Your Roll Number: ....

Department of Mathematics, University of Delhi
M.Sc. Mathematics Examinations, May 2026
Part II Semester IV

MMath18- 401(i): Advanced Group Theory (UPC 223502401)

Time: 3 hours Maximum Marks: 70

Instructions: • Write your roll number on the space provided at the top of this page immediately on receipt of this question paper • **Q 1** is compulsoty • Answer **any four** questions from Q2 to Q7. • Each question carries 14 marks.

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(1) (a) Define presentation of a group G. Give two distinct presentations of a cyclic group of order 10. [$3\frac{1}{2}$ Marks]

(b) Let H char G and $H \leq K \leq G$. Show that K/H char G/H implies that K char G. [$3\frac{1}{2}$ Marks]

(c) Let H be a nontrivial normal subgroup of a p-group G. Show that $H \cap Z(G) \neq 1$. [$3\frac{1}{2}$ Marks]

(d) Let P be normal Sylow subgroup of a finite group G. Show that P is fully invariant in G. [$3\frac{1}{2}$ Marks]

(2) Prove that a characteristically simple finite group is either simple or a direct product of isomorphic simple groups. Give example of an infinite characteristically simple group which is neither simple nor a direct product of isomorphic simple groups. [9+5 Marks]

(3) (a) Let $A \triangleleft A^*$ and $B \triangleleft B^*$ be four subgroups of a group G. Show that $B(B^* \cap A) \triangleleft B(B^* \cap A^*)$, and $A(A^* \cap B) \triangleleft A(B^* \cap A^*)$, and there is an isomorphism $B(B^* \cap A^*)/B(B^* \cap A) \cong A(B^* \cap A^*)/A(A^* \cap B)$. [9 Marks]

(b) Let T be a group of order 12 which is generated by two elements a and b such that $a^6 = 1$ and $b^2 = a^3 = (ab)^2$. Show that T is a semidirect product of $\mathbb{Z}_3$ by $\mathbb{Z}_4$. [5 Marks]

(4) (a) Let the prime decomposition of the cardinality $|G|$ of a finite group G consists of at least three primes. If G has p-compliment for every prime p occuring in the prime decomposition of $|G|$, then show that G possesses a proper normal subgroup. [9 Marks]

(b) Let H be a solvable subgroup of a finite group G and P be a unique sylow subgroup of G. Show that HP is a solvable subgroup of G. [5 Marks]

(5) (a) Let G be a finite p-group. Show that the quotient group $G/\Phi G$ is a vector space over $\mathbb{Z}_p$. [9 Marks]

(b) Show that the homomorphic image of a solvable group is solvable. [5 Marks]

(6) (a) Let G be a group having both chain conditions, and let ϕ and ψ be • normal endomorphisms of G. If $\phi + \psi$ is an endomorphism of G then show that it is nilpotent. [9 Marks]

(b) Let H be normal in G, and both H and the quitioent group G/H have descending chain conditions. Show that G has descending chain condition. [5 Marks]

(7) (a) Determine the free abelian group generated by two elements. [5 Marks]

(b) Show that every two composition series of a group are equivalent. Deduce that every integer greater than 1 can be uniquely expressed as a product of prime numbers. [9 Marks]

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Roll No. ....

Department of Mathematics
University of Delhi, Delhi

M.Sc. Mathematics, Part-II, Semester-IV
Examination, May-June 2026

Paper: **MMATH18 401(ii) Algebraic Number Theory**
Unique Paper Code: 223502402

Time: 3 Hour *Maximum Marks: 70*

Note: • Attempt five questions in all. • Question 1 is compulsory.

1. (a) Prove that every algebraic number is of the form a/b, where a is an algebraic integer and b a nonzero integer. (2)

   (b) Find an integral basis and the discriminant of $\mathbb{Q}(\zeta + \zeta^{-1})$, where ζ is a primitive 5-th root of unity. (3)

   (c) Let $d \equiv 1 \pmod 4$ be a square-free integer different from 1. Prove that $\mathbb{Z}[\sqrt{d}]$ is never a unique factorization domain. (3)

   (d) Prove that the volume of the fundamental domain of an n-dimensional lattice is independent of the choice of basis. (3)

   (e) Find the class number of $\mathbb{Q}(\zeta)$, where ζ is a primitive 5th root of unity. (3)

2. (a) Let $K = \mathbb{Q}(\theta)$ be a number field of degree n and $\theta \in \mathcal{O}_K$, where $\mathcal{O}_K$ is the ring of algebraic integers of K. Prove that

$$\Delta_K(1, \theta, \ldots, \theta^{n-1}) = [\mathcal{O}_K : \mathbb{Z}[\theta]]^2 d_K,$$

   where d_k is the discriminant of K. (9)

   (b) Find the discriminant of $\mathbb{Q}(\theta)$, where $\theta^3 = 54\theta + 150$. (5)

3. (a) Define a quadratic number field and find its integral basis and discriminant. (1+6)

   (b) Find the group of units of the ring of algebraic integers for the imaginary quadratic number field. (7)

4. (a) Prove that the ring of algebraic integers of $\mathbb{Q}(\sqrt{d})$, where $d < 0$ is a square-free integer, is never Euclidean for $d < -11$. (8)

   (b) Solve $X^2 + 2 = Y^3$ for $X, Y \in \mathbb{Z}$. (6)

5. (a) Prove that every principal ideal domain is a Dedekind domain. Does the converse hold? (3+2)

(b) Let K be a number field and $\mathcal{O}_K$ its ring of algebraic integers. Define the norm of an ideal of $\mathcal{O}_K$ and prove that every non-trivial ideal has finite norm. (1+4)

(c) Prove that $P = \langle 2, \sqrt{6} \rangle$ is a prime ideal of $\mathbb{Z}[\sqrt{6}]$ and also find its inverse. (4)

6. (a) State and prove Minkowski's Theorem and deduce The two Squares Theorem. (5+5)

(b) Let $K = \mathbb{Q}(\theta)$, where $\theta \in \mathbb{R}$ and $\theta^3 = 3$. Prove that the image of the additive subgroup of K generated by $1+\theta$ and $\theta^2 - 2$ is a lattice in $\mathbb{L}^{st}$. (4)

7. (a) Define class number of a number field and prove that it is always finite. (2+4)

(b) Give an example, with justification, of a number field whose ring of algebraic integers is a unique factorizaton domain but not a Euclidean domain. (4)

(c) Write down the prime ideal factorization of the ideal $\langle 13 \rangle$ of $\mathbb{Z}[\sqrt{-5}]$. (4)

Roll Number:.........

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics, End-semester Examination, May 2026
Part - II; Semester - IV
MMATH18-401(iii) : **Simplicial Homology Theory**
Unique paper code: 223502403

Time: 3 Hours | Max Marks: 70

Instructions: • All symbols have their usual meaning. • Question No.1 in **Part-A** is compulsory. • Answer any four questions from remaining six questions in **Part-B**.

Part-A, Compulsory:

1. Choose appropriate options. Justifications are not required: (2×7)

(a) Which of the following statements are true:

A. A geometrically independent set with $k+1$ elements passes through a unique hyperplane H^k.

B. Every subset of a geometrically independent set is linearly independent.

C. Barycentric coordinates of a point in a hyperplane generated by a geometrically independent set are unique.

D. The geometric carrier of $Cl(\sigma^n)$ is an open subset of the hyperplane generated by vertices of σ^n.

(b) Let K and L be two simplicial complexes in $\mathbb{R}^m$, m sufficiently large. Then which of the following statements are true:

A. The space $\{(x, \sin(\frac{1}{x})) \mid x \in [a, 1], a > 0\}$ is not a polyhedron.

B. $|K|$ is not locally path connected.

C. p^{th} homology group of a polyhedron is independent of triangulation.

D. If $|K|$ and $|L|$ are homeomorphic, then $H_p(K) \cong H_p(L)$ for all $p \geq 0$.

(c) Let $f, g : \mathbb{S}^n \to \mathbb{S}^n$ be two continuous maps. Then which of the following statements are false:

