Method of Convex Integration and solutions to Differential Relations.

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contents

1 First order differential relations

A first order Ordinary Differential Relation for maps $x : I \to \mathbb{R}^q$, can be viewed as a *differential inclusion* of the form:

$$x'(t) \in F(t, x(t)), \quad \text{for} \quad t \in I, \tag{1}$$

where $t \mapsto F(t, x(t))$ is a set valued map for subsets $F(t, x(t))$ in $\mathbb{R}^q$ for each $t \in I$.
The problem is to find a solution to the above differential inclusion. The solutions that are of interest may be of different nature: To analysts a *Lipschitz map* or more generally an *absolutely continuous map* (which implies differentiability almost everywhere) satisfying the above equation is a solution; on the other hand topologists are interested in solutions that are at least C^1-*regular*.

2 Lipshitz Condition

The work on the above problem in Optimal Control Theory goes back to 1960's. In 1967, A. F. Filippov gave a sufficient condition for the existence of a solution to the above differential inclusion which is known as *Relaxation Theorem*.

Definition

A set valued function F defined on a subset C of $\mathbb{R}^q$ is said to be *Lipschitzean* with constant k if

$$d_H(F(x), F(y)) \leq k\|x - y\| \text{ for all } x, y \in C,$$

where d_H is the Hausdorff distance on subsets on $\mathbb{R}^q$.

Recall that the Hausdorff distance between two compact subsets A, B in a metric space (X, d) is the infimum of all real $r > 0$ for which the closed r-neighborhood of any x in A contains at least one point y of B and vice versa.

3 Philippov's Theorem

The following is a simplified version of Filippov's Theorem :

Theorem

Let F be a set valued map defined on a closed ball $B = B(a; r)$ in $\mathbb{R}^q$ with values compact subsets of $\mathbb{R}^q$ such that F is a Lipshitzian with constant k. Let $I = [-T, T]$ and $x : I \to Int\, B$ be an absolutely continuous function such that

$$x'(t) \in Conv\, F(x(t)) \;\; \text{for all} \;\; t \in I; \;\; x(0) = a. \tag{2}$$

Let $\varepsilon > 0$. Then there exists an absolutely continuous function $y : I \to B$ such that

$$y'(t) \in F(y(t)); \;\; y(0) = a,$$

and such that $\|x(t) - y(t)\| < \varepsilon$.

4 Remarks

- The theorem says that under Lipschitzean condition on F the problem may be substantially simplified when F does not take convex set values.
- However, if F is a convex set valued function then convex hull construction does not enlarge the solution space and the situation is beyond the scope of this result.
- When $q > 1$, the solution to this problem is not unique, in contrast with the initial value problem for functions $\mathbb{R} \to \mathbb{R}$.

5 Piecewise linear solutions

Consider the curve $x(t) = (t, 0)$ in $\mathbb{R}^2$ so that $x'(t) = (1, 0)$ for all t. Now take $A = \{(1, 1), (1, -1)\}$. Then it is easy to obtain a *piecewise linear* function $y(t)$ in an arbitrary neighbourhood of $x(t)$ such that $y'(t) \in F$ for all t.

Lemma

Let $\varepsilon > 0$. If $0 \in Conv\ A$, then there exists a piecewise linear map $f : I \to B(0; \varepsilon) \subset \mathbb{R}^q$ such that $f'(t) \in A$ for all $t \in I$.

Proof

By the given hypothesis, there exists a path $\gamma : I \to A$ such that $\int_0^1 \gamma(t)\,dt = 0$. For each n, define a function $f_n : I \to \mathbb{R}^n$ by

$$f_n(t) = \int_0^t \gamma^n(s)\,ds,$$

where γ^n is the n-fold uniform product of γ with itself. Clearly $f_n'(t) \in A$ for all $t \in I$. If $k/n \leq t < (k+1)/n$ then

$$f_n(t) = \int_0^t \gamma^n(s)\,ds = \sum_{j=1}^{k-1} \int_{j/n}^{(j+1)/n} \gamma(ns-j)\,ds + \int_{k/n}^t \gamma(ns-k)\,ds$$

By a change of variable, $\int_{j/n}^{(j+1)/n} \gamma(ns-j)\,ds = \frac{1}{n}\int_0^1 \gamma(t)\,dt = 0$ and hence $f_n(t) = \frac{1}{n}\int_0^{nt-k} \gamma(s)\,ds$. Therefore, $f_n \to 0$ with respect to the C^0-norm.

