Hilbert-Kunz multiplicity and Hilbert-Kunz function

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Consider a ring, for example,
a quotient of a polynomial ring over a field, say

$$R = \frac{k[X_1, \ldots, X_n]}{(f_1, \ldots, f_m)},$$

where $f_1, \ldots, f_m \in k[X_1, \ldots, X_n]$.

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where $f_1, \ldots, f_m \in k[X_1, \ldots, X_n]$.

Such rings are called **geometric rings** as one can associate a geometric object X_R, called variety, to such a ring R:

$$\begin{aligned} X_R &= \text{the zero set of } \{f_1, \ldots, f_m\} \text{ in } k^n \\ &= \{(a_1, \ldots, a_n) \in k^n \mid f_i(a_1, \ldots, a_n) = 0, \ \forall 1 \leq i \leq n\}. \end{aligned}$$

We attach Zariski topology to such sets, as follows:

$$\text{ideals of } R \longleftrightarrow \quad \text{closed sets of } X_R.$$

$$I \to V(I) = \{(a_1, \ldots, a_n) \in X_R \mid g(a_1, \ldots, a_n) = 0, \ \forall \ g \in I\}.$$

Moreover a maximal ideal of R, given by

$$\mathbf{m} = (X_1 - a_1, \ldots, X_n - a_n) \to x_{\mathbf{m}} = (a_1, \ldots, a_n),$$

which is a point of X_R.

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Therefore in Zariski topology, points are closed sets, infact it is the weakest topology on X_R, where points are closed sets.

But this is the prefered topology in algebraic geometry because

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algebraic geometry is the subject which builds **geometry** from **algebra.**

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Philosophy: A property P of a ring R (or of a variety X_R) is 'good/reasonable' if it is an 'open' property,

This means if P holds at a point $x \in X_R$ then it holds in a Zariski open neighbourhood of x in X_R.

Why ‘open’?
Because an open set is a dense set in X_R, so if P holds on an open set then it holds almost everywhere.
Moreover, intersection of two nonempty open sets is a nonempty open set so
if property P_1 holds on an open set U_1 and P_2 holds on an open set U_2 then properties P_1 and P_2 hold on a nonempty open set.

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Moreover such properties tend to satisfy *local global principal*.

This often reduces the work, to study the property at the local rings (means rings localized at a point), which could be easier to deal with.

For example, let $(R, \mathbf{m})$ be a commutative Noetherian ring of dimension d with the maximal ideal $\mathbf{m}$ of R. Then $R/\mathbf{m}$ is a field and $R/\mathbf{m}^n$ is filtered by finite $R/\mathbf{m}$-modules and therefore has a finite length.

One classical numerical invariant is the *Hilbert-Samuel function* of R at $\mathbf{m}$, namely a function

$$HS(R, \mathbf{m}) : \mathbb{N} \to \mathbb{N},$$

given by $n \mapsto \ell(R/\mathbf{m}^n)$.

It is a polynomial function, *i.e.*, for $n >> 0$,

$$HS(R, \mathbf{m})(n) = e_0 \binom{n+d-1}{d} - e_1 \binom{n+d-2}{d-1} + \cdots + (-1)^d e_d,$$

where

$$e_0 = e_0(R, \mathbf{m}) = \lim_{n \mapsto \infty} \frac{d!}{n^d} HS(R, \mathbf{m})(n)$$

is the *classical multiplicity* of R at $\mathbf{m}$ and is a positive integer.

$e_0(R, \mathbf{m})$ is a numerical invariant characterizing the singularity of X_R around the neigbourhood of the point $x_{\mathbf{m}}$.

For example

1. If X_R is smooth at a point $x_{\mathbf{m}}$ then $e_0(R, \mathbf{m}) = 1$.
   Infact, in general, *if* $(R, \mathbf{m})$ *is an integral domain, then* $e_0(R, \mathbf{m}) = 1$ *if and only if* X_R *is smooth at the point* $x_{\mathbf{m}}$.
2. If R is plane curve with a node at $x_{\mathbf{m}}$ then $e_0(R, \mathbf{m}) = 2$.