A. If f and g are homotopic, then $\deg(f) = \deg(g)$.

B. If $\deg(f) \neq 1$, then f has an antipodal point.

C. If $\deg(f) \neq \pm 1$, then f do not have a fixed point.

D. If f is a constant map, then degree of f must be non-zero.

(d) Let K be a simplicial complex, $v \in K$ be a vertex and σ be a simplex in K. Then which of the following statements are true:

A. Interior of a simplex is open in the hyperplane spanned by the vertices of that simplex.

B. If $v \notin \sigma$, then $(|K| \setminus st(v)) \cap \sigma$ is closed in $|K|$.

C. The geometric carrier of a combinatorial component of K need not be open in $|K|$.

D. The geometric carrier of a combinatorial component of K need not be closed in $|K|$.

(e) Which of the following statements are false :

A. A minimal triangulation of $\mathbb{T}^2$ has 21 edges.

B. 2^{nd} Betti number of $\mathbb{T}^2$ is 1.

C. The Euler's characteristic of $\mathbb{T}^2$ is 2.

D. If $H_n(K)$ is trivial for an n-pseudomanifold K, then K is orientable.

(f) Let K and L be simplicial complexes in $\mathbb{R}^m$. Then which of the following statements are true:

A. The relation of connectedness in K is anti-symmetric.

B. The homologous relation in $Z_p(K)$ is transitive.

C. If $\phi : K \to L$ is a simplicial approximation to $f : |K| \to |L|$, then f is homotopic to $|\phi|$.

D. K may contain infinitely many vertices.

(g) Let K = 2-skeleton of $Cl(\langle v_0, v_1, v_2, v_3\rangle)$ and $L = Cl(\langle a_0, a_1, a_2\rangle)$ be two simplicial complexes. Let $\phi : K \to L$ be a simplicial map defined by $\phi(v_0) = \phi(v_3) = a_0$, $\phi(v_1) = a_1$ and $\phi(v_2) = a_2$. If $\{\phi_p\}_{p\geq 0}$ is the chain map induced by ϕ, then which of the following statements are true:

A. $\phi_2(\langle v_0, v_2, v_3\rangle = \langle a_0, a_2\rangle$.

B. $H_2(K) \cong \mathbb{Z}$.

C. ϕ_0^* is trivial.

D. ϕ_2^* is trivial.

Part-B, Attempt any FOUR questions from the following:

2. (a) Prove that a subset $A = \{v_0, v_1, \ldots, v_k\}$, $k \geq 1$ of $\mathbb{R}^m$ is geometrically independent if and only if all points of A do not lie on a hyperplane of dimension $k-1$. (6)

   (b) Prove that an n-simplex is a compact, connected and a Hausdorff subspace of $\mathbb{R}^m$, m sufficiently large. (8)

3. (a) Let $K = \{\langle v_0\rangle, \langle v_1\rangle, \langle v_2\rangle, \langle v_3\rangle, \langle v_0, v_1\rangle, \langle v_0, v_2\rangle, \langle v_1, v_2\rangle, \langle v_1, v_3\rangle, \langle v_2, v_3\rangle, \langle v_0, v_1, v_2\rangle\}$ be a simplicial complex with orientation $v_0 < v_1 < v_2 < v_3$. Compute $H_p(K)$ for all $k \geq 0$. (6)

   (b) Let U be a non-empty bounded, convex and open subset of $\mathbb{R}^n$, $n \geq 1$. Prove that there is a homeomorphism of $\overline{U}$ with unit disk $\mathbb{D}^n$, which carries the boundary of U onto the unit sphere $\mathbb{S}^{n-1}$. (8)

4. (a) Define the notion star related. Let K and L be simplicial complexes and $f : |K| \to |L|$ be a continuous map. Prove that f has a simplicial approximation if and only if K is star related to L with respect to f. (10)

   (b) Prove that the composition of simplicial approximations of two continuous maps is a simplicial approximation to the compositions of the maps. (4)

5. (a) For each $n \geq 0$, establish that $|K| = |K^{(n)}|$, where K is a simplicial complex. (7)

   (b) Show that the path components of geometrical carrier $|K|$ of a simplicial complex K coincides with the geometric carrier of combinatorial components of K. (7)

6. (a) State and prove Euler's theorem. (8)

   (b) Prove that there are only five simple regular polyhedra. (6)

7. (a) Show that $\mathbb{S}^{n-1}$ is not a retract of $\mathbb{D}^n$, $n \geq 1$. (6)

   (b) Prove that $[0,1]^n$, $n \geq 1$, has fixed point property. (8)

Your Roll Number: ....

Department of Mathematics, University of Delhi
M.Sc. Mathematics Examinations, May-June, 2026
Part II Semester IV

MMATH18- 402(i): Abstract Harmonic Analysis, UPC: 223502405

Time: 3 hours Maximum Marks: 70

Instructions: • Answer **five** questions • **Question 1** is compulsory. Attempt any **four** questions from **Q2** to **Q7** • Each question carries 14 marks. All the symbols used have their usual meaning.

(1) (a) Let G be a compact group and $[\pi] \in \widehat{G}$. Prove that $\chi_\pi = \sum_{i=1}^{d_\pi} \pi_{ii}$. (2)

(b) For $\mu \in M(G)$, prove that $\|\mu^*\| = \|\mu\|$. (3)

(c) Prove that the Dirac (point mass) measure on a locally compact group G is a regular Borel measure. (3)

(d) Let μ be a left Haar measure on locally compact group G. Prove that $\mu(U) > 0$ for any non-empty open subset of G. (3)

(e) Give example (with justification) of a representation which is not a unitary representation. (3)

(2) (a) State and prove Schur's Lemma. (7)

(b) For $n \in \mathbb{N}$, let H_n denote the space of homogeneous polynomials of degree n of two variables. Give example of a representation of $SU(2)$ on H_n. Further, for $0 \leq k \leq n-1$, define $a_k = (1, w^k)^t$, $w = e^{2\pi i/n}$ and $a_n = (0,1)^t$. If $\psi_k(z) = (za_k)^n$, prove that $\{\psi_k\}_{0\leq k\leq n}$ is a basis of H_n. (7)

(3) (a) Let G be a locally compact group. Prove that the Banach algebra $M(G)$ is unital. Further, prove that $M(G)$ is commutative if and only if G is abelian. (7)

(b) Let $\mathcal{E} = span\{\pi_{ij} : \pi \text{ is an irreducible unitary representation of } G, i, j = 1, \ldots d_\pi\}$, G being a compact group. Prove that $\mathcal{E}$ is closed under conjugation . Further, assuming that $\mathcal{E}$ is uniformly dense in $C(G)$, prove that $\mathcal{E}$ is dense in $L^2(G)$. (7)

(4) (a) Let G be any topological group and $f \in C_c(G)$. Prove that f is right uniformly continuous. (7)

(b) Let ϕ be a function of positive type on a locally compact abelian group G. For any $f \in L^1(G)$, prove that $|\int \phi f| \leq \|\widehat{f}\|_\infty$. (7)

(5) (a) Let π be a unitary representation of a locally compact group G and ρ be its integrated representation. Prove that ρ is irreducible if and only if π is irreducible. (7)

(b) Prove that the dual group of $\mathbb{R}$ is (group) isomorphic to $\mathbb{R}$ itself. (7)

(6) (a) Let G be a compact group, $f \in L^2(G)$ and $[\pi] \in \widehat{G}$. Define the Fourier transform of f at π. Prove that (7)

$$f(\cdot) = \sum_{[\pi]\in\widehat{G}} d_\pi tr(\hat{f}(\pi)\pi(\cdot)) \quad \text{in } L^2(G).$$

(b) Prove that the left regular representation of $\mathbb{R}$ has no irreducible subrepresentation. (7)

(7) (a) Let G be a locally compact abelian group and $\mu \in M(G)$. Define the Fourier Stieltjes transform of μ (with justification). Prove that for $\mu, \nu \in M(G)$, (3+4)

$$\widehat{(\mu * \nu)}(\psi) = \widehat{\mu}(\psi)\widehat{\nu}(\psi), \quad \text{for all } \psi \in \widehat{G}.$$

(b) For a compact group G, define $ZL^2(G)$. Prove that $ZL^2(G)$ is the center of the the Banach algebra $L^2(G)$. What is the dimension of $ZL^2(S_3)$? (7)

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No. of Printed Pages **2**

M.A./M.Sc. Mathematics Examination (2026)
Part II, Semester IV
MMATH18-402(ii): Frames and Wavelets
Unique Paper Code-223502406

Time: 3 hr. Maximum Marks: 70

Instructions: • Attempt **FIVE** questions in all. • Q. No. 1 is **COMPULSORY**. • All questions carry equal marks. • Symbols have their usual meaning.

