Your Roll No.......................................

Department of Mathematics
University of Delhi, Delhi

M.Sc. Mathematics Examination, December 2025
Part-I, Semester-I

Paper: **DSC-1 Field Theory**
Unique Paper Code: 235127921101

Time: 3 Hour *Maximum Marks: 90*

Note: • Attempt five questions in all. • Question 1 is compulsory. • All questions carry equal marks.

1. (a) Find the minimal splitting field of $x^5 - 2$ over $\mathbb{Q}$. (4)
   (b) Prove that every algebraically closed field is infinite and perfect. (4)
   (c) Assuming that the Galois group of the extension $\mathbb{Q}(2^{1/3}, \omega)|\mathbb{Q}$, where ω is complex cube root of unity, is isomorphic to S_3, find the fixed fields corresponding to the subgroups $\langle(123)\rangle$ and $\langle(12)\rangle$. (4)
   (d) Give an example of a tower $k \subset F \subset E$ of field extensions such that $E|F$ and $F|k$ are both Galois but $E|k$ is not. (3)
   (c) Construct a finite field with 8 elements. (3)

2. (a) Let k be a field and let $f(x)$ be a polynomial of degree n in $k[x]$. Show that there exists a splitting field F of $f(x)$ over k of degree at most $n!$. (10)
   (b) Let k be a field. Show that a minimal splitting field Ω for the set of all polynomials over k is an algebraic closure of k. (8)

3. (a) Let $k \subset F \subset E$ be a tower of field extensions such that $E|k$ is finite normal. Prove that $E|F$ is always normal but $F|k$ need not be normal. Show that $F|k$ is normal if and only if $\sigma(F) = F$ for every k-automorphism σ of E. (2+3+7)
   (b) Prove that a polynomial over a field k is separable if and only if it is coprime with its derivative. (6)

4. (a) State and prove the Artin's Theorem. (10)
   (b) Let $F|k$ be a field extension of degree n. Prove that the extension $F|k$ is Galois if and only if it is both normal and separable. (8)

5. (a) Prove that for every positive integer n, the n-th cyclotomic polynomial Φ_n is a monic polynomial with integer coefficients and is irreducible over $\mathbb{Q}$. (9)
   (b) Prove that a finite extension of a finite field is Galois with cyclic Galois group. (9)

6. (a) Prove that a finite extension $F|k$ is simple if and only if there are only finitely many subfields between F and k. Deduce that every finite extension of $\mathbb{Q}$ is simple. (10)

(b) Let k be a field containing a primitive n-th root of unity. Prove that an extension $F|k$ of degree n is radical if and only if it is cyclic. (8)

7. (a) Let k be a field of characteristic $p > 0$. Show that the polynomial $x^p - x - a \in k[x]$, $a \neq 0$, is either irreducible or splits completely over k. (5)

(b) Prove that an equation $f = 0$ over a field of characteristic zero is solvable by radicals if the group of the equation is a solvable group. (13)

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.Sc. Mathematics Examinations, December 2025
Part I Semester I

DSC-2: INTRODUCTION TO TOPOLOGY
(UPC-235127921102)

Time: 3 hours | Maximum Marks: 90

Instructions: • Attempt five questions in all. **Question No. 1 is compulsory.** All questions carry equal marks.• The symbols used have their usual meanings.

(1) (a) In the plane $\mathbb{R}^2$, is the set $\{x \times y \,|\, x \leq 0 \text{ and } y \geq 0\}$ closed? Justify your answer. [4]

(b) Give an example to illustrate that a space need not be Hausdorff even if all finite point sets in X are closed. [4]

(c) Is the space $\mathbb{R}_l$ connected? Justify your answer. [2]

(d) Let $Y = \{a, b\}$ be a two point space in the indiscrete topology. Consider $\mathbb{N}$ as a subspace of $\mathbb{R}$ and $X = \mathbb{N} \times Y$. Is X compact? Limit point compact? Justify your answer. [2+3]

(e) Prove that $\mathbb{R}_l$ is not second countable. [3]

