# GROTHENDIECK'S TENSOR NORMS

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# Overview

- Grothendieck's tensor norms
- Operator spaces
- Tensor products of Operator spaces
- Schur tensor product

# Grothendieck's tensor norms

A Banach space is a complete normed space. For Banach spaces $X$ and $Y$, $X \otimes Y = span\{x \otimes y : x \in X, y \in Y\}$, where $x \otimes y$ is the functional on $B(X^* \times Y^*, \mathbb{C})$ given by $x \otimes y(f,g) = f(x)g(y)$ for $f \in X^*$ and $g \in Y^*$.

- For a pair of arbitrary Banach spaces $X$ and $Y$, the norm on $X \otimes Y$ induced by the embedding $X \otimes Y \to B(X^* \times Y^*, \mathbb{C})$ is known as Banach space injective tensor norm. That is , for $u \in X \otimes Y$, the Banach space injective tensor norm is defined to be

$$\|u\|_\lambda = \sup\left\{\left|\sum_{i=1}^{n} f(x_i)g(y_i)\right| : f \in X_1^*, g \in Y_1^*\right\}.$$

# Grothendieck's tensor norms

Question is
How can we norm on $X \otimes Y$?

- $\|x \otimes y\|_\alpha \leq \|x\|\|y\|$, then, for $u = \sum_{i=1}^{n} x_i \otimes y_i$, by triangle's inequality it follows that $\|u\|_\alpha \leq \sum_{i=1}^{n} \|x_i\|\|y_i\|$. Since this holds for every representation of $u$, so we have $\|u\|_\alpha \leq \inf\{\sum_{i=1}^{n} \|x_i\|\|y_i\|\}$.
- For a pair of arbitrary Banach spaces $X$ and $Y$ and $u$ an element in the algebraic tensor product $X \otimes Y$, the Banach space projective tensor norm is defined to be
$$\|u\|_\gamma = \inf\{\sum_{i=1}^{n} \|x_i\|\|y_i\| : u = \sum_{i=1}^{n} x_i \otimes y_i, n \in \mathbb{N}\}.$$
$X \otimes^\gamma Y$ will denote the completion of $X \otimes Y$ with respect to this norm.

# Tensor Products: Properties

For Banach spaces $X_1$, $X_2$, $\tilde{X}_1$, $\tilde{X}_2$ and $X_3$,

- projective, For the quotient maps $Q : X_1 \to \tilde{X}_1$ and $R : X_2 \to \tilde{X}_2$, the corresponding map $Q \otimes R : X_1 \otimes X_2 \to \tilde{X}_1 \otimes \tilde{X}_2$ extends to a quotient map $Q \otimes^{\gamma} R : X_1 \otimes^{\gamma} X_2 \to \tilde{X}_1 \otimes^{\gamma} \tilde{X}_2$.
- commutative, $X_1 \otimes^{\gamma} X_2$ is isometrically isomorphic to $X_2 \otimes^{\gamma} X_1$;
- associative, $X_1 \otimes^{\gamma} (X_2 \otimes^{\gamma} X_3)$ is isometrically isomorphic to $(X_1 \otimes^{\gamma} X_2) \otimes^{\gamma} X_3$;
- If $\phi_i : X_i \to Y_i, i = 1, 2$, are bounded maps between the Banach spaces, then $\phi_1 \otimes \phi_2 : X_1 \otimes X_2 \to Y_1 \otimes Y_2$ extends to a bounded map $\phi_1 \otimes^{\gamma} \phi_2 : X_1 \otimes^{\gamma} X_2 \to Y_1 \otimes^{\gamma} Y_2$ with $\|\phi_1 \otimes^{\gamma} \phi_2\| = \|\phi_1\|\|\phi_2\|$;
- In general, Banach space projective tensor product does not respect subspaces, that is, if $Y_1$ is a subspace of $X_1$, then $Y_1 \otimes^{\gamma} X_2$ is not, in general, a subspace of $X_1 \otimes^{\gamma} X_2$. However, it does for closed $^*$-subalgebras of $C^*$-algebras. Also, for Banach spaces $X$ and $Y$, $X \otimes^{\gamma} Y$ is a closed subspace of $X^{**} \otimes^{\gamma} Y^{**}$.

# Tensor Products: Properties

The key property of this tensor product is that it linearizes bounded bilinear maps just as the algebraic tensor product linearizes bilinear mappings.