6 Example-I

The topological interest lies in the C^1-smooth solutions of the differential inclusion

$$x'(t) \in A, \text{ for } t \in I,$$

where F is some subset of $\mathbb{R}^q$.

- Connectedness of F is necessary to obtain a C^1 solution.

We modify the previous example in the following way: Let

$$A = \{(1, s) \in \mathbb{R}^2 | -\varepsilon \leq s \leq \varepsilon\}$$

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- Connectedness of F is necessary to obtain a C^1 solution.

We modify the previous example in the following way: Let

$$A = \{(1, s) \in \mathbb{R}^2 | -\varepsilon \leq s \leq \varepsilon\}$$

Then, A is connected and $(1, 0)$ belongs to the convex hull of A.

- The sinusoidal curve $y(t) = (t, \varepsilon \sin t)$, $t \in \mathbb{R}$, is a desired solution.
- $y(t)$ can be made to lie in an arbitrary neighbourhood of $x(t)$ with an appropriate choice of ε.

7 C^1-maps on circle

Lemma

Let $A \subset \mathbb{R}^q$ and let $f : S^1 \longrightarrow \mathbb{R}^q$ be a C^1 map such that

$$\phi = f' : S^1 \longrightarrow A.$$

Then 0 belongs to the convex hull of the path-component of A that receives Im ϕ.

Proof. Indeed expressing $\int_{S^1} \phi(s)\, ds$ as the limit of Riemann sums we get

$$0 = \int_{S^1} \phi(s)\, ds = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{2\pi}{n} \phi(s_k) = 2\pi \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \phi(s_k),$$

where $\frac{1}{n}\sum_{k=1}^{n} \phi(s_k) \in \mathrm{Conv}\,(\mathrm{Im}\phi)$ for each n. Thus $0 \in \mathrm{Conv}\, A$.

8 Example-II

The converse of the above is not in general true. To see this consider

$$A = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1, x_1 \geq 0\},$$

so that $0 \in \operatorname{Conv} A$.
If possible, let $f : S^1 \longrightarrow \mathbb{R}^2$ be a C^1 map such that $\phi = \frac{df}{ds}$ maps S^1 into A. Hence $\int_{S^1} \phi(s)\, ds = 0$. Writing $\phi = (\phi_1, \phi_2)$ we get

$$\int_{S^1} \phi_1(s)\, ds = 0 \;\text{ and }\; \int_{S^1} \phi_2(s)\, ds = 0.$$

Since $\phi_1 \geq 0$ we conclude from the first integral that

$$\phi(x) = (0, 1) \text{ for all } x \;\text{ or }\; \phi(x) = (0, -1) \text{ for all } x.$$

However, this contradicts $\int_{S^1} \phi_2(s)\, ds = 0$.

9 One-dimensional convex integration

Theorem

Let $A \subset \mathbb{R}^q$ be such that

- *A is connected and*
- *0 belongs to the interior of the convex hull of A.*

Then there exists a C^1 map $f : S^1 \longrightarrow \mathbb{R}^q$ such that

$$\frac{df}{ds}(S^1) \subset A.$$

Moreover, $\operatorname{Im} f$ can be made to lie in an arbitrary small neighbourhood of 0.