(2) (a) (i) Prove that the collection

$$\mathscr{B} = \{[a, b) \,:\, a, b \in \mathbb{Q}\,; a < b\}$$

is a basis that generates a topology different from the lower limit topology on $\mathbb{R}$. [5]

(ii) Give an example to show that the union of two topologies on X need not be a topology on X. [4]

(b) Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces and let $X = \Pi_{\alpha \in J} X_\alpha$ have the product topology. Let $f : A \to X$ be given by $f(a) = (f_\alpha(a))_{\alpha \in J}$, where $f_\alpha : A \to X_\alpha$ for each α. Show that f is continuous if and only if each f_α is continuous. [9]

(3) (a) Let X be a topological space. Let $\mathscr{C}$ be a collection of open subsets of X such that for each open set U of X and for each $x \in U$, there is an element C of $\mathscr{C}$ such that $x \in C \in U$. Show that $\mathscr{C}$ is a basis for the topology of X. [9]

(b) Let X be a metric space with metric d. Define $\overline{d} : X \times X \to \mathbb{R}$ by $\overline{d}(x, y) = \min\{d(x, y), 1\}$. Show that $\overline{d}$ is a metric that induces the same topology as d. [9]

(4) (a) Prove that every closed subspace of a compact space is compact. Give an example to show that the conclusion fails if the hypothesis of closedness is dropped. [5+4]

(b) Let A be a subspace of the space X and B be a subspace of the space Y. Prove that the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $X \times Y$. [9]

(5) (a) Let X and Y be topological spaces and $f : X \to Y$. Prove the following are equivalent: [10]

(i) f is continuous.

(ii) For every subset A of X, $f(\overline{A}) \subset \overline{f(A)}$.

(iii) For every closed set B of Y, $f^{-1}(B)$ is closed in X.

(b) Prove the continuous image of a path connected space is path connected. Hence or otherwise, prove that the unit sphere $S^{n-1} = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| = 1\}$ of $\mathbb{R}^n$ is path connected, if $n > 1$. [4+4]

(6) (a) Show that the subspace $[a, b]$ of $\mathbb{R}$, where $a < b$, is homeomorphic with $[0, 1]$. Is the subspace $[0, 1]$ of $\mathbb{R}$ homeomorphic with $(2, 3)$? With $[2, 3)$? Justify your answers. [8]

(b) (i) When do you say a topological space X to be locally path connected? [2]

(ii) Let X be a topological space. Show that each path component of X lies in a component of X. If X is locally path connected prove that the components and the path components are the same. [8]

(7) (a) Let X be a set with a simple order relation having at least two elements. Define a basis $\mathscr{B}$ for the order topology on X and show that $\mathscr{B}$ is a basis. [9]

(b) Let X, Y be topological spaces and $f : X \to Y$. If f is continuous prove that for every convergent sequence $x_n \to x$ in X, the sequence $f(x_n)$ converges to $f(x)$ and that the converse holds if X is first countable. [9]

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examinations, Dec. 2025
Part I Semester I

DSC-3: ORDINARY DIFFERENTIAL EQUATIONS
UPC - 235127921103

Time: 3 hours Maximum Marks: 90

Instructions: • Write your roll number on the space provided at the top of this page immediately on receipt of this question paper. • **Q.No.1 is compulsory.** • **Answer 5 questions in all.** • Each question carries 18 marks. • All the symbols have their usual meaning.