(Question No. 1 is Compulsory)

(1) (a) Define optimal frame bounds of a frame. [2 Marks]

(b) Show that if $\{f_k\}_{k=1}^m$ is a tight frame for $\mathbb{C}^n$ with frame operator S and $\|f_k\| = 1$ for all $k = 1, 2, \ldots, m$, then $\|S\| \in [1, \infty)$. [3 Marks]

(c) What can you say about the existence of an ω-independent frame for $\ell^2(\mathbb{N})$ that is not minimal? Explain. [3 Marks]

(d) Give an example, with justification, of a Riesz basis for an infinite-dimensional Hilbert space $\mathcal{H}$ which is not an orthonormal basis for $\mathcal{H}$. [3 Marks]

(e) What is the relation between the Haar scaling function and Haar wavelet? Explain. [3 Marks]

(Answer any FOUR questions from Q. No. 2 to Q. No. 7)

(2) (a) Let $\{f_k\}_{k=1}^m$ be a frame for $\mathbb{C}^n$. Show that there exist finitely many vectors $g_1, g_2, \ldots, g_s$ in $\mathbb{C}^n$ such that $\{f_k\}_{k=1}^m \bigcup \{g_j\}_{j=1}^s$ is a tight frame for $\mathbb{C}^n$. [8 Marks]

(b) Show that the coefficient functionals associated to a Schauder basis for a separable Banach space are linear and continuous. [2+4=6 Marks]

(3) (a) State the Chebyshev frame algorithm. State condition(s) under which it gives faster convergence than the frame algorithm. [4 Marks]

(b) State and prove the frame minus one theorem. [10 Marks]

(4) (a) Find an exact frame for the jth approximate space $V_j \subset L^2(\mathbb{R})$, $j \in \mathbb{Z}$. [4 Marks]

(b) Show that a sequence $\{f_k\}_{k=1}^\infty$ in an infinite-dimensional separable Hilbert space $\mathcal{H}$ is a Bessel sequence with Bessel bound B if and only if the Gram matrix associated to $\{f_k\}_{k=1}^\infty$ defines a bounded operator on $\ell^2(\mathbb{N})$ with norm at most B. [10 Marks]

(5) (a) Show that if Θ is a linear operator on $\mathbb{C}^n$ and $\{f_k\}_{k=1}^m$ is a Parseval frame for $\mathbb{C}^n$, then the trace of Θ is $\sum_{k=1}^m \langle \Theta f_k, f_k \rangle$. [4 Marks]

(b) Let $\{f_k\}_{k=1}^{\infty}$ be a frame for an infinite-dimensional separable Hilbert $\mathcal{H}$ with frame operator S. Show that the dual frames of $\{f_k\}_{k=1}^{\infty}$ are precisely the families [10 Marks]

$$\{g_k\}_{k=1}^{\infty} = \left\{ S^{-1}f_k + h_k - \sum_{j=1}^{\infty} \langle S^{-1}f_k, f_j \rangle h_j \right\}_{k=1}^{\infty},$$

where $\{h_k\}_{k=1}^{\infty}$ is a Bessel sequence in $\mathcal{H}$.

(6) (a) State and prove the Haar reconstruction theorem. [7 Marks]

(b) Let $\{f_k\}_{k=1}^{\infty}$ be a frame for an infinite-dimensional separable Hilbert $\mathcal{H}$ and U be a non-zero bounded linear operator on $\mathcal{H}$ with closed range. Show that $\{Uf_k\}_{k=1}^{\infty}$ is a frame sequence. [7 Marks]

(7) (a) Show that a Riesz basis for an infinite-dimensional separable Hilbert $\mathcal{H}$ is a frame for $\mathcal{H}$. [6 Marks]

(b) Define multiresolution analysis. State and prove the scaling relation of a multiresolution analysis. [3+5=8 Marks]

Roll Number: ....

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI

M.A./M.Sc. Examinations, June-2026

MMATH18-402 (iv): THEORY OF UNBOUNDED OPERATORS

(UPC NO. 223502408)

Time: 3 Hours | Maximum Marks: 70

Instructions: • Attempt **five** questions in all. • Question 1 is compulsory. • All the symbols have their usual meaning unless otherwise specified.

Throughout this paper, H will be a complex Hilbert space and X will be a Banach space.

Q1. (a) Let $T : \mathcal{D}(T) \to \ell^2$, where $\mathcal{D}(T) \subset \ell^2$ consists of all $x = (\xi_j)$ with only finitely many nonzero terms ξ_j and $y = (\eta_j) = Tx = (j\xi_j)$. Show that T is not closed. [3 Marks]

(b) Show that the spectrum of a self-adjoint linear operator consists entirely of approximate eigenvalues. [3 Marks]

(c) If a self-adjoint linear operator $T : \mathcal{D}(T) \to H$ is injective, show that [4 Marks]

(i) the range of T is dense in H,

(ii) T^{-1} is self-adjoint.

(d) Let a linear operator A be the infinitesimal generator of a C_0 semigroup $\{T(t) \mid t \geq 0\}$ satisfying $\|T(t)\| \leq e^{\omega t}$ (for some $\omega \geq 0$). Show that $S(t) = e^{-\omega t}T(t)$, $t \geq 0$ forms a C_0 semigroup with the infinitesimal generator $A - \omega I$. [4 Marks]

Q2. (a) Let $T : \mathcal{D}(T) \to H$ be a densely defined linear operator in H. Moreover, suppose that T is injective and its range $\mathcal{R}(T)$ is dense in H. Prove that T^* is injective, and $(T^*)^{-1} = (T^{-1})^*$. [8 Marks]

(b) Let Y be a reflexive Banach space and let $\{T(t) \mid t \geq 0\}$ be a C_0 semigroup on Y with infinitesimal generator A. Show that the adjoint semigroup $T(t)^*$ of $T(t)$ is a C_0 semigroup on Y^* (the dual of Y) with infinitesimal generator A^* (the adjoint of A). [6 Marks]

Q3. (a) Let $T : \mathcal{D}(T) \to H$ be a densely defined linear operator in H. If T is symmetric, then prove that its closure $\overline{T}$ exists and is unique. [8 Marks]

(b) Define the multiplication operator and show that it is self-adjoint. [6 Marks]

Q4. (a) Let A be a linear operator with dense domain $\mathcal{D}(A)$ in X. If A is dissipative and there exists $\lambda_0 > 0$ such that the range $\mathcal{R}(\lambda_0 I - A) = X$, then show that A is the infinitesimal generator of a C_0 semigroup of contractions on X. [8 Marks]

(b) Let $\{T(t) \mid t \geq 0\}$ be a C_0 semigroup satisfying $\|T(t)\| \leq M$ and let A be its infinitesimal generator. Prove that [6 Marks]

(i) A is closed and $\overline{\mathcal{D}(A)} = X$.

(ii) $(0, \infty) \subseteq \rho(A)$ and for every $\lambda > 0$, $\|R(\lambda : A)^n\| \leq \frac{M}{\lambda^n}$ for $n \in \mathbb{N}$.

Q5. (a) Let $\{T(t) \mid t \geq 0\}$ be a C_0 semigroup and let A be its infinitesimal generator. Show that there exist constants $\omega \geq 0$ and $M \geq 1$ such that $\|T(t)\| \leq Me^{\omega t}$ for $0 \leq t < \infty$. Furthermore, show that for $x \in X$, $\int_0^t T(s)x ds \in \mathcal{D}(A)$. [8 Marks]

(b) Let $\{T(t) \mid t \geq 0\}$ be a C_0 semigroup. If $0 \in \rho(T(t_0))$ for some $t_0 > 0$ then show that $0 \in \rho(T(t))$ for all $t > 0$. [6 Marks]

Q6. (a) Let A be a dissipative operator in X. [8 Marks]

(i) If A is closable then show that $\overline{A}$, the closure of A, is also dissipative.

