Your Roll Number: ....

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.Sc. Mathematics Examinations, May 16, 2026
Part I Semester II

DSC-4: MODULE THEORY

(Unique Paper Code: 235127921201)

Time: 3 Hours — Maximum Marks: 90

Instructions: • Question 1 is compulsory. • Attempt any four questions from question 2 to question 7. • All questions carry equal marks. • Throughout the question paper the word 'ring' or R denotes the ring with unity.

(1) (a) Define semisimple module. Further, give an example of a semisimple module and an example of a module which is not semisimple. [4]

(b) Let $f : M \to M'$ be an R-homomorphism with $Ker(f) = K$. Show that the sequence $0 \to K \xrightarrow{i} M \xrightarrow{f} M' \xrightarrow{\pi} \frac{M'}{Im(f)} \to 0$ is exact, where i is the inclusion map and π is the natural projection map. [5]

(c) Let M, N be modules over a commutative ring R. Define $M \otimes_R N$. Also, state the existence and uniqueness theorem of tensor product of two R-modules. [5]

(d) Define Noetherian and Artinian modules with examples. [4]

(2) (a) Let $f : M \to M'$ be an R-homomorphism and N be a submodule of M such that $N \subseteq Ker(f)$. Show that there exists a unique R-homomorphism $\bar{f} : \frac{M}{N} \to M'$ such that $f = \pi.\bar{f}$, where $\pi : M \to \frac{M}{N}$ is the natural projection map. [9]

(b) Let $\{M_i\}_{i\in\Delta}$ be a family of left R-modules. Define $\prod_{i\in\Delta} M_i$. Show that $\prod_{i\in\Delta} M_i$ is a left R-module and $\coprod_{i\in\Delta} M_i$ is its submodule. [2+3+4]

(3) (a) Let R be a ring and M be an additive abelian group. Show that M is a right R-module if and only if there is a ring homomorphism R to $End(M)$. [9]

(b) Let M be a finitely generated free left R-module and B be any basis. Show that B is finite and $M \cong R^{|B|}$. [4+5]

(4) (a) Let R be a principal ideal domain and M be a torsion module. Show that M is a finite direct sum of p-primary modules for different primes p of R. [9]

(b) Let R be a commutative ring with unity and $M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ be an exact sequence of R-modules. Show that the sequence $M' \otimes N \xrightarrow{f \otimes I} M \otimes N \xrightarrow{g \otimes I} M'' \otimes N \to 0$ is exact, for any R-module N. [9]

(5) (a) Show that every free left R-module is projective. [8]

(b) Let M be a left R-module. Show that M is injective if and only if every R-homomorphism from U to M can be extended to an R-homomorphism from R to M, where U is a left ideal of R. [10]

(6) (a) Let $M = \oplus_{i \in I} S_i$ be a semisimple left R-module, where S_i is simple for all $i \in I$. Show that M is finitely generated if and only if I is a finite set. [6]

(b) Let M be a left R-module and $M = \sum_{i \in I} S_i$, where S_i is simple for all $i \in I$. Show that M is semisimple. [12]

(7) (a) Give an example of a Noetherian module that is not Artinian and an example of an Artinian module module that is not Noetherian. Give the complete justification also. [4+5]

(b) Let N be a submodule of a left R-module M. Show that M is Noetherian if and only if N, $\frac{M}{N}$ are both Noetherian. [9]

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Your Roll Number: ....

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations, May 2026
Part I Semester II
Paper No. DSC 5: Functional Analysis
Unique Paper Code : 235127921202

Time: Three hours | Maximum Marks: 90

Instructions: • Write your roll number on the space provided at the top of this page immediately on receipt of this question paper • Question 1 is compulsory • Attempt 4 questions from the remaining six questions • All symbols have their usual meaning.

Section A

Attempt all parts in this section.

(1) (a) Let c_{00} be the set of all sequences with only finitely many non-zero terms. Show that $c_{00} \subset \ell^1$. Is it also dense in ℓ^1? Justify.