- Let $X$, $Y$ and $Z$ be Banach spaces. Then there exists a natural isometric isomorphism $B(X \times Y, Z) \cong B(X \otimes^{\gamma} Y, Z)$ under which any element $B \in B(X \times Y, Z)$ is mapped to the element $\tilde{B}$ of $B(X \otimes^{\gamma} Y, Z)$ given by $\tilde{B}(x \otimes y) = B(x, y)$ $(x \in X, y \in Y)$.
- In particular, $(X \otimes^{\gamma} Y)^* = B(X \times Y, \mathbb{C})$.

# Tensor Products: Properties

- For Banach algebras $A$ and $B$, $A \otimes^{\gamma} B$ is a Banach algebra,
- It is commutative if and only if both $A$ and $B$ are.
- If $A$ and $B$ are Banach $^*$-algebras, then $A \otimes^{\gamma} B$ is a Banach $^*$-algebra under the natural involution $(a \otimes b)^* = a^* \otimes b^*$ for $a \in A$ and $b \in B$.

# Tensor Products: Properties

For Banach spaces $X_1$, $X_2$, $Y_1$, $Y_2$ and $X_3$,

- If $u_i : X_i \to Y_i$ are isometries for $i = 1, 2$ then the corresponding map $u_1 \otimes u_2 : X_1 \otimes^\lambda X_2 \to Y_1 \otimes^\lambda Y_2$ is an isometry too.
- $X_1 \otimes^\lambda X_2$ is isometrically isomorphic to $X_2 \otimes^\lambda X_1$;
- $X_1 \otimes^\lambda (X_2 \otimes^\lambda X_3)$ is isometrically isomorphic to $(X_1 \otimes^\lambda X_2) \otimes^\lambda X_3$;
- If $\phi_i : X_i \to Y_i, i = 1, 2$, are bounded maps between the Banach spaces, then $\phi_1 \otimes \phi_2 : X_1 \otimes X_2 \to Y_1 \otimes Y_2$ extends to a bounded map $\phi_1 \otimes^\lambda \phi_2 : X_1 \otimes^\lambda X_2 \to Y_1 \otimes^\lambda Y_2$ with $\|\phi_1 \otimes^\lambda \phi_2\| = \|\phi_1\| \|\phi_2\|$;
- In general, Banach space injective tensor product does not respect quotient.

# Tensor Products: Properties

- For Banach algebras $A$ and $B$, $A \otimes^{\lambda} B$ is not a Banach algebra in general.
- Blecher showed that if $A$ and $B$ are unital $C^*$-algebras then $\| \cdot \|_{\lambda}$ is submultiplicative on $A \otimes B$ if and only if $A$ or $B$ is commutative.

- An (concrete) operator space $V$ is a closed subspace of $\mathcal{B}(H)$ together with the natural norms on $M_n(V)$ inherited from $M_n(\mathcal{B}(H)) = \mathcal{B}(H^n)$.
- A normed space $V$ with a sequence of norms

$$\|\cdot\|_n : M_n(V) \to [0, \infty), \quad n \in \mathbb{N}$$

is said to be an (*abstract*) *operator space* if:

(i) $\|v \oplus w\|_{n+m} = \max\{\|v\|_n, \|w\|_m\}$ for all $v \in M_n(V), w \in M_m(V)$, where $v \oplus w$ denotes the matrix $\left(\begin{smallmatrix} v & 0 \\ 0 & w \end{smallmatrix}\right) \in M_{n+m}(V)$.

(ii) $\|\alpha v \beta\|_m \leq \|\alpha\| \|v\|_n \|\beta\|$, for all $\alpha \in \mathbb{M}_{m,n}, \beta \in \mathbb{M}_{n,m}, v \in M_n(V)$.

- An operator $\phi : V \to W$ between operator spaces $V$ and $W$ is said to be *completely bounded* (abbreviated as c.b.) if

$$\|\phi\|_{cb} := \sup\{\|\phi_n\| : n \in \mathbb{N}\} < \infty,$$

where $\phi_n : M_n(V) \to M_n(W)$ is defined by

$$\phi_n((x_{ij})) = (\phi(x_{ij})) \text{ for all } (x_{ij}) \in M_n(V).$$

The set of all completely bounded maps from $V$ into $W$ is denoted by $CB(V, W)$.

- Two operator spaces $V$ and $W$ are said to be *completely isometrically isomorphic* if there exists a completely bounded isometry $\phi : V \to W$ whose inverse is also completely bounded.

# Operator Spaces

**Ruan(1988)**

If $V$ is an abstract operator space, then $V$ is completely isometrically isomorphic to a closed linear subspace of $\mathcal{B}(H)$ for some Hilbert space $H$.

