Weighted Hardy inequalities on grand Lebesgue spaces for monotone functions

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Grand Lebesgue spaces

Iwaniec and Sbordone [6], 1992.

For $p > 1$ and $I = (0, 1)$, we define the grand Lebesgue spaces $L^{p)}(I)$ to be the collection of measurable functions f for which

$$\|f\|_{L^{p)}(I)} := \sup_{0<\varepsilon<p-1} \left(\varepsilon \int_0^1 |f(x)|^{p-\varepsilon}\, dx \right)^{1/p-\varepsilon} < \infty.$$

- The spaces are rearrangement invariant Banach function spaces [2].

- **Note.** $L^p \subseteq L^{p)} \subseteq L^{p-\varepsilon}$ for all $0 < \varepsilon \leq p - 1$, where L^p denote Lebesgue spaces.

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- $f \in L^p_w(I) \Rightarrow fw^{1/p} \in L^p(I)$.
- But it is not so for WGLS's.
- **For example.** Take a weight $w(x) = x^\alpha, \;\; \alpha > 0$ and set $f(x) = x^\beta, \;\; \beta > -\alpha - 1$. Then one may easily check that $f \in L^{p)}_w(I)$. But $fw^{1/p} \notin L^{p)}(I)$, since $(fw^{1/p})^{p-\varepsilon}$ is not integrable in $(0, 1)$.

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Hardy's Inequality (initial form)

In 1920, **Hardy** [5] gave the following inequality

$$\int_0^\infty [Hf(x)]^p x^{-p} dx \leq C \int_0^\infty f^p(x) dx, \quad f \geq 0, \; p > 1, \qquad (1)$$

where $Hf(x) := \int_0^x f(t)dt$, commonly known as the Hardy operator.

- **Landau** [9]

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The modern form of Hardy's inequality reads as:

$$\left(\int_0^\infty [Hf(x)]^q u(x)dx\right)^{1/q} \leq C\left(\int_0^\infty f^p(x)v(x)dx\right)^{1/p}, \quad f \geq 0 \quad (2)$$

where u, v are the weight functions.

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Hardy's inequality for $0 \leq f \downarrow$ functions

- Objective to study the Hardy's inequality for non-increasing functions?

- **Lorentz spaces.** (1960's)
  For $0 < p < \infty$ and w a weight function defined on $(0, \infty)$, Lorentz spaces are the spaces defined as -

$$\Lambda_{p,w} := \{f \ \ measurable : \|f\|_{p,w} := \left(\int_0^\infty f^*(t)^p w(t)dt\right)^{1/p} < \infty, \}$$

  where f^* denotes the decreasing rearrangement of $|f|$ defined by

$$f^*(t) = inf\{\lambda > 0 : \mu(\{x > 0 : |f(x)| > \lambda\}) \leq t\}.$$

- The functional $\|f\|_{p,w}$ is a norm on $\Lambda_{p,w}$ if and only if w is decreasing and $p \geq 1$.

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A requirement

An investigation of the structure of the Lorentz spaces $\Lambda_{p,w}$ required the boundedness of the Hardy averaging operator: $f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)ds$ in these spaces. I.e., the study of the inequality

$$\left(\int_0^\infty \left(\frac{1}{t}\int_0^t f^*(s)ds\right)^q w(t)dt\right)^{1/q} \leq C\left(\int_0^\infty f^{*p}(t)w(t)dt\right)^{1/p}, \quad (3)$$

for **non-increasing functions** f^*.

A finding

We define Hardy maximal function as

$$Mf(x) := \sup_{x \in J} \frac{1}{|J|} \int_J |f(y)| dy, \quad x \in [a,b], \quad J \subseteq [a,b].$$

For this operator we have the equivalence

$$(Mf)^*(t) \approx \frac{1}{t} \int_0^t f^*(s) ds, \tag{4}$$

where the estimate from above is due to **Riesz-Wiener** and the **Stein-Herz** gave the estimate from below.

An equivalent study

From (3) and (4) above it is just evident that: to prove the boundedness $M : \Lambda_{p,v} \to \Lambda_{p,u}$ is equivalent to prove that the weighted Hardy inequality

$$\left(\int_0^\infty \left(\frac{1}{t} \int_0^t f(s) ds \right)^q u(t) dt \right)^{1/q} \leq C \left(\int_0^\infty f^p(t) v(t) dt \right)^{1/p}, \quad (5)$$

holds for all positive **non-increasing functions** f on $(0, \infty)$.

Enough to focus on the functions: $0 \leq f \downarrow$

Arino and Muckenhoupt- 1990.

For $1 \leq q = p < \infty$ and $w(x) \geq 0$, the inequality (5) holds for all non-negative non-increasing functions f on $(0, \infty)$ there is a constant C such that for every $r > 0$, the weight w satisfies the condition

$$\int_r^{\infty} \left(\frac{r}{x}\right)^p dx \leq C \int_0^r w(x)\, dx.$$

This class of weights was given a formal nomenclature by **Boyd**, and is now popularly known in literature as the B_p-class.

Example. Weight $x^{\alpha} \in B_p \Leftrightarrow -1 < \alpha < p - 1.$

Hardy averaging operator studied on WGLS's

Meskhi in 2011, studied inequality (5) on WGLS's and proved that: for $1 < p < \infty$, the inequality

$$\|Af\|_{L_w^{p)}(I)} \leq C\|f\|_{L_w^{p)}(I)}$$

holds for all $0 \leq f \downarrow$ if and only if $w \in B_p$.