(b) Give a non-trivial example of a bounded linear functional on $C[0,1]$ (with the sup norm) and compute its norm.

(c) Is the space $(c_0, \|\cdot\|_\infty)$ reflexive ? Justify.

(d) Let $T : \ell^2 \to \ell^2$ be given by $T(\zeta_1, \zeta_2, \zeta_3, \ldots) = (0, \zeta_1, \zeta_2, \zeta_3, \ldots)$, for all $(\zeta_1, \zeta_2, \zeta_3, \ldots) \in \ell^2$. Show that T is a bounded linear transformation and find $\|T\|$.

(e) Let $M \subset H$ be a non-empty subset of the Hilbert space H. Show that $M^\perp$ is a closed subspace of H.

(f) Let $T : \ell^2 \to \ell^2$ be given by $T(\zeta_1, \zeta_2, \ldots) = (\zeta_1, \frac{\zeta_2}{2}, \frac{\zeta_3}{3}, \ldots)$, for all $(\zeta_1, \zeta_2, \zeta_3, \ldots) \in \ell^2$. Is T a unitary operator?

(3 x 6)

Section B

Attempt four questions in all from this section.

(2) Let $(X, \|\cdot\|)$ be a normed space.

(a) What is meant by a Cauchy sequence in X? Show that every convergent sequence in X is Cauchy. (1+3)

(b) Let $X = \ell^\infty$ and $Y = \{(x_n)_{n=1}^\infty \subset \ell^\infty : \exists n_0 \in \mathbb{N}$ such that $x_n = 0 \, \forall n \geq n_0\}$. For $n \in \mathbb{N}$, let $y_n := (1, 1/2, \ldots, \frac{1}{n}, 0, 0, \ldots)$. Show that $(y_n)_{n=1}^\infty$ is a Cauchy sequence. Is it convergent in Y? Hence or otherwise show that Y is not closed in ℓ^∞. (4+3+2)

(c) Show that X is a Banach space if and only if $\{x \in X : \|x\| = 1\}$ is complete. (5)

(3) Let N, N' be normed spaces.

(a) Show that the set $\mathcal{B}(N, N')$ of all continuous linear transformations from N into N' is a normed space. Prove further that $\mathcal{B}(N, N')$ is a Banach space if N' is a Banach space. (4+5)

(b) Let M be a closed linear subspace of a normed linear space N and define $T : N \to N/M$ be $T(x) := x + M$. Show that T is a bounded linear transformation and $\|T\| \leq 1$. (5)

(c) Let $x_0 \in N, x_0 \neq 0$. Show that there exists a continuous linear functional f_0 on N such that $f_0(x_0) = 1$ and $\|f_0\| = \frac{1}{\|x_0\|}$. (4)

(4) Let N be a complex normed space and N^* be its conjugate space.

(a) For $x \in N$, let $F_x : N^* \to \mathbb{C}$ be given by $F_x(f) = f(x) \forall f \in N^*$. Show that F_x is a continuous linear transformation on N^*. (5)

(b) Show that the map C given by $C(x) = F_x$ is an isometric isomorphism of N into N^{**}. (5)

(c) Prove that if $p \in (1, \infty)$, and $N = \ell^p$, then the map C above is surjective. (8)

(5) Let $(B, \|\cdot\|)$ and $(B', \|\cdot\|')$ be Banach spaces.

(a) Show that $B \times B' = \{(x, y) : x \in B, y \in B'\}$ is a Banach space with respect to the norm given by $\|(x, y)\|_0 := \|x\| + \|y\|', x \in B, y \in B'$. Show further that the graph $G(T)$ of a linear transformation $T : B \to B'$ is closed if and only if whenever $x_n \to x, ((x_n) \in B)$ and $Tx_n \to y$ as $n \to \infty, y = Tx$. (3+4)

(b) State the Closed Graph Theorem and prove it. (1+6)

(c) Let $T : B \to B'$ be a continuous, bijective linear transformation. Show that there exist non negative real numbers a, b such that $a\|x\| \leq \|Tx\|' \leq b\|x\|$ for all $x \in B$. (4)

(6) Let H be a Hilbert space.