# Introduction to Operator Space: Examples

- By Gelfand-Naimark theorem, every $C^*$-algebra is an operator space.
- For Hilbert spaces $H$ and $K$, $\mathcal{B}(H,K)$ is an operator space by the embedding $\mathcal{B}(H,K) \hookrightarrow \mathcal{B}(H \oplus K)$. The matrix norms are given by $M_n(\mathcal{B}(H,K)) \cong \mathcal{B}(H^n, K^n)$.
- Every Banach space $X$ possess a structure of operator space. To see this, consider $\Gamma = X_1^*$, which is compact in the $w^*$-topology by Alogalu Theorem, so that $C(\Gamma) \subseteq \mathcal{B}(\ell_2(\Gamma))$ is a $C^*$-algebra. The isometric embedding $X \hookrightarrow C(\Gamma)$ via $x \to f_x$, where $f_x(g) := g(x)$ for $g \in \Gamma$, equips $X$ with an operator space structure.

# Quantum tensor product-Tensor Products of operator spaces

For operator spaces $V$ and $W$ and $V \otimes W$ their algebraic tensor product.
Assumptions:

- $\| \cdot \|_n$ on $M_n(V \otimes W)$ satisfying Ruan's Theorem,
- $\| \cdot \|_n$ on $M_n(V \otimes W)$ are subcross matrix norms, where an operator space matrix norm $\| \cdot \|_\mu$ on $V \otimes W$ is called a subcross matrix norm if $\|v \otimes w\|_\mu \leq \|v\|\|w\|$ for all $v \in M_p(V)$ and $w \in M_q(W)$, $p, q \in \mathbb{N}$; if in addition, $\|v \otimes w\|_\mu = \|v\|\|w\|$, then $\| \cdot \|_\mu$ is called a cross matrix norm on $V \otimes W$.
- For operator spaces $V$ and $W$, the *operator space projective tensor product* denoted by $V \widehat{\otimes} W$, is the completion of the algebraic tensor product $V \otimes W$ under the norm

$$\|u\|_\wedge = \inf\{\|\alpha\|\|v\|\|w\|\|\beta\| : u = \alpha(v \otimes w)\beta\},\, u \in M_n(V \otimes W),$$

where infimum runs over arbitrary decompositions with $v \in M_p(V)$, $w \in M_q(W), \alpha \in M_{n,pq}, \beta \in M_{pq,n}$ and $p, q \in \mathbb{N}$ arbitrary.

# Quantum tensor product-Tensor Products of operator spaces

The analogy of $\widehat{\otimes}$ and $\otimes^\gamma$ is not completely transparent from the definition; the following universal properties of $\widehat{\otimes}$ confirm the parallelism.

**Theorem**

[Effros-Ruan and Blecher-Paulsen] If $V$, $W$ and $Z$ are operator spaces, then there are natural completely isometric identifications :

$$CB(V\widehat{\otimes}W, Z) \overset{cb}{\cong} JCB(V \times W, Z).$$

In particular, if $Z = \mathbb{C}$, $(V\widehat{\otimes}W)^* \overset{cb}{\cong} JCB(V \times W, \mathbb{C})$.

where a bilinear map $u : V \times W \to Z$ is said to be *jointly completely bounded* (in short, j.c.b.) if the associated maps $u_n : M_n(V) \times M_n(W) \to M_{n^2}(Z)$ given by

$$u_n\big((v_{ij}),(w_{kl})\big) = \big(u(v_{ij}, w_{kl})\big), \ n \in \mathbb{N}$$

are uniformly bounded, and in this case we denote $\|u\|_{jcb} = \sup_n \|u_n\|$.

# Quantum tensor Products

- Given operator spaces $X \subseteq B(H)$ and $Y \subseteq B(K)$, the norm induced on $X \otimes Y$ via the inclusion $X \otimes Y \subseteq B(H \otimes_2 K)$ is known as the operator space injective tensor norm.

# Tensor Products

- For $u \in M_n(X \otimes Y)$, the *Haagerup tensor norm* is defined as

  $$\|u\|_h = \inf\{\|v\|\|w\| : u = v \odot w, v \in M_{n,p}(X), w \in M_{p,n}(Y), p \in \mathbb{N}\},$$

  where $v \odot w = \left( \sum_{k=1}^{p} v_{ik} \otimes w_{kj} \right)_{ij}$.