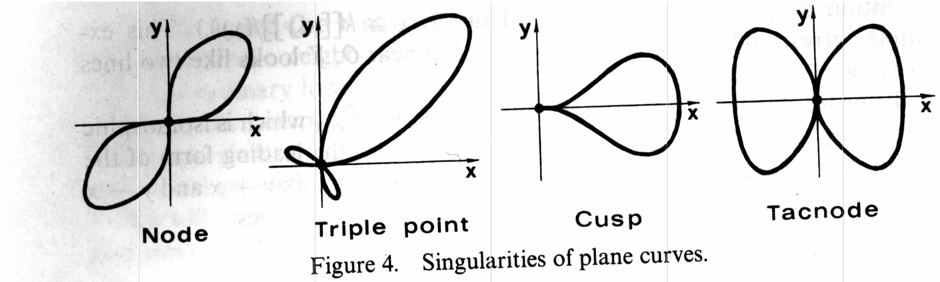

Figure 4. Singularities of plane curves.

$$xy = x^6 + y^6 \text{ (Node)}, \quad x^3 = y^2 + x^4 + y^4 \text{ (Cusp)},$$

$$x^2 = x^4 + y^4 \text{ (Tacnode)}, \quad x^2y + xy^2 = x^4 + y^4 \text{ (Triple Point)}$$

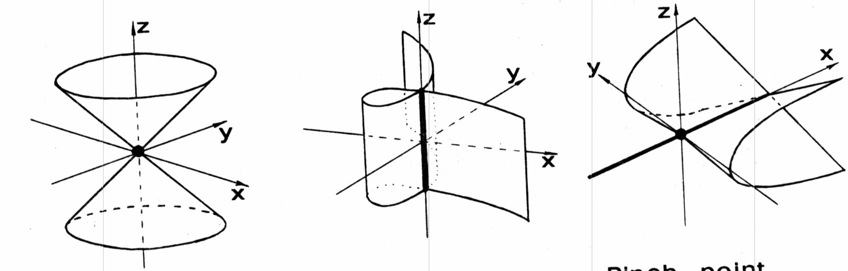

Figure 5. Surface singularities.

$x^2 + y^2 = z^2$ (Conical Double Point), $xy = x^3 + y^3$ (Double line),

$xy^2 = z^2$ (Pinch Point)

In general, larger the multiplicity $e_0(R, \mathbf{m})$, more singular is the variety at the point $x_{\mathbf{m}}$.

These examples also demonstrate that the ‘smoothness’ is an open property. Moreover e_0 is a well behaved invariant in the sense,

1. it does not change after taking a general hyperplane section, and
2. remains constant in a flat family.
3. it has a cohomological interpretation.

Definition: The *characteristic* of a ring R is the least positive integer m such that $m \cdot 1_R = 0$, moreover if there is no such integer then charactristic of the ring is 0.
E.g., $\mathbb{Z}/p\mathbb{Z}$ is a ring of characteristic p, and $\mathbb{Z}$ is of characteristic 0.
Many times it is easier to solve a problem by going to reduction *mod p*.
Recall

Eisenstein's criteria for checking irreducibility

Let $f(X) = a_nX^n + a_{n-1}X^{n-1} + \cdots + a_0 \in \mathbb{Z}[X]$, with $n \geq 2$. Suppose there is a prime number p such that

$$p/a_{n-1}, p/a_{n-2}, \ldots, p/a_0, \quad p \not| a_n, \quad p^2 \not| a_0$$

then $f(X)$ is irreducible in $\mathbb{Q}[X]$.

