psi_i \geq 0$ for all $i = 0, \ldots, N$.

Lemma

Under the assumption (3)-(6), the solution of the discrete problem (11) with the boundary conditions (12) satisfy

$$||Y||_0 \leq ||f||_0 k^{-1} + C_1(||\phi||_{0,\omega_0} + |\gamma|)$$

where $k = min_{0 \leq i \leq N}\{(b_i - c_i),\ b_i\},\ C_1 \geq 1$ *is a positive constant.*

Theorem

If Y_i *is the solution of the discrete problem (14) applied to the problem (1)-(2) then the local truncation error is given by*

$$|\tau_i^N| = L^N(Y_i - y_i) \leq Ch^{min(\beta,1)}.$$

Numerical Experiments

The maximum absolute errors E_ε^N are evaluated using the double mesh principle for the proposed numerical scheme

$$E_\varepsilon^N = \max_{x\in\bar{\Omega}^N} |v_j^N - v_j^{2N}|$$

where v_j^N andv_j^{2N} are the computed solutions by taking N and $2N$ points, respectively. The numerical rates of convergence are computed using the formula

$$r_N = \log_2(E_\varepsilon^N / E_\varepsilon^{2N}).$$

Example 1 :

$$\varepsilon u''(x) + 4(x-0.5)u'(x) - 2y(x) + 0.5u(x-\delta) = x, \quad x \in (0,1)$$

$$u(1) = 1, \quad u(x) = 1 \qquad -\delta \le x \le 0.$$

$\varepsilon \downarrow$	$\delta = 0.1$		$\delta = 0.2$		$\delta = 0.3$	
	$h = 1/512$	$h = 1/1024$	$h = 1/512$	$h = 1/1024$	$h = 1/512$	$h = 1/1024$
10^{-1}	0.71	0.99	0.99	1.49	0.99	1.02
10^{-2}	0.93	1.01	1.02	1.15	1.02	0.84
10^{-4}	1.04	1.11	1.13	1.2	1.14	1.02
10^{-6}	0.78	0.92	0.83	0.96	0.82	0.89
10^{-8}	0.46	0.52	0.51	0.54	0.5	0.49
10^{-10}	0.46	0.51	0.51	0.53	0.49	0.48
10^{-12}	0.46	0.51	0.51	0.53	0.49	0.48
10^{-14}	0.46	0.51	0.51	0.53	0.49	0.48
10^{-16}	0.46	0.51	0.51	0.53	0.49	0.48
10^{-18}	0.46	0.51	0.51	0.53	0.49	0.48
$\min(\beta, 1) = 0.5$						

Table: The order of convergence for example 1 for various values of δ.

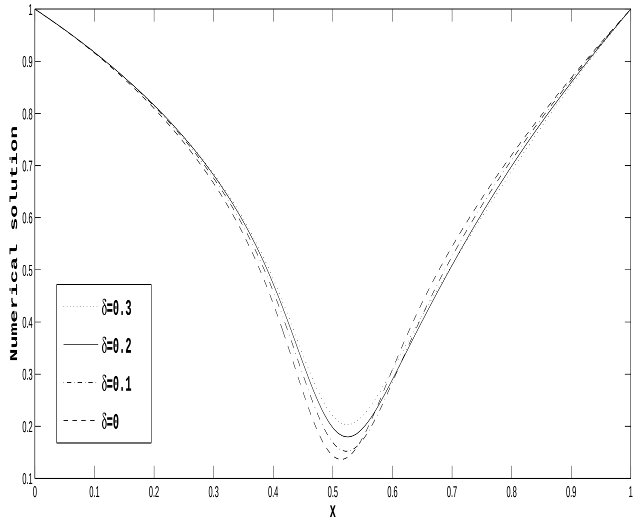

Figure: The numerical solution for example 1 ($\varepsilon = 0.01$).

Example 2:

$$\varepsilon u''(x)+(3(x-0.5)+4(x-0.5)^2)u'(x)-2u(x)+4(x-0.5)^2u(x-\delta)=1, \quad x\in(0,1)$$

$$u(x)=0 \qquad -\delta\le x\le 0, \quad u(1)=1.$$

$\varepsilon \downarrow$	$\delta=0.1$		$\delta=0.2$		$\delta=0.3$	
	$h=1/512$	$h=1/1024$	$h=1/512$	$h=1/1024$	$h=1/512$	$h=1/1024$
10^{-1}	0.96	1.0	1.05	1.06	1.0	0.97
10^{-2}	0.99	1.01	1.03	1.07	1.03	0.99
10^{-4}	1.15	1.13	1.15	1.13	1.15	1.13
10^{-6}	0.85	0.94	0.85	0.94	0.85	0.94
10^{-8}	0.70	0.69	0.7	0.69	0.7	0.69
10^{-10}	0.70	0.68	0.7	0.68	0.69	0.68
10^{-12}	0.70	0.68	0.7	0.68	0.69	0.68
10^{-14}	0.70	0.68	0.7	0.68	0.69	0.68
10^{-16}	0.70	0.68	0.7	0.68	0.69	0.68
10^{-18}	0.70	0.68	0.7	0.68	0.69	0.68
$\min(\beta,1)=0.67$						

Table: The maximum pointwise error and rate of convergence for example 2 for various values of δ.