- **The equivalence** of boundedness of Hardy averaging operator on weighted Lebesgue spaces and weighted grand Lebesgue spaces.

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Our contribution / mentioning the operator studied

We have studied the boundedness of a very general operator on weighted grand Lebesgue spaces - $L^{p)}_w(I)$, from which a number of results available in the literature drops out as special cases. The operator is -

$$T_\psi f(x) := \int_0^x \psi(x, y) f(y)\, dy,$$

ψ being a function from $\mathbb{R}^+ \times \mathbb{R}^+$ to $\mathbb{R}^+$.

- The operator T_ψ was studied on WLS's by **Lai** ([8], 1993), for monotone functions.

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Our contribution/defining a weight class

For $0 < p < \infty$, we denote by $B^{b}_{\psi,\,p}$, the class of weights w for which the inequality

$$\int_r^b \Psi(x,r)^p w(x)\,dx \leq C_1 \int_0^r w(x)\,dx, \quad 0 < r < b,$$

holds for some constant $C_1 > 0$, where $\Psi(x,r) = \int_0^r \psi(x,y)\,dy$ satisfies the following:

P1 $\Psi(x,r) \leq \alpha\ \Psi(x,t)\Psi(t,r)$ for some $\alpha > 0$ and all $0 < r \leq t \leq x$;

P2 $f \downarrow \Rightarrow T_\psi f \downarrow$; and

P3 $\Psi(x,x) \leq D, \ \ x \in (0,b)$ for some constant $D \geq 1$.

Our contribution/One of our Results

Theorem

Let $1 < p < \infty$, *and* **P1, P2, P3** *hold. Then the inequality*

$$\|T_\psi f\|_{L_w^{p)}(I)} \leq C\,\|f\|_{L_w^{p)}(I)} \qquad (6)$$

holds for all non-negative $f \downarrow$ *if and only if* $w \in B^1_{\psi,\,p}$.

- In fact, what we have proven is a little more, that: for non-increasing functions, boundedness of the operator T_ψ on WGLS's is equivalent to its boundedness on WLS's.

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Deduction 1

Corollary

Let $1 < p < \infty$ and ϕ be non-negative locally integrable and $\downarrow$. Then the inequality

$$\|S_\phi f\|_{L_w^{p)}(I)} \leq c\,\|f\|_{L_w^{p)}(I)}$$

holds for all $f \downarrow$ if and only if

$$\int_r^1 \left(\frac{\Phi(r)}{\Phi(x)}\right)^p w(x)\,dx \leq c \int_0^r w(x)\,dx, \quad 0 < r < 1, \qquad (7)$$

where $S_\phi := \frac{1}{\Phi(x)} \int_0^x f(t)\phi(t)dt$, and $\Phi(x) := \int_0^x \phi(t)dt$.

- **Carro and Soria** [3] studied its boundedness on L_w^p-spaces for $0 \leq f \downarrow$ in 1993. **Hint:** Take $\psi(x, y) = \frac{\phi(y)}{\Phi(x)}$.

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Deduction 2

Corollary

Let $1 < q < \infty$ *and consider the operator*

$$A_q f(x) := \frac{1}{x^{1/q}} \int_0^x \frac{f(t)}{t^{1/q'}}\, dt.$$

For $1 < p < \infty$, *the inequality*

$$\|A_q f\|_{L_w^{p)}(I)} \leq c\, \|f\|_{L_w^{p)}(I)}$$

holds for all $f \downarrow$ *if and only if*

$$\int_r^1 \left(\frac{r}{x}\right)^{p/q} w(x)\, dx \leq c \int_0^r w(x)\, dx, \qquad 0 < r < 1. \tag{8}$$

Deduction 2-3

The boundedness of the above operator was studied by **Neugebauer** [11] in 1992. **Hint:** Take $\phi(t) = \frac{1}{qt^{1/q'}}$ in the previous Corollary.

On taking $\psi(t, y) = \frac{1}{t}$ in the above Theorem, it get reduced to the result for the Hardy averaging operator ([1]).

References

- M. A. Arino and B. Muckenhoupt, *Maximal functions on classical Lorentz spaces and Hardy's inequality with weights for non-increasing functions*, Trans. Amer. Math. Soc., 320 (1990),727–735.
- G. Bennet and Sharpley, *Interpolation Operators*, Academic Press,1988 .
- M.J. Carro and J. Soria, *Boundedness of some integral operators*, Can. J. Math., 45(1993), 1155–1166.
- A. Fiorenza, B. Gupta and P. Jain, *The maximal theorem for weighted grand Lebesgue spaces*, Studia Math., 188 (2008), 123–133.
- G.H. Hardy, *Notes on a theorem of Hilbert*, Math. Z., 6 (1920), 314-317.

- T. Iwaniec and C. Sbordone, *On the integrability of the Jacobian under minimal hypotheses*, Arch. Ration. Mech. Anal., 119 (1992), 129–143.
- A. Kufner, L. Maligranda and L.E. Persson, *The Hardy Inequality - About its history and some related results*, Vydavatelsky Sewis, Pilsen, 2007.
- S. Lai, *Weighted norm inequalities for general operators on monotone functions*, Trans. Amer. Math. Soc., 340 (1993), 811–836.
- E. Landau, *Letter to G.H. Hardy*, June 21, 1921.
- A. Meskhi, *Weighted criteria for the Hardy transform under the B_p-condition in grand Lebesgue spaces and some applications*, J. Math. Sci., 178 (2011), 622–636.

C.J. Neugebauer, *Some classical operators on Lorentz space*, Forum Math. 4 (1992), 135–146.

THANK YOU