- For $C^*$-algebras $A$ and $B$, the Haagerup norm of an element $u \in A \otimes B$ takes a simpler and convenient form given by

  $$\|u\|_h = \inf\left\{ \|\Sigma_i \, a_i a_i^*\|^{1/2} \ \|\Sigma_i \, b_i^* b_i\|^{1/2} : \ u = \Sigma_{i=1}^{n} a_i \otimes b_i, n \in \mathbb{N} \right\}.$$

Like the projective tensor product of Banach spaces and the operator space projective tensor product of operator spaces, the Haagerup tensor norm is naturally associated with the completely bounded bilinear maps through the following identifications:

$$CB(V \times W, Z) \overset{cb}{\cong} CB(V \otimes^h W, Z) \text{ and } CB(V \times W, \mathbb{C}) \overset{cb}{\cong} (V \otimes^h W)^*.$$

By a *completely bounded* (in short, c.b.) bilinear map, we mean a bilinear map $u : V \times W \to Z$ for which the associated maps $u_n : M_n(V) \times M_n(W) \to M_n(Z)$ given by

$$u_n\big((v_{ij}), (w_{kl})\big) = \Big(\sum_k u(v_{ik}, w_{kj})\Big),\ n \in \mathbb{N}$$

are uniformly bounded, and in this case we denote $\|u\|_{cb} = \sup_n \|u_n\|$.

# Tensor Products: Properties

For operator spaces $V_1$, $V_2$ and $V_3$, the product $\widehat{\otimes}$ is

- *commutative*, that is, $V_1 \widehat{\otimes} V_2 \stackrel{cb}{=} V_2 \widehat{\otimes} V_1$;
- *associative*, that is, $V_1 \widehat{\otimes} (V_2 \widehat{\otimes} V_3) \stackrel{cb}{=} (V_1 \widehat{\otimes} V_2) \widehat{\otimes} V_3$;
- *functorial*, that is, if $\phi_i : V_i \to W_i, i = 1, 2$, are completely bounded maps between the operator spaces, then $\phi_1 \otimes \phi_2 : V_1 \otimes V_2 \to W_1 \otimes W_2$ extends to a completely bounded map $\phi_1 \widehat{\otimes} \phi_2 : V_1 \widehat{\otimes} V_2 \to W_1 \widehat{\otimes} W_2$ with $\|\phi_1 \widehat{\otimes} \phi_2\|_{\mathsf{cb}} \leq \|\phi_1\|_{\mathsf{cb}} \|\phi_2\|_{\mathsf{cb}}$;
- *projective*, that is, for closed subspaces $W_i \subseteq V_i, i = 1, 2$, the tensor map $V_1 \otimes V_2 \to (V_1/W_1) \otimes (V_2/W_2)$ extends to a complete quotient map $V_1 \widehat{\otimes} V_2 \to (V_1/W_1) \widehat{\otimes} (V_2/W_2)$;
- In general, operator space projective tensor product is not *injective*, that is, for subspaces $W_i \subseteq V_i, i = 1, 2$, the induced map $W_1 \widehat{\otimes} W_2 \to V_1 \widehat{\otimes} V_2$ is not a complete isometry. However, for $C^*$-algebras, it is injective for finite dimensional $C^*$-subalgebras.

# Tensor Products: Properties

- Haagerup tensor product is associative, projective, injective and functorial. However, it is not commutative.

# Algebraic structure of $A\widehat{\otimes}B$

- (Kumar, Itoh) For $C^*$-algebras $A$ and $B$, $A\widehat{\otimes}B$ is a Banach $*$-algebra, and is a $C^*$-algebra if and only if either $A$ or $B$ is $\mathbb{C}$.
- (Kumar) However, the natural involution is an isometry on $A \otimes^h B$ if and only if $A$ and $B$ are commutative.

# Schur tensor product of operator spaces

For operator spaces $V$ and $W$, and elements $x = [x_{ij}] \in M_n(V)$ and $y = [y_{ij}] \in M_n(W)$, we define an element $x \circ y \in M_n(V \otimes W)$ by $x \circ y = [x_{ij} \otimes y_{ij}]$.

- $x \circ y = [e_{11},\ e_{22},\ e_{33}, \cdots, e_{nn}](x \otimes y)[e_{11},\ e_{22},\ e_{33}, \cdots, e_{nn}]^t$
- Each element $u$ in $M_p(V \otimes W)$, $p \in \mathbb{N}$, can be written as $u = \alpha(x \circ y)\beta$ for some $x \in M_n(V)$, $y \in M_n(W)$, $\alpha \in M_{p,n}$, and $\beta \in M_{n,p}$, $n \in \mathbb{N}$, and we define

$$\|u\|_s = \inf\{\|\alpha\|\|x\|\|y\|\|\beta\|\}$$

where infimum is taken over arbitrary decompositions as above. Let $V \otimes_s W = (V \otimes W, \|\cdot\|_s)$, and define the Schur tensor product $V \otimes^s W$ to be the completion of $V \otimes W$ in this norm.