(ii) If $\overline{\mathcal{D}(A)} = X$ then show that A is closable.

(b) Let $Z = BU(0, \infty)$. Define $(T(t)f)(s) = f(t+s)$. Show that $\{T(t) \mid t \geq 0\}$ is a C_0 semigroup of contractions on Z. Also, find its infinitesimal generator. [6 Marks]

Q7. (a) Let A be dissipative with $\mathcal{R}(I - A) = X$. If X is reflexive, prove that $\mathcal{D}(A)$ is dense in X. Does this result also hold for general Banach spaces? Justify your answer. [6+4 Marks]

(b) Show that the Hilbert-adjoint operator of a densely defined linear operator is linear and closed. [4 Marks]

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Your Roll No:......................

M.A/M.Sc. Mathematics, Part-II, Sem-IV (2026)
MMATH18-403(i), Calculus on $\mathbb{R}^n$
Unique Paper Code: 223502409

Time : 3 hours

Maximum Marks : 70

Question No. **1** is compulsory. Attempt any **four** questions from Question Nos. **2** to **7**.
Unless otherwise mentioned, U will be an open subset of $\mathbb{R}^n$.

(1) (a) Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} x^2 \sin\left(\dfrac{1}{x}\right) + y^2 \sin\left(\dfrac{1}{y}\right) & \text{if } xy \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

(i) Show that every directional derivative of f exists at $(0,0)$.

(ii) Determine whether f is differentiable at $(0,0)$.

(iii) Show that the partial derivatives $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ are *not* continuous at $(0,0)$.

[8]

(b) Let $f : U \to \mathbb{R}^m$; $f(x) = (f_1(x), f_2(x), \ldots, f_m(x))$ and $g : U \to \mathbb{R}$ be differentiable at $a \in U$. Define $gf : U \to \mathbb{R}$ as $(gf)(x) = g(x)f(x)$. Prove the product rule [6]

$$\mathrm{D}(gf)(a) = g(a)\,\mathrm{D}f(a) + f(a)^{\top}\mathrm{D}g(a).$$

(2) (a) Let $x(s,t) = s^2 + e^t$ and $y(s,t) = se^t$ and $f(x,y) = x^2y + \sin xy$. Evaluate $\frac{df}{dt}$ at the curve $s = t$. [4]

(b) If $f : U \subset \mathbb{R}^2 \to \mathbb{R}$ satisfies $\dfrac{\partial^2 f}{\partial x\,\partial y} = \dfrac{\partial^2 f}{\partial y\,\partial x}$ everywhere on U, must f be of class C^2? Prove or provide a counterexample. [4]

(c) Let $f : U \to \mathbb{R}$ be a map of class C^k. State and prove Taylor's formula with Lagrange's remainder of order k in a neighborhood of $a \in U$. [6]

(3) (a) State and prove the **Implicit Function Theorem** for $F : \mathbb{R}^{n+m} \to \mathbb{R}^m$ of class C^1. Show that the implicitly defined map φ is unique and of class C^1, and provide a formula for $D\varphi$. [8]

(b) Let $F : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by

$$F(x,y,z) = \left(x^2 + y^2 + z^2 - 3,\ xyz - 1\right).$$

At the point $(1,1,1)$, determine whether one can locally express (y,z) as a C^1 function of x. If so, compute $\dfrac{dy}{dx}$ and $\dfrac{dz}{dx}$ at $x = 1$. [6]

(4) (a) Let ω be a k-form and η an l-form of class C^1 on an open set $E \subset \mathbb{R}^n$. Prove that

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta,$$

and deduce that $d^2\omega = 0$ for every form ω of class C^2. [4]

(b) Let $\omega = P\,dx + Q\,dy$ be a 1-form on $\mathbb{R}^2 \setminus \{0\}$. State necessary and sufficient conditions for ω to be *exact.* Prove that

$$\omega = \frac{-y\,dx + x\,dy}{x^2 + y^2}$$

is *closed* but *not exact* on $\mathbb{R}^2 \setminus \{0\}$. [4]

(c) Let $\omega = \sum_I b_I(x)\,dx_I$ be the standard presentation of a k-form on $\mathbb{R}^n$. If $\omega = 0$ in $\mathbb{R}^n$, prove that $b_I(x) = 0$ for all $x \in \mathbb{R}^n$ and every increasing k-index I. [6]

(5) (a) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. Show that $\omega_T = \det(T) \cdot \omega$ for any n-form ω on $\mathbb{R}^n$. [6]

(b) Let $\Phi : [0,1]^2 \to \mathbb{R}^3$ be given by

$$\Phi(u, v) = (u\cos v,\ u\sin v,\ v),$$

and let $\alpha = z\ dx \wedge dy + x\ dy \wedge dz$. Compute the pullback α_Φ explicitly and evaluate $\displaystyle\int_\Phi \alpha$. [8]

(6) (a) State and prove **Stokes' Theorem** for a smooth k-chain c in $\mathbb{R}^n$. [10]

(b) Derive **Green's Theorem** from Stokes' Theorem: for a compact region $D \subset \mathbb{R}^2$ with smooth positively-oriented boundary ∂D,

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy = \oint_{\partial D} P\,dx + Q\,dy.$$

Hence, compute the area enclosed by the *astroid* $\gamma(t) = (\cos^3 t, \sin^3 t)$, $t \in [0, 2\pi]$. [4]

(7) (a) Give the complete statement of the Partition of Unity theorem. [2]

(b) Construct *explicitly* a C^∞ partition of unity $\{\varphi_1, \varphi_2, \varphi_3\}$ on $\mathbb{R}$ subordinate to the open cover

$$\{(-\infty, 2),\ (0, 3),\ (1, \infty)\}.$$

Verify each defining property rigorously and sketch the graphs of the three functions. [12]

— END OF QUESTION PAPER —

M.A./M.Sc. Mathematics, Part-II, Semester IV
Examination, May-June 2026

Paper: MMATH 18-403(ii). Differential Geometry
Unique Paper Code 223502410

Time: 3 Hours Maximum Marks: 70

INSTRUCTIONS: Attempt **five** (5) questions in all. **Question No. 1 is compulsory**. All questions carry equal marks.

1. (a) Evaluate $\nabla_{\mathbf{v}}\mathbf{X}$ where $\mathbf{X}(x_1, x_2) = (x_1, x_2, x_1x_2, -x_2^2)$, $\mathbf{v} = (1, 1, 0, 1)$. [2]
   (b) Compute the line integral $\int_\alpha (x_2dx_1 - x_1dx_2)$, where $\alpha(t) = (2cost, 2sint), 0 \leq t \leq 2\pi$. [2]
   (c) Let $f : U \longrightarrow \mathbb{R}$ be a smooth function, U open in $\mathbb{R}^{n+1}$ and $p \in f^{-1}(c)$. Show that the gradient of f at p is orthogonal to all vectors tangent to $f^{-1}(c)$ at p. [3]
   (d) Show that velocity vector field along a parametrized curve α on a surface S is *Levi-Civita parallel* or *parallel* if and only if α is a *geodesic* on S. [3]
   (e) Let S be an oriented 2-surface in $\mathbb{R}^3$ and let $\{\mathbf{v}_1, \mathbf{v}_2\}$ be orthonormal basis for the tangent space S_p, consisting of eigenvectors of the weingarten map L_p. If $\mathbf{v} = (cos\theta)\mathbf{v}_1 + (sin\theta)\mathbf{v}_2 \in S_p$ show that the normal curvature of S at p in the direction of $\mathbf{v}$ is given by $\kappa(\mathbf{v}) = \kappa(\mathbf{v}_1)cos^2\theta + \kappa(\mathbf{v}_2)sin^2\theta$. [4]

2. (a) Let $\alpha : I \longrightarrow S$ be a parametrized curve in an n-surface S, let $t_0 \in I$ and $\mathbf{v} \in S_{\alpha(t_0)}$. Show that there exists a unique vector field $\mathbf{V}$, tangent to S along α, which is parallel and has $\mathbf{V}(t_0) = \mathbf{v}$. [9]
   (b) Define *exact 1-form* on an open set $U \subset \mathbb{R}^{n+1}$ with an example. Show that the integral of an exact 1-form w over a closed curve α is zero. [3+2]