10 Application

Let $h_1 = \sum_{i=1}^{q} y_i^2$ and $h_2 = \sum_{i=1}^{q} \lambda_i^2 y_i^2$ be two Euclidean metrics on $\mathbb{R}^q$, $q \geq 4$. Let

$$g_1 = ds^2 \text{ and } g_2 = c^2 ds^2$$

be two Riemannian metrics on S^1, where s is the arc-length function on S^1. If $f : S^1 \longrightarrow \mathbb{R}^q$ is a C^1-immersion such that $f^* h_i = g_i$ for $i = 1, 2$ then

$$\|\frac{df}{ds}\|_1 = 1 \quad \text{and} \quad \|\frac{df}{ds}\|_2 = c,$$

where $\|.\|_i$ denote the norm relative to the metric h_i, $i = 1, 2$.
This implies that $\frac{df}{ds} \in A$, where A is given by

$$A = \{(y_1, \ldots, y_q) \in \mathbb{R}^q : \sum y_i^2 = 1 \text{ and } \sum \lambda_i^2 y_i^2 = c^2\}.$$

If $r_{\pm}(c^2 h_1 - h_2) \geq 2$, A is connected and the interior of the convex hull of A contains the origin. Thus, by the above proposition, there exists an immersion $f : S^1 \longrightarrow \mathbb{R}^q$ such that $f^* h_i = g_i, i = 1, 2$ when $r_{\pm}(c^2 h_1 - h_2) \geq 2$.

There does not exist any such isometric immersion if $q \leq 3$. For, if there is a map $f : S^1 \longrightarrow \mathbb{R}^q$ such that $f^*h_1 = ds^2 = f^*h_2$, then $f^*(h_1 - h_2) = 0$. If $h_1 - h_2$ is a non-degenerate indefinite form then either $r_+ = 1, r_- = 2$ or $r_+ = 1, r_- = 2$. In either of these two cases, A is a disjoint union of two circles none of which contains the origin in its convex hull, thereby ruling out the existence of C^1-immersion with the desired isometry property.

11. General framework

More generally, let $\tilde{A}$ be an *open* subset of $\mathbb{R}\times\mathbb{R}^q\times\mathbb{R}^q$. For each $(t,x)\in\mathbb{R}\times\mathbb{R}^q$, define

$$A(t,x)=\{y\in\mathbb{R}^q|(t,x,y)\in\tilde{A}\}.$$

Suppose that there exists a C^1 function $f_0 : I\to\mathbb{R}^q$ and a C^0 function $\varphi_0 : I\to\mathbb{R}^q$ such that

- $(t,f_0(t),\varphi_0(t))\in\tilde{A}$ for all $t\in I$ and
- $f_0'(t)\in \text{Conv } A(t,f_0(t))$ for all $t\in I$.

If $A(t,f_0(t))$ is path-connected for all $t\in I$, then there exists a C^1-function $f : I\to R^q$ arbitrarily C^0-close to f_0 such that

- $(t,f(t),f'(t))\in\tilde{A}$ and
- f coincides with f_0 on the boundary points.

Special Case: Take $\tilde{A}=\mathbb{R}\times\mathbb{R}^q\times A$ and $f_0=0$.

12 Convex integration: Language of jets

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We shall transform this into a differential inclusion in one variable.
We split $\mathbb{R}^n$ as $\mathbb{R}^{n-1} \times \mathbb{R}$. Denote the coordinates on $\mathbb{R}^{n-1}$ by $x_1, \ldots, x_{n-1}$ and the coordinate on the second factor $\mathbb{R}$ by t.

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$$J_f(x) = (J_f^{\perp}(x), \partial_t f(x)),$$

where

$$J_f^{\perp}(x) = (x, f(x), \partial_1 f(x), \ldots, \partial_{n-1} f(x)) \in J^{\perp}(\mathbb{R}^n, \mathbb{R}^q) = \mathbb{R}^n \times \mathbb{R}^{nq}.$$

Let $p^{\perp} : J^1(\mathbb{R}^n, \mathbb{R}^q) \to J^{\perp}(\mathbb{R}^n, \mathbb{R}^q)$ denote the canonical projection.