(1) (a) Does there exist unique solution of an IVP $y' = a(x)y^2 + b(x)y + c(x), y(0) = 0$, where $a(x), b(x)$ and $c(x)$ are continuous on $|x| \leq \alpha$. Give justification for your answer. [3 Marks]

(b) Prove that if ψ is a fundamental matrix of the system $Y' = AY$, then so is $C\psi$, where C is a constant non-singular matrix. [3 Marks]

(c) State the existence and uniqueness theorem for an IVP $y_1' = f_1(x, y_1, y_2, y_3); y_2' = f_2(x, y_1, y_2, y_3); y_3' = f_3(x, y_1, y_2, y_3);$ $y_1(x_0) = c_1^0, y_2(x_0) = c_2^0, y_3(x_0) = y_3^0$. [3 Marks]

(d) Find the condition for the existence of alternate zeros of functions $a\ cosx + b\ sinx$ and $c\ cosx + d\ sinx$ on R. [3 Marks]

(e) Write $\epsilon - \delta$ definition of stability of a critical point of a non-linear autonomous system $\dot{X} = F(X), X \in R^n$. [3 Marks]

(f) Reduce the autonomous non-linear system $\dot{x} = f(x, y)$, $\dot{y} = g(x, y)$ with suitable assumptions into associated linear system using method of perturbations. [3 Marks]

(2) (a) State and prove the continuity theorem for the solution of $\frac{dy}{dx} = f(x, y), y(x_0) = y_0$. [7 Marks]

(b) Find the system of three first order equation equivalent to the third order equation $y''' = x^2 + yy' + (y'')^2$. Explain precisely why or why not there exist a unique solution of the given equation under the conditions $y(0) = 1, y'(0) = -3, y''(0) = 0$. [7 Marks]

(c) Find the solution of $Y' = F(x, Y), Y(0) = Y_0$, where $Y = (y_1, y_2), F = (y_2, -y_1), Y_0 = (0, 1)$ by method of successive approximations. [5 Marks]

(3) (a) State Picard's existence theorem for the solution of initial value problem $\frac{dy}{dx} = f(x, y), y(x_0) = y_0$. [4 Marks]

(b) Define the Green's function $G(x, \xi)$ for a two point BVP. [3+5 Marks]

Determine the Green's function for the BVP $y'' = -x$, $y(0) = 0, y(1) + y'(1) = 2$ and hence solve it.

(c) Find the general solution of the system $y_1' = y_1 + y_2 - 2y_3$, $y_2' = -y_1 + 2y_2 + y_3, y_3' = y_2 - y_3$ by the method of eigenvalues. [6 Marks]

(4) (a) Prove that the eigenfunctions corresponding to distinct eigenvalues of Strum-Liouville system are orthogonal with respect to the weight function $s(x)$ in $[a, b]$. [7 Marks]

(b) Define limit cycle of an autonomous non-linear system. Find the limit cycle of the system $\dot{x} = y + \frac{x(1-x^2-y^2)}{\sqrt{x^2+y^2}}, \dot{y} = -x + \frac{y(1-x^2-y^2)}{\sqrt{x^2+y^2}}$. [6 Marks]

(c) Define a well posed problem for IVP $\dot{X} = F(X, t), Y(t_0) = Y_0$. Define critical point of a non-linear plane autonomous system as spiral point and node. [5 Marks]

(5) (a) Define Lyapunov function for a non-linear system $\dot{X} = F(X)$. [4 Marks]

(b) Prove that if there exist a Lyapunov $V(X)$ for non-linear system $\dot{X} = F(X)$ in an open region Ω, then the origin is stable. [7 Marks]

(c) Determine the nature and stability of the critical point of the spring mass system with damping given by $m\ddot{x} + c\dot{x} + kx = 0$, where m, c and k are positive constant. [7 Marks]

(6) (a) Show that linear combination of solutions $\phi_1, \phi_2, ...\phi_n$ of the system $Y' = A(t)Y$, where $A(t)$ is a $n \times n$ matrix, is also a solution of $Y' = A(t)Y$. If ϕ_j satisfies $\phi_j(t_0) = E_j$, where E_j are linearly independent vectors then prove that ϕ_j are linearly independent. [5 Marks]

(b) If ψ is a fundamental matrix of the associated homogenous system $Y' = AY$, then find solution of the non-homogenous system $Y' = A(t)Y + G(t)$ satisfying $\phi(t_0) = E$. [7 Marks]

(c) Define exponential matrix. Show that if A is a $n \times n$ constant matrix, then $\psi(t) = e^{tA}$ is a fundamental matrix of the system $Y' = AY$ on J. Find the exponential matrix of $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. [6 Marks]

