Frame System in Banach Spaces

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Basics

- A sequence $\{x_k\}_{k=1}^{\infty}$ in $\mathcal{H}$ is a frame for $\mathcal{H}$ if $\exists\ A, B > 0$ such that

$$A||x||^2 \leq \sum_{k=1}^{\infty} | < x, x_k > |^2 \leq B||x||^2, \quad \forall x \in \mathcal{H}.$$

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- A and B $\rightarrow$ frame bounds.
- For frame $\{x_n\}$ in $\mathcal{H}$, $T : l^2(\mathbb{N}) \rightarrow \mathcal{H}$,

$$T(\{c_k\}) = \sum_{k=1}^{\infty} c_k x_k.$$

is pre- frame operator.

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- Frame operator $S = TT^* : \mathcal{H} \to \mathcal{H}$ defined as

$$S(x) = \sum_{k=1}^{\infty} < x, x_k > x_k \quad x \in \mathcal{H}.$$

- Let $\{x_n\}$ be a frame for $\mathcal{H}$ with frame operator $\mathcal{S}$. Then

$$x = \sum_{k=1}^{\infty} < x, S^{-1}x_k > x_k, \quad x \in \mathcal{H}.$$

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- Let $\{x_n\}$ in $\mathcal{E}$ and $\{f_n\}$ in $\mathcal{E}^*$. Then $(\{f_n\}, \{x_n\})$ is an atomic decomposition of $\mathcal{E}$ with respect to $\mathcal{E}_d$, if

  (a) $\{f_n(x)\} \in \mathcal{E}_d$, $\forall x \in \mathcal{E}$.

  (b) $A||x||_E \leq ||\{f_n(x)\}||_{E_d} \leq B||x||_E, \quad x \in \mathcal{E}$

  (c) $x = \sum_{k=1}^{\infty} f_k(x)x_k$, $\forall x \in \mathcal{E}$.

  $A, B \to$ atomic bounds.

Banach Frame

Let $\{f_n\}$ in $\mathcal{E}^*$ and $\mathcal{S} : \mathcal{E}_d \to \mathcal{E}$. Then $(\{f_n\}, \mathcal{S})$ is Banach frame for $\mathcal{E}$ w.r.to $\mathcal{E}_d$ if

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(a) $\{f_n(x)\} \in \mathcal{E}_d$, $\forall x \in \mathcal{E}$.

(b) $A\|x\|_{\mathcal{E}} \leq \|\{f_n(x)\}\|_{\mathcal{E}_d} \leq B\|x\|_{\mathcal{E}}, \quad x \in \mathcal{E}$.

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(a) $\{f_n(x)\} \in \mathcal{E}_d$, $\forall x \in \mathcal{E}$.

(b) $A\|x\|_{\mathcal{E}} \leq \|\{f_n(x)\}\|_{\mathcal{E}_d} \leq B\|x\|_{\mathcal{E}}, \quad x \in \mathcal{E}$.

(c) $\mathcal{S}$ is bounded linear operator s.t.

$$\mathcal{S}(f_n(x)) = x \quad x \in \mathcal{E}.$$

$A, B \to$ frame bounds.

(i) Tight frame if $A = B$

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(iii) Exact frame if there exists no reconstruction operator $\mathcal{S}_0$ such that $(\{f_n\}_{n \neq i}, \mathcal{S}_0)$ $(i \in \mathbb{N})$ is a Banach frame for $\mathcal{E}$.

Examples

- For $\mathcal{E} = c_0$ and $\{f_n\}$ in $\mathcal{E}^*$ defined as

$$f_n(x) = \xi_n, \ (n \in \mathbb{N}) \quad x = \{\xi_n\} \in \mathcal{E}.$$

Then there exist $\mathcal{E}_d = \{\{f_n(x)\} : x \in \mathcal{E}\}$ with norm

$$||\{f_n(x)\}||_{\mathcal{E}_d} = ||x||_{\mathcal{E}}$$

and $S : \mathcal{E}_d \to \mathcal{E}$ defined by

$$S(\{f_n(x)\}) = x, \quad x \in \mathcal{E}.$$

Thus $(\{f_n\}, S)$ is a Banach frame for $\mathcal{E}$ w. r. to $\mathcal{E}_d$.