3. (a) Given an orientation $\mathbf{N}$ for an n-surface $S = f^{-1}(c)$ in $\mathbb{R}^{n+1}$ deduce the *Weingarten map* L_p. Does L_p depend on the orientation of S? Determine the Weingarten map for the n-sphere $x_1^2 + ... + x_{n+1}^2 = 9$ in $\mathbb{R}^{n+1}$ oriented by the inward normal vector field $\mathbf{N}(q) = \left(q, -\frac{q}{\| q \|}\right)$. [3+6]
   (b) Let $\alpha : I \longrightarrow \mathbb{R}^2$ be a regular parametrized curve in $\mathbb{R}^2$, I open in $\mathbb{R}$, whose image lies above the x axis; i.e.,

$y(t) > 0$ for all t where $\alpha(t) = (x(t), y(t))$. Show that $\phi : I \times \mathbb{R} \longrightarrow \mathbb{R}^3$ define by $\phi(t, \theta) = (x(t), y(t)cos\theta, y(t)sin\theta)$ is a parametrized 2-surface in $\mathbb{R}^3$. [5]

4. (a) Let $S = f^{-1}(c)$ be an n-surface in $\mathbb{R}^{n+1}$ where $f : U \to \mathbb{R}$ is such that $\nabla f(q) \neq 0$ for all $q \in S$. Suppose $g : U \to \mathbb{R}$ is a smooth function and $p \in S$ is an extreme point of g on S. Show that there is a real number λ such that $\nabla g(p) = \lambda \nabla f(p)$. What happen if S is compact ? Determine the extreme values on the unit circle S^1 in $\mathbb{R}^2$ for the function $g : \mathbb{R}^2 \to \mathbb{R}$ given by $g(x_1, x_2) = 4x_1^2 + 4x_1x_2 + x_2^2$. [5+4]

   (b) Let $C = f^{-1}(c)$ be a plane curve oriented by $\mathbf{N} = \dfrac{\nabla f}{\| \nabla f \|}$ and $p \in C$. Define curvature $\kappa(p)$ of C at p. If α is a unit speed parametrized curve in C show that $\kappa(\alpha(t)) = \ddot{\alpha} \cdot \mathbf{N}(\alpha(t))$. [2+3]

5. (a) Define *second fundamental form* for an n-surface S at $p \in S$. Show that on each compact oriented n-surface S in $\mathbb{R}^{n+1}$ there exists a point p such that the second fundamental form at p is definite. [2+7]

   (b) Define *cylinder* over a given *parametrized n-surface* in $\mathbb{R}^{n+k}, k \geq 0$. Show that the cylinder over a parametrized n-surface in $\mathbb{R}^{n+k}$ is a parametrized $(n+1)$-surface in $\mathbb{R}^{n+k+1}$. [2+3]

6. (a) Show that a connected n-surface S in $\mathbb{R}^{n+1}$ has exactly two unit normal vector fields, one negative of the other. [5]

   (b) Show that a parametrized curve α is a geodesic in the unit n-sphere in $\mathbb{R}^{n+1}$ if and only if it is of the form $\alpha(t) = cos(at)\mathbf{e}_1 + sin(at)\mathbf{e}_2$ for some orthonormal pair of unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ in $\mathbb{R}^{n+1}$. [9]

7. (a) Show that for each 1-form w on an open set $U \subset \mathbb{R}^{n+1}$ there exist unique functions $f_i : U \longrightarrow R$, $1 \leq i \leq n+1$ such that $w = \sum_{i=1}^{n+1} f_i dx_i$. If $f : U \longrightarrow \mathbb{R}$ is a smooth function show that $df = \sum_{i=1}^{n+1} \dfrac{\partial f}{\partial x_i} dx_i$. [5+2]

   (b) For the parametrized torus $\varphi : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $\varphi(\theta, \phi) = ((a + bcos\phi)cos\theta, (a + bcos\phi)sin\theta, bsin\phi)$, $a > b > 0$, determine the *principal curvatures* and the *Gaussian curvature* K. [7]

Your Roll Number: ....

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations, May 2026
Part- II Semester- IV

MMATH18-403(iii): Topological Dynamics
(Unique Paper Code 223502411)

Time: **3 hours** Marks: **70**

Instructions: • All notations used are standard • **Question no. 1** is compulsory • Attempt any **four** questions from the remaining six questions.

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(1) Do as directed.

(a) If metric spaces (X, d) and (Y, ρ) are homeomorphic and X admits an expansive homeomorphism then can we say Y will also admit an expansive homeomorphism? Justify your claim. [3 Marks]

(b) Consider $X = \{\frac{1}{n}, 1 - \frac{1}{n} : n \in \mathbb{N}\}$ under usual metric. Construct a generator for (X, f) where $f : X \to X$ is the right shift operator. [3 Marks]

(c) Is Sarkovskii's theorem true for any continuous map $f : \mathbb{S}^1 \to \mathbb{S}^1$? Justify your claim. [2 Marks]

(d) Prove that if a $k \times k$ matrix A with entries in $\{0, 1\}$ is irreducible then $A \vee (A * A) \vee \cdots \vee \underbrace{(A * A * \cdots * A)}_{n-times} = J$, for some $n \in \mathbb{N}$. [3 Marks]

(e) Can we say $f \times g : [0, 1] \times \mathbb{S}^1 \to [0, 1] \times \mathbb{S}^1$ defined by $(f \times g)(t, e^{i\theta}) = (\cos^{95} t, e^{i(\theta + 2\sqrt{2}\pi)})$ has POTP under the maximum metric on $[0, 1] \times \mathbb{S}^1$? Justify your claim. [3 Marks]

(2) (a) Let A be a $k \times k$ matrix with entries in $\{0, 1\}$ such that no row/column of A is full of zeros. If $\sigma|_{X_A}$ is topologically transitive then prove that A is an irreducible matrix. What can you say about the digraph associated with A? Justify your answer. [9 Marks]

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = -\frac{1}{2}(x^3 + x)$. Do the graphical analysis of f and draw the phase portrait of the orbit of $x = \frac{1}{2}$. Find the fixed points of f, if any. [5 Marks]

(3) (a) Prove that if continuous map $f : \mathbb{R} \to \mathbb{R}$ has a periodic point of prime period 3 then it has periodic points of all prime periods $n \geq 1$. [9 Marks]

(b) For the logistic map $F_\mu : \mathbb{R} \to \mathbb{R}$ defined by $F_\mu(x) = \mu x(1 - x)$, prove that $\lim_{n \to \infty} F_\mu^n(x) = -\infty$ if [5 Marks]

(i) $\mu > 1$, $x < 0$ and
(ii) $\mu > 1$, $x > 1$.

(4) (a) Let $f : \mathbb{R} \to \mathbb{R}$ be a C^1-map and p be a repelling fixed point of [5 Marks]

P.T.O.

f. Prove that there exists an open interval U about p such that for any $x \in U, x \neq p$ there exists a $k \in \mathbb{N}$ such that $f^k(x) \notin U$.

(b) Prove that $\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ admits no expansive homeomorphism. Does there exist an expansive homeomorphism on the closed unit disc $\mathbb{D}^2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$? Justify. [9 Marks]

(5) (a) Let (X, d) be a metric space and $h : X \to X$ be a homeomorphism. Prove that h is expansive with expansive constant δ if and only if [5 Marks]

(i) h δ-separates orbits and

(ii) for any orbit $O(p)$ and integer n such that $h^n(p) \neq p$, there exist integers r and s with $r - s = n$ satisfying $d(h^r(p), h^s(p)) > \delta$.