(7) (a) State and prove Strum-comparison theorem. Show that every solution of $y'' + x^2y = 0$ has infinitely many zeros on $[1, \infty]$. [6 Marks]

(b) Derive the adjoint operator, self-adjoint form and Lagrange identity for a linear operator $L[y] = a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$. [6 Marks]

(c) Find the eigenvalues and eigenfunction of the system $y'' + y' + (1 + \lambda)y = 0, y(0) = 0, y(1) = 0$. [6 Marks]

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Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations, December 2025
Part I Semester I

DSE-1 (ii) (UPC-235137921102) : Numerical Analysis

Maximum Marks: 90

Time: 3 hours

Instructions: • Question 1 is compulsory • Attempt **any Four** questions from remaining questions. • Each question carries 18 marks.

(1) (a) Use Neville's method to approximate $\sqrt{3}$ with the function $f(x) = 3^x$, and values $x_0 = -2,\ x_1 = -1,\ x_2 = 0,\ x_3 = 1$, and $x_4 = 2$. [4 Marks]

(b) Determine the multiplicity of the root $x = 1$ of the polynomial $P(x) = x^5 - 2x^4 + 4x^3 - x^2 - 7x + 5 = 0$ by performing synthetic division. Also, use synthetic division to compute the values $P'(2)$ and $P''(2)$. [4 Marks]

(c) Use Crout factorization to solve the linear system [5 Marks]

$$3x_1 + x_2 = -1,$$
$$2x_1 + 4x_2 + x_3 = 7,$$
$$2x_2 + 5x_3 = 9.$$

(d) Construct an algorithm that computes the roots of the quadratic equation $ax^2+bx+c = 0$ and that avoids as many roundoff error problems as possible. Test your algorithm by computing the roots of the quadratic equation $0.2x^2 - 47.91x + 6 = 0$. Use 4 decimal digit rounding arithmetic in your calculations. [5 Marks]

(2) (a) Let $x_0 < x_1 < \cdots < x_n$ be knots and let $s(x)$ be cubic spline satisfying $s(x_i) = y_i$ for $i = 0, \ldots, n$. If $h_i = x_{i+1} - x_i$ and $M_i = s''(x_i)$, then show that [9 Marks]

$$\frac{h_{i-1}}{6}M_{i-1} + \frac{h_{i-1} + h_i}{3}M_i + \frac{h_i}{6}M_{i+1} = \frac{y_{i+1} - y_i}{h_i} - \frac{y_i - y_{i-1}}{h_{i-1}},\ i = 1, \ldots, n-1.$$

Moreover if $s(x)$ satisfy $s'(x_0) = y_0'$ and $s'(x_n) = y_n'$ derive the linear system of equations satisfied by the unknowns $M_0, M_1, \ldots, M_n$ in the form $AM = D$, and show that the linear system has a unique solution.

(b) Let $B_i(x)$ is a cubic spline with knots $x_i, \ldots, x_{i+4}$. Then show that [9 Marks]

(a) $B_i(x) = 0$ outside of $x_i < x < x_{i+4}$, (b) $0 \leq B_i(x) \leq 1$ for all x,

(c) $\displaystyle\sum_{i=-3}^{n-1} B_i(x) = 1$ for $x_0 \leq x \leq x_n$,

(3) (a) Use Muller's method to find the approximations to within 10^{-4} to all the zeros of the polynomial $x^3 - x - 1 = 0$, by taking initial approximations $x_0 = 0,\ x_1 = 1$, and $x_2 = 2$. [9 Marks]

(b) Use Taylor's method of order four to approximate the solutions for the initial-value problem $y' = 1 + (t - y)^2,\ 2 \leq t \leq 3,\ y(2) = 1$, with $h = 0.5$. [9 Marks]