13 C^1-solutions of open relations

For every $b \in J^{\perp}(\mathbb{R}^{n-1} \times \mathbb{R}, \mathbb{R}^q)$ the fibre over b in $J^1(\mathbb{R}^n, \mathbb{R}^q)$ is a q-dimensional affine space. Define

$$\mathcal{R}_b = \{\alpha \in J^1(\mathbb{R}^n, \mathbb{R}^q) : \alpha = (b, v) \in \mathcal{R}\} \cong \{v \in \mathbb{R}^q : (b, v) \in \mathcal{R}\}$$

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Theorem

Let $\mathcal{R}$ be a subset of $J^1(\mathbb{R}^n, \mathbb{R}^q)$ satisfying the following conditions:

- *$\mathcal{R}$ is open;*
- *$\mathcal{R}_b$ is connected for all $b \in J^{\perp}(\mathbb{R}^n, \mathbb{R}^q)$;*
- *Convex hull of $\mathcal{R}_b$ is equal to the principal subspace over b for all $b \in J^{\perp}(\mathbb{R}^n, \mathbb{R}^q)$.*

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If there are functions $f, \varphi : \mathbb{R}^n \to \mathbb{R}^q$ such that $(j_f^\perp, \varphi)$ is a section of $\mathcal{R}$ then f can be $C^\perp$ approximated by a C^1 solution of $\mathcal{R}$.

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- *$(j^{\perp} f, \varphi)$ is a section of $\mathcal{R}$ and*
- *$J_f(x)$ lies in the convex hull of $\mathcal{R}_{b(x)}$ for all $x \in \mathbb{R}^n$, where $b(x) = J^{\perp} f(x)$*

then f can be $C^{\perp}$ approximated by a C^1 solution of $\mathcal{R}$.

14 Open ample relations

Let $\mathcal{R} \subset J^1(\mathbb{R}^n, \mathbb{R}^q)$ be a first order relation.

Definition

$\mathcal{R}$ is said to be *ample* in the coordinate direction t if the convex hull of each pathcomponent of $\mathcal{R}_b$ is all of $J^1_b(\mathbb{R}^n, \mathbb{R}^q)$. $\mathcal{R}$ is said to be ample if it is ample in each coordinate direction.

Theorem

Let $\mathcal{R} \subset J^1(\mathbb{R}^n, \mathbb{R}^q)$ be an open relation which is ample in each coordinate direction. Then $\mathcal{R}$ satisfies the (relative) h-principle.

Example: Smale-Hirsch immersion theorem.

15 Applications

Theorem

(Geiges and Gonzalo) Let M be a closed orientable 3-manifold. Then M admits a triple of pointwise independent contact forms α_1, α_2, α_3 *with pointwise linearly independent Reeb vector fields.*

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Theorem

(McDuff) Given M with $\dim M = 2n+1$ and $\omega \in \Omega^2(M)$ with $\omega^n \neq 0$ there exits a $\omega' \in \Omega^2(M)$ such that $d\omega' = 0$ and $(\omega')^n \neq 0$. Moreover, one may prescribe the cohomology class of ω'.

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Theorem

(McDuff) Given M with $\dim M = 2n$ and differential forms $\alpha \in \Omega^1(M)$ and $\beta \in \Omega^2(M)$ with $\alpha \wedge \beta^{n-1}$ there exists a 1-form α' such that $\alpha' \wedge (d\alpha')^{n-1} \neq 0$.