- ℓ^∞/c_0 does not have any Banach frame.

Few Results on Banach Frames

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(2) A Banach space with an atomic decomposition is separable. However, the converse need not be true.

(3) Every separable Banach space has a normalized tight and exact Banach frame.

(4) Let $\mathcal{E}$ be a Banach space having a Banach frame. Then, $\mathcal{E}$ has a normalized tight and exact Banach frame.

Frames in Subspaces of Banach Spaces

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- Let $(\{f_n\},\ \mathcal{S})(\{f_n\} \subset \mathcal{E}^*, \mathcal{S} : \mathcal{E}_d \to \mathcal{E})$ be a Banach frame for $\mathcal{E}$ and $\mathcal{G}$ be a closed subspace of $\mathcal{E}$. Then, $\exists\ \mathcal{G}_d$ and $T : \mathcal{G}_d \to \mathcal{G}$ such that $(\{f_n|_{\mathcal{G}}\},\ \mathcal{T})$ is a Banach frame for $\mathcal{G}$ w.r.to $\mathcal{G}_d$

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- Let $(\{f_n\},\ \mathcal{S})(\{f_n\} \subset \mathcal{E}^*, \mathcal{S} : \mathcal{E}_d \to \mathcal{E})$ be a Banach frame for E. Then, for every $\{n_k\}$, $\exists$ a closed subspace $\mathcal{G}$ of $\mathcal{E}$ and $T : \mathcal{G}_d \to \mathcal{G}$ such that $(\{f_{n_k}|_{\mathcal{G}}\},\ \mathcal{T})$ is Banach frame for $\mathcal{G}$ w.r.t $\mathcal{G}_d$.

Frames in Subspaces of Banach Spaces

- Let $(\{f_n\}\ \mathcal{S})(\{f_n\} \subset \mathcal{E}^*,\ \mathcal{S}{:}\mathcal{E}_d$ to $\mathcal{E})$ be a exact Banach frame for $\mathcal{E}$. Then, for every $\{n_k\}$, $\exists$ a closed subspace $\mathcal{G}$ of $\mathcal{E}$ and $T : \mathcal{G}_d$ to $\mathcal{G}$ such that $(\{f_{n_k}|_{\mathcal{G}}\}\ \mathcal{T})$ is exact Banach frame for $\mathcal{G}$ w.r.to $\mathcal{G}_d$

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- Let $(\{f_n\}\ \mathcal{S})(\{f_n\} \subset \mathcal{E}^*$, $\mathcal{S}$:$\mathcal{E}_d$ to $\mathcal{E})$ be a exact Banach frame for $\mathcal{E}$. Then, for every $\{n_k\}$, $\exists$ a closed subspace $\mathcal{G}$ of $\mathcal{E}$ and $T : \mathcal{G}_d$ to $\mathcal{G}$ such that $(\{f_{n_k}|_{\mathcal{G}}\}\ \mathcal{T})$ is exact Banach frame for $\mathcal{G}$ w.r.to $\mathcal{G}_d$
- Let $\mathcal{G} \subset \mathcal{E}$ such that $\mathcal{G}$ and E/G have Banach frames. Then E also have a Banach Frame.

For $\mathcal{E} = \ell^\infty$ and $\mathcal{G} = c_0$. $\mathcal{E}$ and $\mathcal{G}$ have Banach frame, however $\mathcal{E}/\mathcal{G}$ has no Banach frame.