PROOF. Let $f(X) = g(X)h(X)$, for some nonconstant polynomials $g(X), h(X) \in \mathbb{Z}[X]$.
Consider the canonical map $\mathbb{Z}[X] \to \mathbb{Z}/p\mathbb{Z}[X]$. Then

$$f(X) \mapsto \bar{a}_nX^n = \bar{g}(X)\bar{h}(X).$$

Now

$$\mathbb{Z}/p\mathbb{Z}[X] \text{ UFD} \implies \bar{g}(X) = \bar{g}_mX^m, \quad \bar{h}(X) = \bar{h}_{n-m}X^{n-m}.$$

This implies p divides the constant coefficients of $g(X)$ and $h(X)$
Hence p^2 divides the constant coefficient of $f(X)$, which contradicts the hypothesis.

In characteristic p, we also have the Frobenius map, namely

$$F : R \to R \text{ given by } x \mapsto x^p,$$

this is a ring homomorphism as $(x+y)^p = x^p + y^p$.

Now consider a 'char p' invariant of a ring which relates to 'char p' features of the underlying ring.

Definition: For a commutative local ring $(R, \mathbf{m})$ of characteristic $p > 0$, we define *Hilbert-Kunz function* $HK(R, \mathbf{m}) : \mathbb{N} \to \mathbb{N}$, as

$$HK(R, \mathbf{m})(p^n) = \ell(R/\mathbf{m}^{[p^n]}),$$

where

$$\begin{aligned} \mathbf{m}^{[p^n]} &= \text{the ideal generated by } \{x^{p^n} | x \in \mathbf{m}\} \\ &= F^n(\mathbf{m})R, \end{aligned}$$

where $F^n : R \to R$ is the n-th iterated Frobenius map, given by, $x \mapsto x^{p^n}$, and

$$e_{HK}(R, \mathbf{m}) = \lim_{q \mapsto \infty} \frac{1}{q^d} HK(R, \mathbf{m})(q)$$

is called *Hilbert-Kumz multiplicity*.

Monsky (1980's) proved:

$$HK(R,\mathbf{m})(q) = e_{HK}(R,\mathbf{m})q^d + O(q^{d-1}), \text{ where } q = p^n$$

where $e_{HK}(R,\mathbf{m}) \in \mathbb{R}^+$, and there is a constant C such that

$$|O(q^{d-1})| \leq Cq^{d-1}.$$

One can easily see that

$$\frac{1}{d!}e(R,\mathbf{m}) \leq e_{HK}(R,\mathbf{m}) \leq e(R,\mathbf{m}).$$

Theorem

(Monsky) If $\dim R = 1$,

$$HK(R)(q) = e_0(R, \mathbf{m})q^n + \Delta_n,$$

$q = p^n$*, where* Δ_n *is a periodic function of n, for* $n >> 0$.

Open question (Monsky 1980's): *Is* $e_{HK}(R)$ *a rational number?*

We recall some examples for which $e_{HK}(R)$ or $HK(R)$ has been computed, by various people K. Pardue, R.Buchweitz-C.Chen, P.Monsky, C.Hans-Monsky, A. Conca, Eto, W. Bruns, Watanabe-Yoshida etc.

1. $R =$ a polynomial ring over a field.
2. $R = k[X, Y, Z]/(f)$ a plane curve. Then
   1. R a nodal plane curve.
   2. R an elliptic plane curve and char $k \neq 2$, if R an elliptic plane curve and char $k = 2$.
3. Diagonal hypersurfaces.
4. monomial ideals and binomial hypersurfaces.
5. monoid rings, toric ring.
6. More recently, trinomial plane curves.
7. (N.Fakhruddin, V.T.) R a homogeneous cordinate ring of

1. an elliptic curve X with respect to any line bundle $\mathcal{L}$ of degree ≥ 3, or
2. a full flag variety X embedded by an anticanonical line bundle $\mathcal{L}$, Infact here we have

$$HK(X, \mathcal{L})(q) = e_{HK}q^d + C_1(n)q^{d-1} + \cdots + C_d(n),$$

where $q = p^n$ and $C_i(n)$ are periodic functions of n.
3. (–) Hirzebruch surface $X = F_a$, for $a \geq 1$, with respect to any ample line bundle $\mathcal{L} = \mathcal{O}(cD_1 + dD_4)$

$$HK(X, \mathcal{L})(q) = e_{HK}q^3 + P(a, c, d)q^2 + C_1(a, c, d)q + C_2(a, c, d),$$

where P is a polynomial in a, c, d and C_1 and C_2 are a periodic and doubly periodic functions invoving a, c and d.