(a) What is meant by an orthonormal set in H. Give an example of an orthonormal set in an infinite dimensional Hilbert space. (3)

(b) Show that if $\{e_1, e_2, \ldots, e_n\}$ is a finite orthonormal set in H then for any $x \in H$, $\sum_{i=1}^{n} |\langle x, e_i\rangle|^2 \leq \|x\|^2$ and $x - \sum_{i=1}^{n} \langle x, e_i\rangle e_i$ is orthogonal to e_j for each j. (3+3)

(c) Define a complete orthonormal set and show that $\sum_{i=1}^{n} |\langle x, e_i\rangle|^2 = \|x\|^2$ for every $x \in H$, if and only if $\{e_1, e_2, \ldots, e_n\}$ is a complete orthonormal set in H. (1+8)

(7) Let H be a Hilbert space. For any $y \in H$, define f_y on H by $f_y(x) = \langle x, y\rangle$ for all $x \in H$.

(a) Show that $f_y \in H^*$ for all $y \in H$ and $\|f_y\| = \|y\|$. Further, prove that for every $f \in H^*$, there exists a unique $y \in H$ such that $f = f_y$. (3+5)

(b) Let T be a continuous linear transformation on H. Define clearly the adjoint $T^* : H \to H$ of T as a composition of maps and use this definition to show that $\langle Tx, y\rangle = \langle x, T^*y\rangle$, for all $x, y \in H$. (4)

(c) Let $\mathcal{B}_{sa}(H)$ be the set of all self-adjoint operators in $\mathcal{B}(H)$. Show that $\mathcal{B}_{sa}(H)$ is a real Banach space containing the identity operator. Further, show that $\mathcal{B}_{sa}(H)$ is a partially ordered set. (6)

Your Roll Number: ....

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examinations, May-June 2026
Part I Semester II

DSC-6 : COMPLEX ANALYSIS, UPC: 235127921203

Time: 3 hours | Maximum Marks: 90

Instructions: • Attempt **FIVE** questions in all. • **Q.No.1** is compulsory. • Each question carries 18 marks. • All the symbols have their usual meaning unless otherwise specified.

(1) (a) Show that a branch of logarithm function is analytic and its derivative is z^{-1}. [3 Marks]

(b) Define cross-ratio. Prove that a Möbius transformation takes circles onto circles. [3 Marks]

(c) Let $f(z) = \frac{1}{z(z-1)(z-2)}$. Give the Laurent expansion of $f(z)$ in the annulus $\text{ann}(0; 1, 2)$. [3 Marks]

(d) Does there exist an analytic function $f : D \to D$ with $f(\frac{1}{2}) = \frac{3}{4}$ and $f'(\frac{1}{2}) = \frac{2}{3}$, where D is the open unit disc? Justify your answer. [3 Marks]

(e) Let $\gamma(t) = e^{it}, t \in [0, 2\pi]$. Find $\int_\gamma z^n dz$ for every integer n. [3 Marks]

(f) Let G be an open set and $f : G \to \mathbb{C}$ be analytic and one-one. Show that $f'(z) \neq 0$ for all $z \in G$. [3 Marks]

(2) (a) Determine an analytic function that maps right half plane onto a unit disc. [6 Marks]

(b) If G is open and connected and $f : G \to \mathbb{C}$ is differentiable with $f'(z) = 0, \forall z \in G$ then prove that f is constant. [6 Marks]

(c) Explain the correspondence between unit sphere and the extended complex plane. [6 Marks]

(3) (a) Let G be open in $\mathbb{C}$ and let γ be a rectifiable path in G with initial and end points α and β respectively. If $f : G \to \mathbb{C}$ is a continuous function with primitive $F : G \to \mathbb{C}$, then prove that $\int_\gamma f = F(\beta) - F(\alpha)$. [9 Marks]