(4) (a) Let $\{\varphi_n(x) \mid n \geq 0\}$ be an orthogonal family of polynomials on (a, b) with weight function $w(x)$. With such a family we always assume implicitly that $\deg \varphi_n = n$, $n \geq 0$. If $f(x)$ is a polynomial of degree m, then show that [9 Marks]

$$f(x) = \sum_{n=0}^{m} \frac{(f, \varphi_n)}{(\varphi_n, \varphi_n)} \varphi_n(x)$$

(b) Find the least square polynomial of degree 2 for the data [9 Marks]

x	1.0	1.1	1.3	1.5	1.9	2.1
$P(x)$	1.84	1.96	2.21	2.45	2.94	3.18

(5) Prove that for each $n \geq 1$, there is a unique numerical integration formula $I_n = \sum_{j=1}^{n} w_{j,n} f(x_{j,n})$ [18 Marks] $= \int_a^b w(x)f(x)dx = I(f)$ of degree of precision $2n-1$, where $w(x)$ is non-negative and integrable on $[a, b]$. Moreover, if $f(x)$ is $2n$ times continuously differentiable function on $[a, b]$, then show that formula for $I_n(f)$ and its error is given by

$$\int_a^b w(x)f(x)\,dx = \sum_{j=1}^{n} w_j f(x_j) + \frac{\gamma_n}{A_n^2(2n)!} f^{(2n)}(\eta)$$

for some $a < \eta < b$, where nodes $\{x_j\}$ are the zeros of $\varphi_n(x)$, and the weights $\{w_j\}$ are given by

$$w_j = -\frac{a_n \gamma_n}{\varphi_n'(x_j)\,\varphi_{n+1}(x_j)}, \qquad j = 1, \ldots, n.$$

(6) (a) Let $x_0, x_1, \ldots, x_n$ be distinct real numbers, and let f be a given real valued function with $n+1$ continuous derivatives on the interval $I_t = \mathcal{H}\{t, x_0, \ldots, x_n\}$, with t some given real number. Then show that there exists $\xi \in I_t$, such that [9 Marks]

$$f(t) - \sum_{j=0}^{n} f(x_j) l_j(t) = \frac{(t - x_0) \cdots (t - x_n)}{(n+1)!} f^{(n+1)}(\xi).$$

(b) For matrix $A = \begin{bmatrix} 3 & 2 & -2 \\ -3 & -1 & 3 \\ 1 & 2 & 0 \end{bmatrix}$, perform three iterations of the power method to find its eigenvalue and corresponding Eigenvector by taking $\mathbf{x}^{(0)} = [1 \;\; 0 \;\; 0]^T$. [9 Marks]

(7) (a) Consider the initial value problem $y' = f(t, y), a \leq t \leq b, y(a) = \alpha$. If $t = t_i + sh$ and $\mu_i \in (t_{i+1-m}, t_{i+1})$, show that the explicit m-step Adams–Bashforth method can be written as [9 Marks]

$$y(t_{i+1}) = y(t_i) + h\left[f(t_i, y(t_i)) + \tfrac{1}{2}\nabla f(t_i, y(t_i)) + \tfrac{5}{12}\nabla^2 f(t_i, y(t_i)) + \cdots\right]$$
$$+ h^{m+1} f^{(m)}(\mu_i, y(\mu_i))(-1)^m \int_0^1 \binom{-s}{m} ds.$$

(b) Find the value of $y(1)$ by applying the Adams fourth-order predictor-corrector method for the initial-value problem $y' = y - t^2 + 1, 0 \leq t \leq 2$, with $y(0) = 0.5$, $y(0.2) = 0.8292933$, $y(0.4) = 1.2140762$, $y(0.6) = 1.6489220$ and $h = 0.2$. [9 Marks]

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.Sc. Mathematics Examinations, December 2025
Part I Semester I

DSE-1(i): MATRIX ANALYSIS
(Unique Paper Code: 235137921101)

Time: 3 hours | Maximum Marks: 90

Instructions: • Attempt five questions in all. • **Question 1 is compulsory.** • All questions carry equal marks. • The symbols used have their usual meanings.