We note that the above examples (except the last three) are hypersurfaces of special types or monomial rings. Hence one is able to use combinatorial techniques, grobner bases etc.

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Why is e_{HK} interesting?

Main reason: $e_{HK}(R)$ is a subtler invariant than $e(R)$ and it reveals more information about the char p features of the ring R.

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1. If R s a domain then $e_{HK}(R, \mathbf{m}) = 1 \iff R$ is smooth at $\mathbf{m}$.
2. $$e_{HK}(R) < 1 + (1/d!) \implies R \text{ is } F-\text{rational},$$

   where d is the dimension of R.
   We recall that F-rationality is a substitute for rational singularity in char p, as the problem of existence of a resolution of singularity, for the varieties in char p, is still open.

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3. Conjecture (Watanabe-Yoshida): For every nonregular ring R of dimension d and of char p,
   1. $e_{HK}(R) \leq e_{HK}(A_{p,d})$, where $A_{p,d}$ is a quadratic d-dimesional hypersurface in char p,
      $$A_{p,d} = \mathbb{F}_p[[X_0, \ldots, X_d]]/(X_0^2 + \cdots + X_d^2).$$
   2. if equality holds then $R \cong A_{p,d}$ analytically (upto base change by a field).

Let $R = \oplus_{m \geq 0} R_m$ be a standard graded ring of dimension d over a field of characateristic $p > 0$.

Let $\mathbf{m} = \oplus_{m \geq 1} R_m$

Then $X = Proj\ R$ is a projective variety.

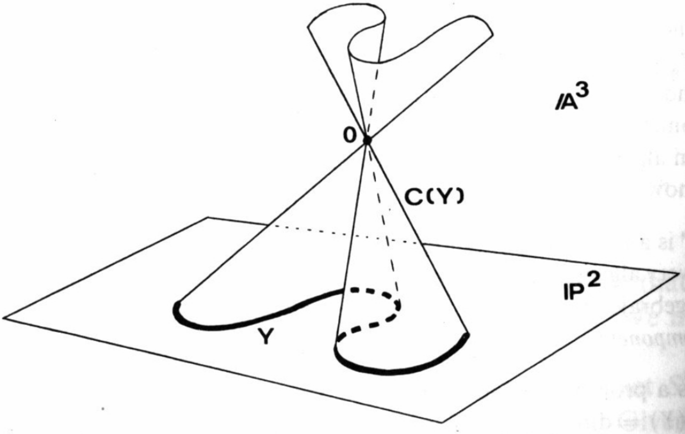

Figure 1. The cone over a curve in $\mathbf{P}^2$.

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Now

$$HK(R, \mathbf{m})(q) = \ell(\frac{R}{\mathbf{m}^{(q)}}) = \ell(R_0) + \ell(R_1) + \cdots + \ell(R_{q-1})$$

$$+\sum_{m\geq 0} \ell(\frac{R_{m+q}}{Im\ R_1^{(q)} \otimes R_m}).$$

Consider the canonical short exact sequence

$$0 \to V \to H^0(X, \mathcal{O}_X(1)) \otimes \mathcal{O}_X \to \mathcal{O}_X(1) \to 0.$$

Note that V is a vector bundle on X, (i.e., if $\mathcal{O}_X$ is the sheaf of rings then V is a sheaf of free modules on $\mathcal{O}_X$).
Let $F^s : X \to X$ be the s-th iterated Frobenius map.

Then

$$0 \to H^0(X, F^{s*}(V)(m)) \to R_1^{[q]} \otimes R_m \xrightarrow{\phi_{m,q}} R_{m+q} \to H^1(X, F^{s*}(V)(m)) \to 0,$$

where

$$HK(R, \mathbf{m})(q) = \ell(R_0) + \ell(R_1) + \cdots + \ell(R_{q-1}) + \sum_{m \geq 0} \ell(coker\ \phi_{m,q}).$$

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Thus the computation of e_{HK} is reduced to the computation of the cohomologies of a vector bundle whose rank and degree we know.