(b) State and prove Cauchy's Estimate. Evaluate $\int_\gamma \frac{z^2+1}{z(z^2+4)} dz$ where $\gamma(t) = re^{it}, 0 \leq t \leq 2\pi$, for all possible values of r, $0 < r < 2$ and $2 < r < \infty$. [9 Marks]

(4) (a) Let G be a connected open set and let $f : G \to \mathbb{C}$ be an analytic function, and if $\{z \in G : f(z) = 0\}$ has a limit point in G then show that there is a point a in G such that $f^{(n)}(a) = 0$ for each $n \geq 0$. [6 Marks]

(b) If $\gamma : [0, 1] \to \mathbb{C}$ is a closed rectifiable curve and $a \notin \{\gamma\}$, then prove that $\frac{1}{2\pi i} \int_\gamma \frac{dz}{z-a}$ is an integer. [6 Marks]

(c) Let G be an open subset of the plane and $f : G \to \mathbb{C}$ be an analytic function. If γ is a closed rectifiable curve in G such that $n(\gamma; w) = 0$ for all w in $\mathbb{C} - G$, then prove that $\int_\gamma f = 0$. [6 Marks]

(5) (a) Using *Residue Theorem* evaluate $\int_0^\infty \frac{dx}{(x^2+a^2)^2}, a > 0$ [6 Marks]

(b) If f has an essential singularity at $z = a$, then show that for every $\delta > 0$, $\overline{\{f[ann(a; 0, \delta)]\}} = \mathbb{C}$. [6 Marks]

(c) Let $f : D \to D$ be a one-one analytic map of disc $D = \{z : |z| < 1\}$ onto itself and suppose $f(a) = 0$. Then prove that there is a complex number c with $|c| = 1$ such that $f = c\phi_a$, where $\phi_a(z) = \frac{z-a}{1-\bar{a}z}$. [6 Marks]

(6) (a) Define the notions of homotopy, homotopic to zero and simply connected set. If γ is a closed rectifiable curve in G such that $\gamma \sim 0$, then show that $n(\gamma; w) = 0$ for all w in $\mathbb{C} - G$. Is the converse true? Justify your answer. [9 Marks]

(b) Let G be a region and suppose that f is a non constant analytic function on G. Then show that for any open set U in G, $f(U)$ is open. [9 Marks]

(7) (a) State and prove Maximum Modulus theorem (First Version). Can the assumption of connectedness in this theorem be dropped? Justify your answer. [9 Marks]

(b) Define Meromorphic function. State and prove Argument principle. [6 Marks]

(c) State and prove Liouville's theorem. [3 Marks]

Roll No. ....

Department of Mathematics
University of Delhi, Delhi

M.Sc. Mathematics, Part-I, Semester-II
Examination, May-June 2026

Paper: **DSE 3(i) Algebraic Number Theory**
Unique Paper Code: 235137920005

Time: 3 Hour *Maximum Marks: 90*

Note: • Attempt five questions in all. • Question 1 is compulsory.

1. (a) Prove that for every algebraic number θ there exists a non-zero integer r such that $r\theta$ is an algebraic integer. (3)

   (b) Let $\alpha \in \mathbb{Z}[\sqrt{10}]$ be such that α divides both 2 and $\sqrt{10}$ in $\mathbb{Z}[\sqrt{10}]$. Prove that α is a unit in $\mathbb{Z}[\sqrt{10}]$. (3)

   (c) Factor $1 - 7i$ into a product of irreducibles in the ring $\mathbb{Z}[i]$. (4)

   (d) Let L be a n-dimensional lattice in $\mathbb{R}^n$ and $\mathcal{F}$ its fundamental domain. Prove that for $l_1 \neq l_2 \in L$, the sets $\mathcal{F} + l_1, \mathcal{F} + l_1$ are disjoint. (4)

   (e) Does there exist a number field with class number more than 1? Justify. (4)

2. (a) Define an integral basis of a number field K and show that it is a $\mathbb{Q}$-basis of K. Also, prove that every number field has an integral basis. (4+8)