**Note:** If $u \in M_p(V \otimes W)$ there exist $n \in \mathbb{N}$, $v \in M_n(V)$, $w \in M_n(W)$, $\alpha \in M_{p,n}$ and $\beta \in M_{n,p}$ such that $u = \alpha(v \circ w)\beta$.

# Schur tensor product of operator spaces

**Theorem**

For operator spaces $V$ and $W$, $\|\cdot\|_s$ is an operator space matrix norm on $V \otimes W$.

Given operator spaces $V$, $W$ and $Z$, a bilinear map $\varphi : V \times W \to Z$ is said to be Schur bounded bilinear map if the associated maps $\varphi_n : M_n(V) \times M_n(W) \to M_n(Z)$ given by

$$\varphi_n\big((v_{ij}), (w_{ij})\big) = \big(\varphi(v_{ij}, w_{ij})\big), \ n \in \mathbb{N}$$

are uniformly bounded, and in this case we denote $\|\varphi\|_{sb} = \sup_n \|\varphi_n\|$.

**Proposition**

If $V$, $W$ and $X$ are operator spaces, then there is a natural isometric identification

$$CB(V\widehat{\otimes}W, X) = SB(V \times W, X).$$

The above identification yields a new formula for the Schur norm :

$$\|u\|_s = \sup\{|\varphi(u)| : \varphi \in SB(V \times W, \mathbb{C}), \|\varphi\| \leq 1\}.$$

# Tensor Products: Properties

For operator spaces $V_1$, $V_2$ and $V_3$, the product $\widehat{\otimes}$ is

- *commutative*, that is, $V_1 \otimes^s V_2 \stackrel{cb}{=} V_2 \otimes^s V_1$;
- *functorial*, that is, if $\phi_i : V_i \to W_i, i = 1, 2$, are completely bounded maps between the operator spaces, then $\phi_1 \otimes \phi_2 : V_1 \otimes V_2 \to W_1 \otimes W_2$ extends to a completely bounded map $\phi_1 \otimes^s \phi_2 : V_1 \otimes^s V_2 \to W_1 \otimes^s W_2$ with $\|\phi_1 \otimes^s \phi_2\|_{\mathsf{cb}} \leq \|\phi_1\|_{\mathsf{cb}} \|\phi_2\|_{\mathsf{cb}}$;

# Tensor Products: Properties

- *projective*, that is, for closed subspaces $W_i \subseteq V_i, i = 1, 2$, the tensor map $V_1 \otimes V_2 \to (V_1/W_1) \otimes (V_2/W_2)$ extends to a complete quotient map $V_1 \otimes^s V_2 \to (V_1/W_1) \otimes^s (V_2/W_2)$;
- Like the operator space projective tensor product, the Schur tensor product is not *injective*, that is, for subspaces $W_i \subseteq V_i, i = 1, 2$, the induced map $W_1 \otimes^s W_2 \to V_1 \otimes^s V_2$ is not a complete isometry. However, it behaves well for completely complemented subspaces. In particular, if $E$ and $F$ are finite dimensional $C^*$-subalgebras of the $C^*$-algebras $A$ and $B$, respectively. Then $E \otimes^s F$ is a closed $^*$-subalgebra of $A \otimes^s B$. Also, for von Neumann algebras $M$ and $N$, $Z(M) \otimes^s Z(N)$ is a closed $^*$-subalgebra of $M \otimes^s N$.

# Tensor Products: Properties

- *Associative*, For finite dimensional operator spaces $X$, $Y$ and any operator space $Z$, $X \otimes^s (Y \otimes^s Z)$ is bi-continuously isomorphic to $(X \otimes^s Y) \otimes^s Z$.
- For commutative $C^*$-algebras $A$ and $C$ with identity and a finite dimensional $C^*$-algebra $B$, $A \otimes^s (B \otimes^s C)$ is bi-continuously isomorphic to $(A \otimes^s B) \otimes^s C$.

# Schur tensor product of completely contractive Banach algebras

**Theorem**

For completely contractive Banach algebras $A$ and $B$, $A \otimes^s B$ is a Banach algebra, and it is $^*$-algebra provided both $A$ and $B$ have completely isometric involution. Furthermore, if $A$ and $B$ possess bounded approximate identities then $A \otimes^s B$ has a bounded approximate identity.

- $A \otimes^s B$ is a $C^*$-algebra if and only if either $A = \mathbb{C}$ or $B = \mathbb{C}$.
- $A \otimes^s B$ is an operator algebra if and only if either $A$ or $B$ is finite dimensional.

# Related Books


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THANK YOU