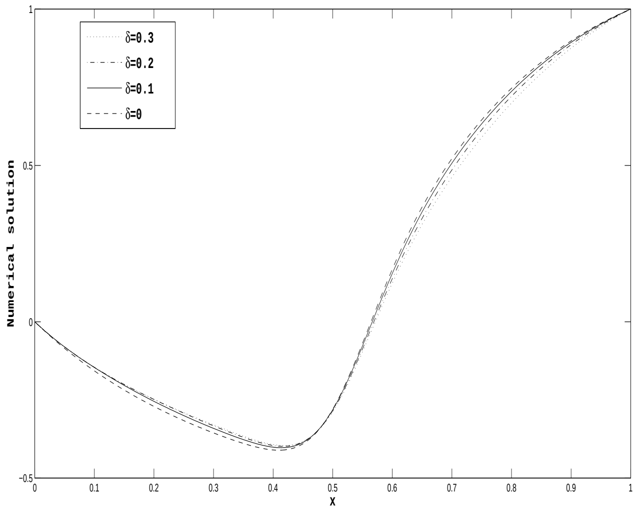

Figure: The numerical solution for example 2 ($\varepsilon = 0.01$).

Example 3:

$$\varepsilon u''(x) + 2xu'(x) - 2u(x) + u(x-\delta) = 0, \quad x \in (-1,1)$$

$$u(1) = 1, \quad u(x) = 1 \qquad -\delta - 1 \leq x \leq -1.$$

$\varepsilon \downarrow$	$\delta = 0.1$		$\delta = 0.2$		$\delta = 0.3$	
	$h = 1/512$	$h = 1/1024$	$h = 1/512$	$h = 1/1024$	$h = 1/512$	$h = 1/1024$
10^{-1}	1.00	1.0	0.98	1.0	0.99	1.0
10^{-2}	1.05	1.03	1.07	1.04	0.98	0.99
10^{-4}	0.99	1.07	1.10	1.0	1.15	1.15
10^{-6}	0.93	0.95	0.94	0.96	0.95	0.97
10^{-8}	0.93	0.95	0.94	0.96	0.95	0.97
10^{-10}	0.93	0.95	0.94	0.96	0.95	0.97
10^{-12}	0.93	0.95	0.94	0.96	0.95	0.97
10^{-14}	0.93	0.95	0.94	0.96	0.95	0.97
10^{-16}	0.93	0.95	0.94	0.96	0.95	0.97
10^{-18}	0.93	0.95	0.94	0.96	0.95	0.97
$\min(\beta, 1) = 1.0$						

Table: The maximum pointwise error and rate of convergence for example 3 for various values of δ.

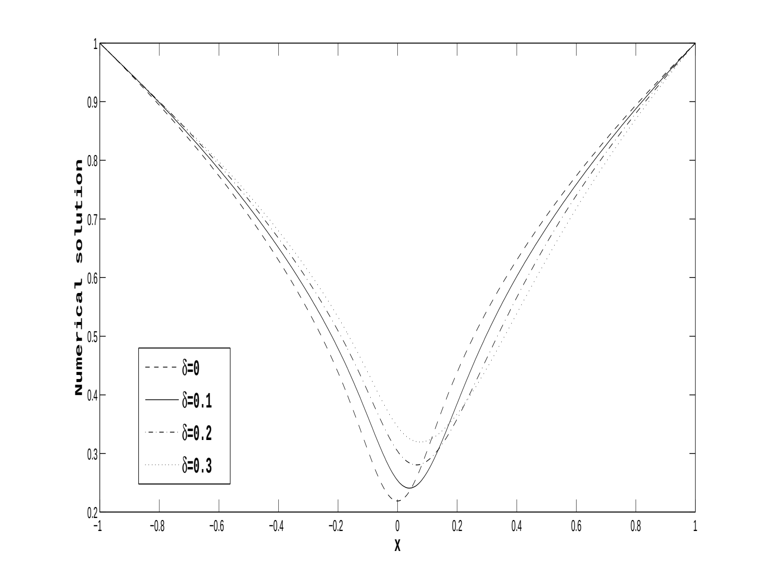

Figure: The numerical solution for example 3 ($\varepsilon = 0.01$).

Summary and conclusions

- Singularly perturbed differential-difference equation exhibiting turning point behaviour and having negative shift in the reaction term is considered.
- A priori estimates are obtained on the solution and its derivatives for the considered problem.
- Finite difference scheme based on Il'in-Allen Southwell fitting is constructed.
- Interpolation is used to tackle the delay term.
- Numerical experiments are carried out to support the theoretical estimates and illustrate the effect of delay on the layer behaviour of the solution.
- It is seen that the rate of convergence is independent of the value of the delay argument.
- It is observed that interior layer is maintained but layer get shifted as the value of the delay argument changes and the amount of shift in the layer depends upon the value of delay argument as well as on the value of the coefficients of the term containing delay.
- Graphs of the solution are plotted for various values of δ for the considered examples to demostrate the effect of the delay argument on the boundary layers.

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