   (b) Find the discriminant and integral basis of $\mathbb{Q}(\theta)$, where $\theta^3 = 36\theta + 78$. (6)

3. (a) For an odd prime p, define the p-th cyclotomic extension and find its discriminant and integral basis. (12)

   (b) Find the group of units of the ring of algebraic integers $\mathcal{O}_K$ of the number field $K = \mathbb{Q}(\sqrt{d})$, where $d < -3$ is a square-free integer. (6)

4. (a) Prove that the ring of algebraic integers $\mathcal{O}_K$ of a number field K is norm Euclidean if and only if for every $\epsilon \in K$, there is a $\gamma \in \mathcal{O}_K$ such that $|N_{K/\mathbb{Q}}(\epsilon - \gamma)| < 1$. (6)

   (b) Prove that the equation

$$y^2 + 4 = z^3$$

   has only the integer solutions $y = \pm 11, z = 5$ or $y = \pm 2, z = 2$. (12)

5. (a) Prove that every ideal and hence every fractional ideal of $\mathcal{O}_K$, the ring of integers of an algebraic number field K, is generated by at most two elements. (12)

(b) Let K be an algebraic number field and $\mathcal{O}_K$ its ring of integers. If a non-zero proper ideal I of $\mathcal{O}_K$ has prime ideal factorization $I = P_1^{a_1} P_2^{a_2} \cdots P_r^{a_r}$, then prove that $N(I) = \prod_{i=1}^{r} N(P_i^{a_i})$. (6)

6. (a) State and prove The Four Squares Theorem. (2+10)

(b) Define a lattice and prove that every discrete subgroup of $(\mathbb{R}^n, +)$ is a lattice. (1+5)

7. (a) Let K be an algebraic number field of degree $n = s + 2t$, $\mathcal{O}_K$ its ring of integers and $\sigma : K \to L^{s,t}$ the natural embedding. Prove that the for any non-zero ideal I of $\mathcal{O}_K$, $\sigma(I)$ is a lattice in $L^{s,t}$. Also, find the volume of the fundamental domain of the lattice $\sigma(I)$. (6+6)

(b) Define class-group and class-number of an algebraic number field and prove that class-number is always finite. (2+4)

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.Sc. Mathematics Examinations, May 2026
Part I Semester II

DSE-3(ii): GENERAL TOPOLOGY
(UPC-235137920006)

Time: 3 hours Maximum Marks: 90

Instructions: • Attempt **five** questions in all. **Question No. 1 is compulsory**. All questions carry equal marks.• The symbols used have their usual meanings.

(1) (a) When do you say a topological space to be completely regular? Prove that a subspace of a completely regular space is completely regular. [2+3]

(b) Prove that the Tietz extension theorem implies the Urysohn's lemma. [3]

(c) State Urysohn's metrization theorem. Give an example, with brief justification, of a topological space X that satisfies the hypotheses of the Urysohn's metrization theorem. [1+3]

(d) Let $\mathcal{A}$ be a locally finite collection of subsets of a space X. Prove that [1+3]

$$\overline{\cup_{A\in\mathcal{A}} A} = \cup_{A\in\mathcal{A}} \overline{A}.$$

(e) Give an example to show that a paracompact subspace of a Hausdorff space X need not be closed in X. [2]

(2) (a) When do you say a topological space X to be regular? Prove that the space $\mathbb{R}_K$ is Hausdorff but not regular, where $K = \{\frac{1}{n} : n \in \mathbb{N}\}$. [2+7]

(b) (i) State Urysohn's lemma. [2]

(ii) If X is a connected normal space having at least two points, prove that X is uncountable. Give an example to show that the conclusion fails if X is not connected. [5+2]

(3) (a) When do you say a topological space X to be normal? Prove that every compact Hausdorff space is normal. [2+7]

(b) Prove that $\mathbb{R}^\omega$ (in the product topology) is not lofcally compact. [4]

(c) Prove that the collection $\mathcal{A} = \{(0, \frac{1}{n}) : n \in \mathbb{N}\}$ is locally finite in $(0, 1)$ but not locally finite in $\mathbb{R}$. [5]