(b) Prove that the shift map $\sigma : [0,1]^{\mathbb{Z}} \to [0,1]^{\mathbb{Z}}$ has POTP. Is σ expansive? Justify your claim. [9 Marks]

(6) (a) Let (X, d) be a compact metric space and $f : X \to X$ be continuous. Prove that for any $x \in X$, $\omega(x) = \bigcap_{m \geq 0} \left(\overline{\bigcup_{n \geq m} \{f^n(x)\}} \right)$. Conclude that $\omega(x)$ is a non-empty, compact subset of X. [7 Marks]

(b) Prove that an irrational rotation on the unit circle $\mathbb{S}^1$ is a minimal homeomorphism. Is minimality a dynamical property for homeomorphisms on $\mathbb{S}^1$? Justify your claim. [7 Marks]

(7) (a) Prove that if (X, d) is a compact metric space and $f : X \to X$ is expansive with expansive constant e then for every $\gamma > 0$ there exists an $n \in \mathbb{N}$ such that for every $x \in X$ and for all $n \geq N$ we have $f^n(W_e^S(x, d)) \subseteq W_\gamma^S(f^n(x), d)$. [5 Marks]

(b) Prove that if (X, d) is a compact metric space and $f : X \to X$ is a topologically Anosov homeomorphism then f is topologically stable in the class of self-homeomorphisms of X. [9 Marks]

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examinations, May-June 2026
Part II Semester IV

MMATH18-404(i): ADVANCED FLUID DYNAMICS
UPC 223502412

Time: 3 Hours | Maximum Marks: 70

Instructions: • Attempt **FIVE** questions in all. • Q.No.1 is compulsory. Each question carries 14 marks. • All the symbols have their usual meaning.

(1) (a) Explain need of different forms of second law of thermodynamics. [3 Marks]

(b) Derive the relation $\frac{T_0}{T} = 1 + \frac{1}{2}(\gamma - 1)M^2$, for isentropic flow of an ideal gas, T_0 is temperature of the gas at rest. [3 Marks]

(c) Write the Maxwell's electromagnetic field equation for the conducting medium in motion. [3 Marks]

(d) Derive the law of conservation of charge for its continuous distribution. [2 Marks]

(e) Show that the shearing stress τ_0 on a wall of length l and breadth b for a fluid flow of density ρ, viscosity μ and uniform stream speed U is proportional to $U^{\frac{3}{2}}$. [3 Marks]

(2) (a) Reduce the Navier Stokes equation of motion into non-dimensional form. Explain physically the obtained non-dimensional numbers. [7 Marks]

(b) State the principle of conservation of energy for a fluid flow. Derive the total energy equation for the fluid in integral and differential form. Write the physical interpretation of each term in the equation. [7 Marks]

(3) (a) Derive shock jump condition for conservation law $u_t + \phi(u)_x = 0, x \in R, t > 0$. Write the mass, momentum and energy equation in differential equation form for one dimensional motion of an inviscid gas and hence derive the corresponding shock conditions. [9 Marks]

(b) Prove that the shock wave is compressive in nature. [5 Marks]

(4) (a) Derive the differential equation for electric and magnetic field in a conducting fluid under the Pre-Maxwell's equation and explain them physically [6 Marks]

(b) Define specific heat of a gas. Derive relation between specific heats at constant volume and pressure for a gas. Explain adiabatic constant, absolute zero temperature and Kelvin scale. [8 Marks]

(5) (a) Derive wave equations for speed and density of an ideal gas due to a small disturbance using system of one-dimensional gas dynamic equations for unsteady isentropic flow. [6 Marks]

(b) Derive the Prandtl's boundary layer equations along with the boundary conditions for two dimensional viscous incompressible fluid flow over a slender body. Write the Prandtl's boundary layer equations and boundary conditions in term of stream function $\psi(x, y)$. [8 Marks]

(6) (a) Derive the equation of motion of a non-viscous conducting fluid $\frac{\partial(\rho\vec{v})}{\partial t} + \rho(\vec{v}.\nabla)\vec{v} = \overrightarrow{f(ex)} + \frac{\mu}{4\pi}(\vec{H}.\nabla)\vec{H} - \nabla(p + \frac{\mu\vec{H}^2}{8\pi}) - \vec{v}\nabla.(\rho\vec{v})$. Write the corresponding equation for viscous and incompressible fluid. [7 Marks]

(b) Show that the Lorentz's force per unit volume of a conducting fluid is equivalent to the hydro-static pressure $\frac{\mu H^2}{8\pi}$ and a tension $\frac{\mu H^2}{4\pi}$ per unit area along the line of force. [7 Marks]

(7) (a) Derive momentum integral equation for steady, two dimensional boundary layer flow of incompressible fluid. [6 Marks]

(b) Define the momentum and energy thickness of a boundary layer and find the expression for each of them on a flat plate. Compute and compare the two thicknesses over a flat plate for linear velocity distribution given by [8 Marks]

$$u(y) = \begin{cases} \frac{U_\infty}{\delta} y, & 0 < y < \delta \\ U_\infty, & \delta < y < \infty, \end{cases}$$

where U_∞ is uniform stream velocity.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examinations, May 2026
Part II Semester IV,UPC 223502413

MMATH18-404(ii): COMPUTATIONAL METHODS FOR PDES

Time: 3 hours — Maximum Marks: 70

Instructions: • Attempt **5** questions in all • Question number **1** is compulsory and attempt any **4** questions from the remaining • Each question carries 14 marks • Scientific non-programming calculator for doing numerical calculations is allowed for this examination • Notations and acronyms have their usual meaning.

(1) (a) If $v_{l,m}$ is a discrete function defined on a grid on the unit square with $v_{l,m} = 0$ on the boundary, then prove that [5 Marks]

$$\|v\|_\infty \leq \frac{1}{8}\|\nabla_h^2 v\|_\infty.$$

(b) The difference scheme $u^{n+1} = Qu^n, n \geq 0$ is stable with respect to the norm $\|\ .\ \|$ if and only if there exists positive constants $\triangle x_0$ and $\triangle t_0$ and non-negative constants K and β so that [5 Marks]

$$\| Q^{n+1} \| \leq Ke^{\beta t}$$

for $0 \leq t = (n+1)\triangle t, \quad 0 < \triangle x \leq \triangle x_0, \quad \text{and } 0 < \triangle t \leq \triangle t_0.$

(c) Define: Truncation error. Order of accuracy, Consistency and Stability of a finite difference scheme. [4 Marks]

(2) (a) Find the solution of the initial value problem [7 Marks]

$$u_t + au_x = f(t,x), \text{ where } a \text{ is positive}$$

with $u(0,x) = 0$ and $f(t,x) = \begin{cases} 1 \ if \ -1 \leq x \leq 1 \\ 0 \ otherwise. \end{cases}$

(b) Consider the system [7 Marks]

$$\begin{pmatrix} u^1 \\ u^2 \end{pmatrix}_t + \begin{pmatrix} a & b \\ b & a \end{pmatrix} \begin{pmatrix} u^1 \\ u^2 \end{pmatrix}_x = 0$$

on the interval [0,1], with a equal to 0 and b equal to 1 and with boundary conditions u^1 equal to 0 at both left and right boundaries. Show that if the initial data are given by $u^1(0,x) = x$ and $u^2(0,x) = 1$, then find the solution for $1 \leq t \leq 2$.

(3) (a) Find the truncation error and analyze the Von-Neumann stability of the leapfrog scheme for the equation $u_t + au_x = 0$, where a is a constant. [3+4 Marks]

(b) Write FTCS and Crank-Nicolson schemes for the equation $u_t + au_x + bu_y = 0$, where a and b are constants. Analyze the stability of the Crank-Nicolson scheme. [4+3 Marks]

(4) (a) Find the order of accuracy of the Du Fort-Frankel scheme for the heat conduction equation $u_t = bu_{xx} + f(t, x)$, where b is a positive constant. Also, determine the stability of this scheme by using the Von-Neumann stability analysis. [3+4 Marks]

(b) Use the Du Fort-Frankel scheme to determine the numerical solution of the initial boundary value problem [7 Marks]

$$u_t = u_{xx}$$

subject to the initial conditions

$$u(x, 0) = sin(\pi x),\ 0 < x < 1$$

and the boundary conditions $u(0, t) = u(1, t) = 0,\ t > 0$. Use $h = \frac{1}{3}$ and $k = \frac{1}{36}$ to determine the solution upto two time levels.