Banach Frame System

- Let $(\{f_n\}, \mathcal{S})(\{f_n\} \subset \mathcal{E}^*, \mathcal{S} : \mathcal{E}_d \to \mathcal{E})$ is a Banach frame for $\mathcal{E}$ w.r.to $\mathcal{E}_d$. Let $\{\phi_n\}$ in $\mathcal{E}^{**}$ be such that

$$\phi_i(f_j) = \delta_{i,j}, \quad i, j \in \mathcal{N}.$$

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$$\phi_i(f_j) = \delta_{i,j}, \quad i, j \in \mathcal{N}.$$

  If $\exists$ $\mathcal{E}_d^*$ and $\mathcal{T} : {\mathcal{E}^*}_d \to \mathcal{E}^*$ such that $(\{\phi_n\}, \mathcal{T})$ is a Banach frame for $\mathcal{E}^*$ w.r.to $\mathcal{E}_d^*$, then

$$((\{f_n\}, \mathcal{S}), (\{\phi_n\}, \mathcal{T}))$$

  is a Banach frame system for E.
- $(\{\phi_n\}, \mathcal{T})$ is called an admissible Banach frame to $(\{f_n\}, \mathcal{S})$.

Examples of Banach frame System

Let $\mathcal{E} = l^1$ and $\{f_n\} \subset \mathcal{E}^*$. Define $\{g_n\} \subset \mathcal{E}^{**}$ by

$$g_n(x) = \xi_n, \quad x = \{\xi_n\} \in \ell^\infty, \, n \in \mathbb{N}.$$

Let $\mathcal{E}_d = \{\{f_n(x)\} : x \in \mathcal{E}\}$ and $(\mathcal{E}^*)_d = \{\{g_n(f) : f \in \mathcal{E}^*\}$. Then $\mathcal{E}_d$ and $(\mathcal{E}^*)_d$ are associated Banach space with norms given by $||\{f_n(x)\}||_{\mathcal{E}_d} = ||x||_{\mathcal{E}}, \, x \in \mathcal{E}$ and $||\{g_n(f)\}||_{(\mathcal{E}^*)_d} = ||f||_{\mathcal{E}^*}, \, f \in \mathcal{E}^*$ respectively. Define $S : \mathcal{E}_d \to \mathcal{E}$ by $S(\{f_n(x)\}) = x, \, x \in \mathcal{E}$ and $T : (\mathcal{E}^*)_d \to \mathcal{E}^*$ by $T(\{g_n(f)\}) = f, \, f \in \mathcal{E}^*$. Then $((\{f_n\}, \mathcal{S}), (\{g_n\}, \mathcal{T}))$ is a Banach frame system for $\mathcal{E}$.

Examples of Banach frame System

ℓ^∞ has no Banach frame system however it posseses a Banach frame.

Towards Uniqueness

As $[f_n] \neq \mathcal{E}^*$, let $e \in \mathcal{E}^* \setminus [f_n]$, $\exists$ $g_0 \in \mathcal{E}^{**}$ such that

$$g_0(e) \neq 0$$

and

$$g_0([f_n]) = 0.$$

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Define $\{\phi_n\} \subset \mathcal{E}^{**}$ by

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Then $(\{\phi_n\}, T_0)$ is another admissible Banach frame to $(\{f_n\},\ \mathcal{S})$, where

$$T_0 : (\mathcal{E}^*)_d \to \mathcal{E}^*$$

by $T_0(\{g_n(f)\}) = f,\ f \in \mathcal{E}^*$

Theorems

Theorem

Let$((\{f_n\}, \mathcal{S}), (\{g_n\}, \mathcal{T}))$ be a Banach frame system for $\mathcal{E}$. Then, $(\{g_n\}, \mathcal{T})$ is the unique admissible Banach frame to $(\{f_n\},\ \mathcal{S})$ if and only if $[f_n] = \mathcal{E}^*$

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Theorem

If $(\{f_n\},\ \mathcal{S})$ is an exact Banach frame for $\mathcal{E}$ such that

$$\bigcap_{n=1}^{\infty}\widetilde{[f_i]}_{i=n+1}^{\infty}=\{0\}.$$

Then, $\mathcal{E}$ has a Banach frame system.

Theorems

Theorem

If $((\{f_n\}, \mathcal{S}), (\{\phi_n\}, \mathcal{T}))$ is a Banach frame system for $\mathcal{E}$ such that each ϕ_n is $weak^*$ continuous, then

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Theorem

If $\mathcal{E}^*$ has a Banach frame $(\{\phi_n\}, \mathcal{T})$, where ϕ_n is $weak^*$ continuous for each $n \in \mathbb{N}$, then $\mathcal{E}$ has a Banach frame system.

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