(1) (a) State the Schur Decomposition theorem for square matrices. Use it to prove that if all the eigenvalues of $A \in M_n(\mathbb{C})$ are zero, then $A^n = 0$. [4]

(b) Find the LU Decomposition for the matrix $A = \begin{pmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{pmatrix}$. [4]

(c) Verify Perron's theorem for the matrix $A = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}$ by exhibiting a vector $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ such that $Ax = \rho(A)x$ and $x_i > 0$, $i = 1, 2$. [4]

(d) If $A, B \in M_n(\mathbb{C})$, show that $\mathrm{tr}(A \otimes B) = \mathrm{tr}(A)\mathrm{tr}(B)$. Give an example in $M_2(\mathbb{C})$ to show that $\mathrm{tr}(A \circ B) \neq \mathrm{tr}(A)\mathrm{tr}(B)$. [2+2]

(e) Explain why the l_1-norm: $||A||_1 = \sum_{i,j=1}^{n} |a_{ij}|$ on $M_n(\mathbb{C})$ is not an induced matrix norm? [2]

(2) (a) Let $A \in M_n(\mathbb{C})$ have eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Prove that A is normal if and only if there exists a unitary matrix $U \in M_n(\mathbb{C})$ such that $U^*AU = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$. [10]

(b) Find the invariant factors, elementary divisors, characteristic and minimal polynomials, and Jordan canonical form of the matrix $A = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 2 \\ 3 & 0 & 1 \end{pmatrix}$. [8]

(3) (a) (i) State the Singular Value Decomposition theorem. Use it to prove that if $A \in M_n(\mathbb{C})$, then $A = PU$, where $P = (AA^*)^{1/2}$ is uniquely determined and $U \in M_n(\mathbb{C})$ is a unitary matrix.

(ii) Define $|||A|||_2 = \sigma_1(A)$, the largest singular value of A. Show that $|||.|||_2$ is induced by the l_2-norm on $\mathbb{C}^n$. [6+8]

(b) Let $A \in M_n(\mathbb{C})$ and $|||.|||$ be a matrix norm on $M_n(\mathbb{C})$. If $|||I - A||| < 1$, show that A is non-singular and $A^{-1} = \sum_{k=0}^{\infty} (I - A)^k$. [4]

(4) (a) Let $A \in M_n(\mathbb{C})$ and let $|||.|||$ be a matrix norm on $M_n(\mathbb{C})$. Prove that $\rho(A) \leq |||A|||$. Use it to prove the Gelfand formula: [3+5]

$$\rho(A) = \lim_{k \to \infty} |||A^k|||^{1/k}.$$

(b) For $A = [a_{ij}] \in M_n(\mathbb{C})$, prove that the eigenvalues of A lie in the union

$$\bigcup_{i=1}^{n}\{z \in \mathbb{C} : |z - a_{ii}| \leq R_i'(A)\},$$

where $R_i'(A)$ denotes the deleted absolute row sums of A. Deduce that if $D = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \in M_n(\mathbb{C})$, $E = [e_{ij}] \in M_n(\mathbb{C})$ and $\hat{\lambda}$ is an eigenvalue of $D + E$, then $|\hat{\lambda} - \lambda_i| \leq |||E|||_\infty$ for some $i \in \{1, 2, \ldots, n\}$. [6+4]

(5) (a) Let $A \in M_n(\mathbb{C})$ be positive semidefinite and let $x \in \mathbb{C}^n$. Show that if $x^*Ax = 0$, then $Ax = 0$. Give an example (with justification) to illustrate that the result may fail to hold if A is Hermitian, but not positive semidefinite. [5+3]

(b) Let $A \in M_n(\mathbb{C})$ be Hermitian and positive semidefinite, rank$(A) = r$ and $k \in \{2, 3, \ldots\}$. Prove that there exists (i) a unique positive semidefinite matrix $B \in M_n(\mathbb{C})$ such that $B^k = A$; and (ii) polynomial p with real coefficients such that $B = p(A)$. [10]