(5) (a) Draw a two dimensional polar coordinate finite difference grid and use it to derive an explicit scheme for the Poisson's equation in spherical polar coordinates. Also determine the order of accuracy and truncation error of the explicit scheme. [3+4 Marks]

(b) Elucidate the Alternating Direction Implicit(ADI) scheme. Find the ADI schemes with different step lengths for two and three dimensional parabolic equations. [2+5 Marks]

(6) (a) Let $u(x, y)$ be the solution to $\nabla^2 u = f$ on the unit square with Dirichlet boundary conditions and let $v_{l,m}$ be the solution to $\nabla_h^2 v = f$ with $v_{l,m} = u(x_l, y_m)$ on the boundary. Then [6 Marks]

$$\|u - v\|_\infty \leq ch^2 \|\partial^4 u\|_\infty,$$

where $\|\partial^4 u\|_\infty$ is the maximum magnitude of all the fourth derivatives of u over the interior of the square.

(b) What is a finite element? Elucidate triangular and rectangular elements used for a two-dimensional region. Find a two parameter approximate solution of the boundary value problem [4+4 Marks]

$$\nabla^2 u = x^2 - 1,\ \ |x| \leq 1,\ \ |y| \leq 1$$
$$u = 0 \text{ on the boundary}$$

using the least square method.

(7) (a) Derive the Galerkin finite element method for the equation $A(x, y)u_{xx} + 2B(x, y)u_{xy} + C(x, y)u_{yy} + D(x, y)u_x + E(x, y)u_y + F(x, y)u + G(x, y) = 0, (x, y) \in R$ subject to the Dirichlet condition $u(x, y) = 0, (x, y) \in S$ and hence deduce it for the poisson equation $\nabla^2 u = g(x, y)$. [4+3 Marks]

(b) Find the solution of the boundary value problem [7 Marks]

$$\nabla^2 u = -1,\ \ |x| \leq 1,\ \ |y| \leq 1$$
$$u = 0,\qquad |x| = 1,\ \ |y| = 1$$

by using the Galerkin finite element method with $h = \frac{1}{2}$ and triangular mesh.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examinations, May-June 2026
Part II Semester IV
MATH18-404(iii): Dynamical Systems
(Unique Paper Code: 223502414)

Time: 3 hours | Maximum Marks: 70

Instructions: • Attempt **five** questions in all. • **Question 1 is compulsory.**
• All questions carry equal marks. • All notation are standard.

(1) (a) Give a necessary and sufficient condition for a linear system of differential equations $\dot{x} = Ax$ in $\mathbb{R}^n$ to have a non-constant periodic solution. Justify your answer.

(b) Prove that the exponential operator on the space $M_n(\mathbb{R})$ of $n \times n$ matrices with real entries satisfies $\|e^A\| \leq e^{\|A\|}$, for all $A \in M_n(\mathbb{R})$, where $\|.\|$ denotes any matrix norm.

(c) Let $f \in C^1(\mathbb{R}^n)$. Suppose that the vector field f points inward towards the bounded domain at each point on the unit sphere $S = \{x \in \mathbb{R}^n : \|x\| = 1\}$. Prove that the solution $x(t)$ of the system $\dot{x} = f(x)$ with $x(0) \in S$ exists for all $t \geq 0$.

(d) State the Stable Manifold Theorem.

[4+4+4+2]

(2) (a) Consider the linear system of differential equations $\dot{x} = Ax$ in $\mathbb{R}^n$. Compute the smallest n so that there exists an A such that each of the following four functions is a solution to the given system:

$$tx_1, \quad 10te^{-2t}\sin(6t)x_2, \quad 5t^3e^{-9t}x_3, \quad 9t^2e^{-2t}\cos(6t)x_4,$$

where x_1, x_2, x_3, and x_4, are nonzero vectors in $\mathbb{R}^n$.
For the smallest n, give the real Jordan form of the matrix A.

(b) Let $A \in M_n(\mathbb{R})$ with all its eigenvalues real and lying in the interval $(-1, 1)$. Prove that the fixed point 0 is a sink for the linear map $f(x) = Ax$, for all $x \in \mathbb{R}^n$. Deduce that for the affine map $g(x) = Ax + b$, for $x \in \mathbb{R}^n$ and a fixed vector $b \in \mathbb{R}^n$, its unique fixed point x_0 is a sink.

[6+8]

(3) (a) Compute all values of $\mu \in \mathbb{R}$ for which the origin is asymptotically stable for the planar system:

$$\begin{aligned} \dot{x} &= -5x + \mu y + y^3 + x^5 \\ \dot{y} &= -2x + 2y + y^3 + y^5. \end{aligned}$$

(b) Solve the planar system

$$\dot{y} = -y, \quad \dot{z} = 2z + y^2.$$

Show that the successive approximations $\Phi_k \to \Phi$ and $\Psi_k \to \Psi$ as $k \to \infty$, for all $(y, z) \in \mathbb{R}^2$.
Compute H_0 and the homeomorphism $H : \mathbb{R}^2 \to \mathbb{R}^2$. Compute the stable and unstable manifolds of the system at the origin. Plot the phase portraits for the linearized system at the origin and for the nonlinear system.

[6+8]

Please turn over

(4) (a) For a real Jordan block J, prove that $e^{\mathrm{tr}(J)} = \det\left(e^J\right)$, where tr(.) and det(.) denote the trace and determinant of the matrix, respectively.
Deduce that, for a real square matrix A, $e^{\mathrm{tr}(A)} = \det\left(e^A\right)$.

(b) Prove that any two systems of linear differential equations in $\mathbb{R}^n$ both having the origin as a globally strongly attracting equilibrium point are topologically conjugate to each other.

[7+7]

(5) (a) Consider the planar system

$$\dot{x} = -2x - y^2, \quad \dot{y} = -y - x^2.$$

Using the Lyapunov function $V(x, y) = x^2 + y^2$, determine the largest possible disk centered at the origin on which the origin is asymptotically stable.

(b) Define semiconjugacy between two maps $f : I \to I$ and $g : J \to J$, for closed and bounded intervals I and J in $\mathbb{R}$.
Show that the following two maps $f, g : [0, 1] \to [0, 1]$ given by

$$f(x) = \begin{cases} 2x, & 0 \le x \le 0.5, \\ 2 - 2x, & 0.5 < x \le 1 \end{cases}, \quad g(x) = 4x(1 - x), \quad x \in [0, 1],$$

are semiconjugate.

[7+7]

(6) (a) Prove that a closed orbit in a planar system meets a local section in at most one point.

(b) Consider the planar system:

$$\dot{x} = y - \mu x - x^2, \quad \dot{y} = -\mu y - x + x^2,$$

where μ is the bifurcation parameter.
Determine all values of μ at which Hopf bifurcation occurs. At each bifurcation value for Hopf bifurcation, check whether it is subcritical or supercritical.

[7+7]

(7) (a) Consider the planar system

$$\dot{x} = x^2 y - 3x^5, \quad \dot{y} = -y + x^2.$$

Give an approximation of the local center manifold at the origin.
Plot the phase portrait of the system near the origin also showing $E^c, E^s, W^c_{loc}(0)$.

(b) Consider the following planar system in polar coordinates:

$$\dot{r} = r(1 - r), \quad \dot{\theta} = 1.$$

Compute all ω-limit sets and α-limit sets of the system.
Compute the Poincare map along the periodic solution $r = 1$.
Discuss the stability of this periodic solution using the Poincare map.

[7+7]

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Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations, May–2026
(Part II, Semester IV)
MMATH18–405(i): Cryptography
(UPC NO. 223503401)

Time: 2 Hours | Maximum Marks: 35

Instructions: • Question 1 is compulsory.
• Attempt **FIVE** questions in all.
• All symbols have their usual meanings unless otherwise specified.

(1) Discuss the following:

(a) Determine whether the affine cipher [2 Marks]

$$E(x) = 7x + 5 \pmod{26}$$

is invertible. If yes, find the decryption function.