(4) (a) (i) When do you say a topological space X to be locally compact? [1]

(ii) Suppose X_1 and X_2 are locally compact Hausdorff spaces, Y_1 and Y_2 are their respective one-point compactifications and $f : X_1 \to X_2$ is a homeomorphism. Prove that f extends to a homeomorphism from Y_1 to Y_2. [7]

(b) Let X be a completely regular space. Prove that there exists a compactification Y of X having the property that every bounded continuous map $f : X \to \mathbb{R}$ extends to a continuous map of Y into $\mathbb{R}$. [10]

(5) (a) Let $X = [0,1] \cup [2,3]$ and $Y = [0,2]$ be subspaces of $\mathbb{R}$ and define $p : X \to Y$ as

$$p(x) = \begin{cases} x, & \text{if } x \in [0,1] \\ x-1, & \text{if } x \in [2,3]. \end{cases}$$

Show that p is a quotient map but p is not an open map. Show also that the restriction of p to a subspace of X need not be a quotient map. [6+3]

(b) Prove that every regular space with a countable basis is normal. [9]

(6) (a) Let $p : X \to Y$ be a quotient map, A be a subspace of X that is saturated with respect to p and let $q : A \to p(A)$ be the map obtained by restricting p. [9]

(i) If A is either open or closed in X, prove that q is a quotient map.

(ii) If p is either an open or a closed map, prove that q is a quotient map.

(b) Prove that every paracompact Hausdorff space is regular. [9]

(7) (a) Let A be a closed subspace of a normal space X and $f : A \to [-r, r]$ be a continuous function. Prove that there exists a continuous function $g : X \to \mathbb{R}$ such that [6]

$$|g(x)| \leq \frac{r}{3}, \ \forall x \in X$$

and

$$|g(a) - f(a)| \leq \frac{2r}{3}, \ \forall a \in A.$$

(b) (i) What do you mean by a partition of unity on a topological space? [3]

(ii) If X is a paracompact Hausdorff space and $\{U_\alpha\}_{\alpha \in J}$ is an indexed open covering of X, show that there exists a partition of unity dominated by $\{U_\alpha\}$. [9]

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Your Roll Number: ....

Department of Mathematics, University of Delhi
M.Sc. Mathematics End-Term Examinations, May-June 2026
Part I Semester II
DSE- 4(i): Fourier Analysis; UPC: 235137920007

Time: 3 Hours | Maximum Marks: 90

Instructions: • Answer **FIVE** questions. • **Question 1** is compulsory. Attempt any **Four** questions from **Q2** to **Q7**. • Each question carries 18 marks. All the symbols used have their usual meaning.

(1) (a) Prove that (4)

$$D_N(u) = \frac{\sin(N + \frac{1}{2})u}{2\pi \sin \frac{1}{2}u}.$$

(b) For $z = (2, i, 1), w = (1, 0, 2i)$, calculate $\widehat{(z * w)}$. (4)

(c) If $f \in C^2(\mathbf{T})$ with the Fourier coefficients (c_n), prove that $|c_n| \leq M/n^2$ for some constant M, $n \neq 0$. (4)

(d) Prove that the standard basis for $L^2(\mathbb{Z}_N)$ is time-localized but not frequency-localized. (3)

(e) For a signal $z \in L^2(\mathbb{Z}_N)$, prove that $\widehat{z^*}(m) = \overline{\hat{z}(m)},\ m \in \mathbb{Z}$. (3)

(2) (a) Give a characterization of all signals w in $L^2(\mathbb{Z}_N)$ such that

$$\{M_0 w, M_1 w, \ldots, M_{N-1} w\}$$

is an orthonormal basis for $L^2(\mathbb{Z}_N)$. (9)

(b) Let $A : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$ be a linear operator. Prove that A is a convolution operator if and only if it is a Fourier multiplier. Further, if A is a convolution operator, then prove that the matrix $(A)_F$ of A with respect to the Fourier basis F of $L^2(\mathbb{Z}_N)$ is diagonal. (6+3)