We can also compute cokernel of $\phi_{m,q}$ as the cokernel of the following Frobenius twisted map for $\mathcal{L} = \mathcal{O}_X(1)$,

$$\begin{array}{ccc} R_1^{[q]} \otimes R_m & \xrightarrow{\phi_{m,q}} & R_{m+q} \\ \downarrow & & \downarrow \\ H^0(X, \mathcal{L}) \otimes H^0(X, F_*^s \mathcal{L}^m) & \longrightarrow & H^0(X, F_*^s \mathcal{L}^{m+q}) \end{array}$$

In the last three examples, we use this Frobenius twisted map and the following

1. Atiyah's and Oda's classification of vector bundles for elliptic curves, and
2. the result of Anderson-Haboush: $F_*(\mathcal{L}(p-1)\rho)$ is a trivial bundle for G/B.
3. A result of Lasoń-Michalek for toric varieties: The vector bundle $F_*^s\mathcal{L}$ splits as sum of explicit line bundles.

Atiyah's result is in characteristic 0 we need to modify it for char p.
In the first two examples, the coker $\phi_{m,q}$ is of maximal dimension except at one place.
So question is, when does this happen? Should one look for such bundles?

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In the case R is a dimension 2 graded rings, *i.e.*, when $X = \text{Proj } R$ is a projective curve,
we can carry out the computation in terms of the known invariants like degree and rank of V, **provided** *V and $F^{s*}(V)$ were semistable for $s \geq 0$.*

Definition A vector bundle W on X is *semistable* if for any subbundle $W' \subset W$, we have $\mu(W') \leq \mu(W)$, where

$$\mu(W) = \frac{\deg W}{\operatorname{rank} W}.$$

Lemma For a semistable bundle W of rank r and curve of genus g, we have

1. $h^0(X, W(m)) = 0$, if $\deg W(m) < 0$ and
2. $h^1(X, W(m)) = 0$, if $\deg W(m) > r(2g-2)$,
3. $h^0(X, W(m)) \leq rg$, if $0 \leq \deg W \leq r(2g-2)$.

Are syzygy bundle V and its Frobenius pull backs $F^{s*}(V)$ semistable?

No, but every bundle is filtered by semistable bundles, called the Harder-Narasinhan filtration.
But

$$F^{s*}(\text{HN filtration}) \text{ need not be the HN filtration of } F^{s*}(V), \quad \text{for} \;\; s \geq 0.$$

However, it is, for $s >> 0$ by a (not so old) result of A.Langer

This gives us a well defined notion called *HK slope* of V, as

$$\mu_{HK}(V) := \frac{1}{p^s}\sum_i \mu_i(F^{s*}(V))^2 r_i(F^{s*}(V)), \text{ where } r_i = \text{rank}\frac{E_i}{E_{i-1}} \quad .$$

Theorem

(Brenner, V.T.) If R is a standard graded two dimensional ring over a field of char $p > 0$, then

$$e_{HK}(R) = \frac{\deg X}{2}(\mu_{HK}(V) - embdim(R)).$$

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In particular for a standard graded 2 dimensional ring e_{HK} is a rational number.
This generalizes the result of Monsky, namely if $\dim R = 1$ then $e_{HK} = e_0$,
Question is still open for nongraded 2 dimensional rings.

However, this formula does not help in computing e_{HK},
as the construction of HN filtration, of Frobenius pull backs of a bundle, is rather hard.

Infact e_{HK} gives information about the Frobenius semistabiltiy behaviour of V.

Theorem

(V.T.) For R as above,

$$e_{HK}(R) \geq \frac{\deg\ X}{2}\left[1 + \frac{1}{(embdim(R)) - 1}\right].$$

Moreover equality holds if and only if V is strongly semistable.