18. C^1-solutions of closed relations

Theorem

Let $\mathcal{R}$ be a subset of $J^1(\mathbb{R}^n, \mathbb{R}^q)$ satisfying the following conditions:

- $\mathcal{R} \to J^\perp$ *is a fibre bundle;*
- $\mathcal{R}_b$ *is connected and locally path-connected for all* $b \in J^\perp(\mathbb{R}^n, \mathbb{R}^q)$*;*
- $\mathcal{R}_b$ *is nowhere flat.*

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- $\mathcal{R}_b$ *is nowhere flat.*

If there are functions $f, \varphi : \mathbb{R}^n \to \mathbb{R}^q$ such that

- $(j^\perp f, \varphi)$ *is a section of $\mathcal{R}$ and*
- $J_f(x)$ *lies in the interior of the convex hull $Conv(\mathcal{R}_{b(x)})$ for all $x \in \mathbb{R}^n$, where $b(x) = j^\perp f(x)$.*

then f can be $C^\perp$ approximated by a C^1 solution of $\mathcal{R}$.

19 PDE as relations

A first order partial differential equation for functions $f : \mathbb{R}^n \longrightarrow \mathbb{R}^q$ is of the form:

$$\Psi(x, f(x), J_f(x)) = 0, \tag{3}$$

where Ψ is a vector valued continuous map.

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$$\Psi(x, f(x), J_f(x)) = 0, \qquad (3)$$

where Ψ is a vector valued continuous map.
We can now write the equation (3) as an inclusion of the form

$$\partial_t f(x) \in \mathcal{R}(j_f^{\perp}(x)),$$

where $x \mapsto \mathcal{R}(j_f^{\perp}(x))$ is now a set valued map with values subsets of $\mathbb{R}^q$ for each $x \in \mathbb{R}^n$. The set function $\mathcal{R}$ can be defined as

$$\mathcal{R}(j_f^{\perp}(x)) = \{v \in \mathbb{R}^q : \Psi(j_f^{\perp}(x), v) = 0\}.$$

20 Applications to PDE

Consider the following PDE for functions $f = (f_1, f_2) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$:

$$\Phi_1\left(\frac{\partial f_1}{\partial u_1}, \frac{\partial f_2}{\partial u_1}\right) = \Phi_2\left(u_1, u_2, f_1, f_2, \frac{\partial f_1}{\partial u_2}, \frac{\partial f_2}{\partial u_2}\right),$$

where $\Phi_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is real analytic function without critical point and $\Phi_2 : \mathbb{R}^6 \longrightarrow \mathbb{R}$ is any continuous functions.
Take $\tau = \{u_1 = \text{const.}\}$. Then we can rewrite the above equation as

$$\Phi_1\left(\frac{\partial f_1}{\partial u_1}, \frac{\partial f_2}{\partial u_1}\right) = \Phi_2(j_f^\perp).$$

Hence f is a solution of the relation $\mathcal{R}$ given by

$$\mathcal{R} = \{(b, \alpha_1(x), \alpha_2(x)) : \Phi_1(\alpha_1(x), \alpha_2(x)) = \Phi_2(b)\}.$$

Thus the sets $\mathcal{R}_b$ are the level sets of the function Φ_1.
Consider the function $\Phi_1(x_1, x_2) = x_1 + x_2^3$.

- $\Phi_1^{-1}(a)$, $a \in \mathbb{R}$, is a connected curve in $\mathbb{R}^2$
  $\Rightarrow$ there exists a section $J^\perp \longrightarrow \mathcal{R}$.

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- $\Phi_1^{-1}(a)$, $a \in \mathbb{R}$, is a connected curve in $\mathbb{R}^2$
  $\Rightarrow$ there exists a section $J^{\perp} \longrightarrow \mathcal{R}$.
- Convex hull of $\Phi_1^{-1}(a)$ is all of $\mathbb{R}^2$ for all $a \in \mathbb{R}^2$.

The previous theorem implies that the solutions of the above PDE are dense in the space of C^0-maps $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$.

16 Nash-Kuiper theorem

The following theorem can be obtained by an application of convex integration.

Theorem

(M, g) be a Riemannian manifold of dimension n. Let h denote the canonical metric on $\mathbb{R}^q$. If $q > n$ and M admits a smooth immersion in $\mathbb{R}^q$, then there exists a C^1-immersion $f : M \to \mathbb{R}^q$ such that $f^*h = g$.

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