(3) (a) Let $N = 2M$, where M is a positive integer and $w \in L^2(\mathbb{Z}_N)$. Prove that $\{R_0 w, R_2 w, \ldots, R_{2M-2} w\}$ is an orthonormal set with M distinct signals in $L^2(\mathbb{Z}_N)$ if and only if (9)

$$|\hat{w}(m)|^2 + |\hat{w}(m + M)|^2 = 2,\ m = 0, 1, \ldots, M - 1.$$

(b) State and prove Fourier Inversion Formula. Use this to prove that (6+3)

$$\|z\|^2 = \frac{1}{N}\|\hat{z}\|^2, \quad z \in L^2(\mathbb{Z}_N).$$

(4) (a) Let $T : L^2(\mathbb{Z}_4) \to L^2(\mathbb{Z}_4)$ be defined by

$$T(z) = (2z(0) - z(1), iz(1) + 2z(2), z(1), 0).$$

(i) Is $(T)_S$, the matrix of T with respect to the standard basis, a circulant matrix?

(ii) Show that T is not translation invariant.

(iii) Is the vector $(1, i, -1, -i)$ an eigenvector of T? (9)

(b) Let f be a function that is absolutely Riemann integrable over $[-\pi, \pi]$. Prove that the Fourier coefficients of f tend to zero as $|n| \to \infty$. (9)

(5) (a) Let f be an odd function with period 2, that satisfies $f(t) = t(1-t)$ for $0 \leq t \leq 1$. Find the Fourier series of f. Use this to prove that (5+4)

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3} = \frac{\pi^3}{32}.$$

(b) Let $\{e_m\}_{m=0}^{N-1} \in L^2(\mathbb{Z}_N)$ be defined as $e_m(n) = \frac{1}{\sqrt{N}} e^{2\pi i m n/N}, n = 0, 1, \ldots, N-1$. Prove that (5+4)

(i) $\{e_m\}$ is an orthonormal basis of $L^2(\mathbb{Z}_N)$.

(ii) $z = \sum_{m=0}^{N-1} (z, e_m) e_m$ for every $z \in L^2(\mathbb{Z}_N)$.

(6) (a) Prove that the Fejer Kernel is a positive summation kernel. (9)

(b) Let Ω_N denote the matrix of the Fourier transform $\mathcal{F} : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$ and $\mathcal{G} : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$ denote the inverse Fourier transform. Prove that

(i) The matrix of $\mathcal{G}$ is $\frac{1}{N}\overline{\Omega_N}$.

(ii) $\mathcal{G}$ is the inverse of $\mathcal{F}$. (9)

(7) (a) Define time frequency localized basis for $L^2(\mathbb{Z}_N)$, the mother wavelet and the father wavelet. Let $\phi \in L^2(\mathbb{Z}_4)$ be given by $\widehat{\phi} = (\sqrt{2}, 1, 0, 1)^t$. Find a father wavelet $\psi \in L^2(\mathbb{Z}_4)$ such that $\{R_0\phi, R_2\phi, R_0\psi; R_2\psi\}$ is an orthonormal basis for $L^2(\mathbb{Z}_4)$. (9)

(b) Let f have the Fourier coefficients c_n. Prove the following:

(i) For $a \in \mathbb{Z}$, the function $t \to e^{iat} f(t)$ has the Fourier coefficients $\{c_{n-a}\}$.

(ii) If f is real valued, then $\bar{c_n} = c_{-n}$ for all n. (9)

Your Roll Number: ....

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations, May- June 2026
Part I Semester II

DSE-4 (ii): INTEGRAL EQUATIONS
(UPC 235137920008)

Time: 3 hours | Maximum Marks: 90

Instructions: • Question no. 1 is compulsory. • Answer any four questions from Question 2-7. • All notations have usual meaning.

(1) (a) Form an integral equation corresponding to the IVP: [4 Marks]
$y'' + y = 0; y(0) = 0, y'(0) = 1$.