(b) For the Vigenère cipher with key $K = $ `8 12 6 10`, encrypt the plaintext `RING`. [2 Marks]

(c) Describe the Hill cipher. Encrypt the plaintext [3 Marks]

$$HELP$$

using the key matrix

$$K = \begin{pmatrix} 3 & 3 \\ 2 & 5 \end{pmatrix} \pmod{26}.$$

(2) (a) What is a probability distribution? [2 Marks]

(b) Using CBC mode with block length 4, initialization vector [5 Marks]

$$IV = 1101,$$

and permutation

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix},$$

encrypt the plaintext

$$1110010110101100.$$

(3) What is a public key cryptosystem? Explain the Diffie–Hellman key exchange protocol. Show the complete algorithm for how two users establish a common secret key. Also Discuss the importance of the discrete logarithm for the security of the protocol. [2+5 Marks]

(4) Explain the ElGamal encryption scheme. [2+2+3 Marks]
Let

$$p = 23, \qquad g = 5, \qquad a = 6., b=3.$$

Compute the public key corresponding to the private key a. Encrypt the message $m = 13$, also show the decryption of the obtained ciphertext.

(5) Describe the RSA cryptosystem in detail. [2+2+2+1 Marks]
Take

$$p = 17, \qquad q = 11, \qquad e = 7.$$

(a) Compute the public and private keys.
(b) Encrypt the message $m = 88$.
(c) Decrypt the ciphertext obtained above.
(d) Explain why factorization of $n = pq$ is crucial for breaking RSA.

(6) Let [2+2+3 Marks]

$$n = 561.$$

(a) Show that 561 is composite.

(b) Verify that

$$2^{560} \equiv 1 \pmod{561}.$$

(c) Factorize 561 and determine whether it is a Carmichael number with reason.

(7) What are the plaintext, ciphertext, and key space in DES algorithm? Suppose in a DES round, the left and right halves are received [1+2+2+2 Marks]

$$L_0 = 10101100, \qquad R_0 = 11001010.$$

Suppose the round function output is

$$f_{K_1}(R_0) = 01100101.$$

(a) Compute the outputs

$$L_1, \ R_1.$$

(b) Explain why the Feistel structure permits decryption using the same algorithm.

(c) Explain the role of S-boxes in DES?

— **END OF QUESTION PAPER** —

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Final Examination, 2026
Part II Semester IV

MMATH18 405(ii): SUPPORT VECTOR MACHINES

Unique Paper Code: 223503402

Time: 2 hours Maximum Marks: 35

Instructions: • Attempt five questions in all. • Question 1 is compulsory.
• Answer any four questions from Q. 2 to Q. 7. • All questions carry equal marks.
• All notations have standard meaning.

(1) (a) Define a regression problem. When is it said to be a linear regression problem? [2 Marks]

(b) Define a kernel function on $\mathbb{R}^n \times \mathbb{R}^n$. What do you mean by Gram matrix of the kernel function with respect to l points in $\mathbb{R}^n$? [2 Marks]

(c) If f is a real-valued function defined on $\mathbb{R}^n$ then prove that the set [3 Marks]

$$S = \{x \in \mathbb{R}^n : f(x) \leq c\}$$

is convex for any $c \in \mathbb{R}$.

(2) Consider the convex programming problem (CP) [7 Marks]

$$\text{Minimize } f_0(x)$$
$$\text{subject to } f_i(x) \leq 0, i = 1, ..., m,$$
$$h_i(x) = a_i^T x - b_i = 0, i = 1, ..., p,$$

where $f_0, f_i, i = 1, ..., m$ are continuous convex functions on $\mathbb{R}^n$ and $a_i \in \mathbb{R}^n, b_i \in \mathbb{R}, i = 1, ..., p$. Show that the solution set of (CP) is a closed convex set. Also, show that a local solution of (CP) is a global solution.

(3) For a linearly separable problem with training set $T = \{(x_1, y_1), ..., (x_l, y_l)\} \in (\mathbb{R}^n \times \{-1, 1)\})^l$, prove that there exists a solution (w^*, b^*) to the the following optimization problem [7 Marks]

$$\text{Min } \frac{1}{2}\|w\|^2$$
$$\text{subject to } y_i((w.x_i) + b) \geq 1, i = 1, ..., l,$$

such that $w^* \neq 0$ and there exists $j \in \{i : y_i = 1\}$ satisfying

$$(w^*.x_j) + b^* = 1.$$

(4) Let Φ be a nonlinear map from $\mathbb{R}^n$ to a Hilbert space $\mathcal{H}$ and $C > 0$. Write the dual of the following regression problem (P_{ξ^*}) [7 Marks]

$$\text{Minimize } \tfrac{1}{2}\|W\|^2 + C\textstyle\sum_{i=1}^{l}(\xi_i + \xi_i^*)$$
$$\text{subject to } (W.\Phi(x_i)) + b - y_i \leq \varepsilon + \xi_i, i = 1, ..., l,$$
$$y_i - (W.\Phi(x_i)) - b \leq \varepsilon + \xi_i^*, i = 1, ..., l,$$
$$\xi_i^{(*)} \geq 0, i = 1, ..., l,$$

corresponding to the training set $T = \{(x_1, y_1), ..., (x_l, y_l)\}$.

(5) (a) Consider a hard $\bar{\varepsilon}$-band hyperplane. If $(\bar{w}, \bar{b}, \bar{\eta})$ is the solution to the problem [4 Marks]

$$\begin{aligned} &\text{Minimize } \tfrac{1}{2}\|w\|^2 + \tfrac{1}{2}\eta^2 \\ &\text{subject to } (w.x_i) + \eta(y_i + \bar{\varepsilon}) + b \geq 1, i = 1, ..., l, \\ &(w.x_i) + \eta(y_i - \bar{\varepsilon}) + b \leq -1, i = 1, ..., l, \end{aligned}$$

then show that $\bar{\eta} \neq 0$. Furthermore, let $\varepsilon = \bar{\varepsilon} - \frac{1}{\bar{\eta}}$, and ε_{inf} be the optimal value of the problem

$$\begin{aligned} &\text{Minimize } \varepsilon \\ &\text{subject to } -\varepsilon \leq y_i - ((w.x_i) + b) \leq \varepsilon, i = 1, ..., l, \end{aligned}$$

then show that $\varepsilon_{\text{inf}} \leq \varepsilon < \bar{\varepsilon}$.

(b) Show that for a training set $T = \{(x_1, y_1), ..., (x_l, y_l)\}$, $x_i \in \mathbb{R}^n, y_i \in \{1, -1\}, i = 1, ..., l$, a hyperplane $y = (w.x) + b$ is a hard $\bar{\varepsilon}$-band of hyperplane if and only if the sets $D^+ = \{(x_i^T, y_i + \bar{\varepsilon})^T, i = 1, \ldots, l\}$ and $D^- = \{(x_i^T, y_i - \bar{\varepsilon})^T, i = 1, \ldots, l\}$ locate on both sides of the hyperplane, respectively, and all of the points in D^+ and D^- do not touch this hyperplane. [3 Marks]

(6) (a) Define the Kronecker product $A \otimes A'$ of the matrix $A = (a_{ij}) \in \mathbb{R}^{m \times m}$ and the matrix $A' = (a'_{ij}) \in \mathbb{R}^{n \times n}$. If the adjacent matrices of the graphs $G = (V, E), G' = (V', E')$ are A and A' respectively, and $\lambda \in (0, 1)$ is the weight factor then what is the kernel based on the paths of the graphs G and G'? What is the value of this graph kernel for sufficiently small λ? [3 Marks]

(b) Give the algorithm for C-support vector classification in terms of a kernel function and a positive penalty parameter C. [4 Marks]

(7) For each input $x_i \in \mathbb{R}^n$ of the training set $T = \{(x_1, y_1), ..., (x_l, y_l)\}$, $y_i \in \{1, -1\}$ the value $g_i = g(x_i)$ is computed by the function $g(x)$ and $p_i = p_i(c_1, c_2) = \frac{1}{1+\exp(c_1 g_i + c_2)}$ is the probability of it belonging to the positive class. Show that the unconstrained problem to find the optimal probability function with variables c_1 and c_2 is given by [7 Marks]

$$\text{Minimize } -\sum_{i=1}^{l}\{t_i \log(p_i(c_1, c_2)) + (1 - t_i)\log(1 - p_i(c_1, c_2))\}$$

where

$$t_i = \frac{y_i + 1}{2}, \begin{cases} 1, & \text{if } y_i = 1, \\ 0, & \text{if } y_i = -1. \end{cases}$$