In the case of plane curves e_{HK} gives a numerical characterization of the Frobenius semistablity behaviour of the syzygy bundle.

Theorem

(V.T.) Let X be a nonsingular plane curve of degree $d > 1$. Let V be the syzygy bundle given by the canonical map

$$0 \to V \to H^0(X, \mathcal{O}_X(1)) \otimes \mathcal{O}_X \to \mathcal{O}_X(1) \to 0.$$

Then one of the following holds:

1. *V is strongly semistable. In this case $e_{HK}(X) = 3d/4$.*
2. *V is not semistable. Then $e_{HK}(X) = \frac{3d}{4} + \frac{l^2}{4d}$, where $0 < l < d$ and l is an integer congruent to d (mod 2).*
3. *V is semistable but not strongly semistable. Let $s \geq 1$ be the number such that $F^{(s-1)*}V$ is semistable and $F^{s*}V$ is not semistable. Then*

$$e_{HK}(X) = \frac{3d}{4} + \frac{l^2}{4dp^{2s}},$$

where l is an integer congruent to $pd \pmod 2$ with $0 < l \leq 2g - 2$, so that in particular $0 < l \leq d(d-3)$.

Consider the example,

$$R_p = k[X,Y,Z]/(x^4+y^4+z^4), \text{ where char } k = p.$$

$e_{HK}(R_p)$ is computed by Han-Monsky,
Now applying our numerical characterization to this example and its syzygy bundle V_p, we have

1. V_p is strongly semistable if $p \equiv \pm 1(8)$, or char k is zero.
2. V_p is semistable but $F^* V_p$ is not semistable if $p \equiv \pm -3(8)$.

Conclusion: The semistability of Frobenius pull backs does not behave well under ‘reduction mod p’. Though semistability itself is the open property.
However

Theorem

(V.T.) Let $f : X_A \to \mathrm{Spec}\ A$ *be a family of smooth projective curves, where* A *is a finitely generated* $\mathbb{Z}$*-algebra. Let* $\mathcal{O}_{X_A}(1)$ *be a* f*-very ample sheaf on* X_A*, then*

1. $$\lim_{s \mapsto s_0} \mu_{HK}(V_s) = a_{HK}(V_{s_0}),$$

   where s_0 *is the generic point of* $\mathrm{Spec}\ A$ *and* s *is the closed point of* $\mathrm{Spec}\ A$*. In particular*

2. $$\lim_{s \mapsto s_0} e_{HK}(X_s) = \frac{\deg\ X_{s_0}}{2}(a_{HK}(V_{s_0}) - \mathrm{embdim}\ \mathcal{O}_{X_{s_0}}).$$

Statement (1) of the above theorem holds for families of higher dimensional projective varieties.

In particular though Frobenius semistability does not behave well under *reduction mod p*, we have

$$\lim_{p\mapsto\infty} \mu_{HK}(V_p) = a_{HK}(V).$$

What about e_{HK} in higher dimensions?

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Remark: (Monsky's conjecture)

1. If $R = \frac{\mathbf{Z}}{2}[X, Y, Z, u, v]/(H + uv)$, then
$$e_{HK}(R) = \frac{4}{3} + \frac{5}{14\sqrt{7}}$$
2. If $R = k[X_1, \ldots, X_9]/(f)$, then $e_{HK}(R) \in \mathbf{R}^+$ is transcendental.

Finally it is proved by

Theorem

(Brenner) There exists a three dimensional ring such that $e_{HK}(R, \mathbf{m})$ is not rational.

Basically, he shows that there is a module of finite length over a quartic three dimensional hypersurface has irrational HK-multiplicity.

Huneke-Monsky-Macdormatt proved (under some mild conditions) that

$$HK(R, \mathbf{m})(q) = e_{HK}(R, \mathbf{m})q^d + \beta(R)q^{d-1} + f(n),$$

where $f(n) = O(q^{d-2})$,. Moreover there exists cases where $f(n) \neq \nu(R)q^{d-2} + O(q^{d-3})$.

Thank You!