(b) Define the Singular integral equations. [2 Marks]

(c) State the Volterra's population model. [4 Marks]

(d) Show that the integral equation $2\phi(x) = \int_0^1 e^{\frac{x+t}{2}}(1+\phi^2(t))dt$ has no real solutions. [4 Marks]

(e) Find the resolvent kernel for Volterra-type integral equation with the following kernel $K(x,t) = \frac{1+x^2}{1+t^2}$. [4 Marks]

(2) (a) Solve the following integral equation [6 Marks]

$$\phi(x) = x - \int_0^x \sinh(x-t)\phi(t)dt.$$

(b) Solve the following integro-differential equation: [7 Marks]

$$\phi''(x) - 2\phi'(x) + \phi(x) + 2\int_0^x \cos(x-t)\phi''(t)dt + 2\int_0^x \sin(x-t)\phi'(t)dt = \cos x; \phi(0) = \phi'(0) = 0.$$

(c) Solve the following integral equation by first reducing to second kind [5 Marks]

$$\int_0^x e^{x-t}\phi(t)dt = \sin x.$$

(3) (a) Prove that the solution of Volterra integral equation $\phi(x) = f(x)+\lambda\int_0^x K(x,t)\phi(t)dt$ is given by $\phi(x) = f(x)+\lambda\int_0^x R(x,t;\lambda)f(t)dt$, where $R(x,t;\lambda)$ is the resolvent kernel of the kernel $K(x,t)$. [6 Marks]

(b) Show that when $f(x) = C\sqrt{x}$ the solution of Abel's problem will be straight lines. [6 Marks]

(c) Solve [6 Marks]

$$2\phi(x) = \int_0^x \phi(t)\phi(x-t)dt - \sinh x.$$

(4) (a) Using Fredholm determinants, find resolvent kernel of [5 Marks]

$$k(x,t) = x^2t - xt^2, \qquad 0 \le x \le 1,\ 0 \le t \le 1.$$

(b) Find the iterated kernels of the kernel $k(x,t) = x - t$; with $a = -1$, $b = 1$. [5 Marks]

(c) Find the characteristic number and eigenfunctions of the homogeneous integral equation with the kernel [8 Marks]

$$K(x,t) = \begin{cases} \sin(x+\frac{\pi}{4})\sin(t-\frac{\pi}{4}), & 0 \le x \le t, \\ \sin(t+\frac{\pi}{4})\sin(x-\frac{\pi}{4}), & t \le x \le \pi. \end{cases}$$

(5) (a) Using the Hilbert-Schmidt method, solve $\phi(x) = 1+\lambda\int_0^\pi K(x,t)\phi(t)dt$, where [10 Marks]

$$K(x,t) = \begin{cases} \sin x \cos t, & 0 \le x \le t, \\ \sin t \cos x, & t \le x \le \pi. \end{cases}$$

(b) Show that the integral equation [8 Marks]

$$\phi(x) = \lambda \int_0^\infty \phi(t)\sin xt dt.$$

has characteristic numbers $\lambda = \pm\sqrt{2/\pi}$ of infinite multiplicity and find associated eigenfunctions.

(6) (a) Define Green's function for the 4^{th} order linear BVP with variable coefficient. Reduce the following BVP into integral equation $y'' - \lambda y = x^2, y(0) = y(\frac{\pi}{2}) = 0$. [4+8 Marks]

(b) Show that integral equation: [6 Marks]

$$\phi(x) - \lambda \int_0^1 (3x-2)t\phi(t)dt = 0$$

has no characteristic numbers and eigenfunctions.

(7) (a) Define the Fredholm minor and determinants. Show that the integral equation [4+7 Marks]

$$\phi(x) = f(x) + \frac{1}{x}\int_0^{2\pi} \sin(x+t)\phi(t)dt$$

possesses no solution for $f(x) = x$, but it has infinitely many solution when $f(x) = 1$.

(b) Prove that the characteristic numbers of symmetric kernels are real. [7 Marks]

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