(6) (a) Let A and B be positive semidefinite matrices in $M_n(\mathbb{C})$. Prove that there exists an invertible matrix P such that P^*AP and P^*BP are both diagonal matrices. Use it to deduce that $\det(A + B) \geq \det(A) + \det(B)$. [7+3]

(b) If $A = [a_{ij}] \in M_n(\mathbb{C})$ is positive definite, prove the Hadamard's inequality: $\det(A) \leq \prod_{i=1}^{n} a_{ii}$. By considering the matrix $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$, show that A is neither positive definite nor satisfy Hadamard's inequality. [5+3]

(7) (a) Let $A \in M_m(\mathbb{C})$ and $B \in M_n(\mathbb{C})$ with eigenvalues λ_i $(i = 1, 2, \ldots, m)$ and μ_j $(j = 1, 2, \ldots, n)$ respectively. Show that the eigenvalues of $A \otimes B$ are $\lambda_i\mu_j$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$. Moreover, verify this statement by considering the matrices $A = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 0 \\ -2 & -1 \end{pmatrix}$ in $M_2(\mathbb{C})$. [5+3]

(b) If $A \in M_n(\mathbb{C})$ is doubly stochastic, show that there exists a permutation σ on $\{1, 2, \ldots, n\}$ such that

$$\prod_{i=1}^{n} a_{i\sigma(i)} \neq 0.$$

Use this result to prove that a matrix $A \in M_n(\mathbb{C})$ is doubly stochastic if and only if it is a convex combination of permutation matrices. [3+7]

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Your Roll Number: ....

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.Sc. Mathematics Examinations, December 24, 2025
Part I, Semester I
DSE-2(i): ADVANCED GROUP THEORY
(Unique Paper Code: 235137921103)

Time: 3 Hours Maximum Marks: 90

Instructions: • Question 1 is compulsory. • Attempt any four questions from question 2 to question 7. • All questions carry equal marks.

(1) (a) Show that every finite group has a composition series. [4.5]

(b) Let $p < q$ be prime numbers and n be any natural number. Show that every group of order pq^n is solvable. [4.5]

(c) Let $H \subseteq K$ be subgroups of a finite group G. Show that

$$[G : H] = [G : K][K : H].$$

[4.5]

(d) Give an example of a group satisfying ascending chain condition but not descending chain condition and an example of a group satisfying descending chain condition but not ascending chain condition. [4.5]

(2) (a) Let $A \lhd A^*$ and $B \lhd B^*$ be four subgroups of a group G. Show that $A(A^* \cap B) \lhd A(A^* \cap B^*)$, $B(A \cap B^*) \lhd B(A^* \cap B^*)$, and

$$\frac{A(A^* \cap B^*)}{A(A^* \cap B)} \cong \frac{B(A^* \cap B^*)}{B(A \cap B^*)}.$$

[9]

(b) Assume that $G = H_1 \times H_2 \times \cdots \times H_n = K_1 \times K_2 \times \cdots \times K_m$, where each H_i and K_j is simple. Prove that $m = n$ and there is a permutation $\pi \in S_n$ with $K_{\pi(i)} \cong H_i$ for all i. [4]

(c) State Jordan Holder theorem and use it to prove the fundamental theorem of arithmetic. [1+4]

(3) (a) Let G be a group such that $G/Z(G)$ is finite. Show that G' is finite. [12]

(b) Show that every minimal normal subgroup of a finite solvable group is elementary abelian. [6]

(4) (a) Show that a characteristically simple finite group is either simple or a direct product of isomorphic simple groups. [7]
(b) Let G be a finite group having a p-complement for every prime p. Show that G is solvable. [7]
(c) Define lower central series and upper central series. [4]

(5) (a) Let G be a nilpotent group. Show that if A is maximal abelian normal subgroup of G, then $A = C_G(A)$. [7]
(b) Let G be a finite group. Show that G is nilpotent if and only if $G/\Phi(G)$ is nilpotent. [7]
(c) Let G be a group and $X \subseteq G$. Let $< X >^G$ denotes the smallest normal subgroup of G that contains X. Let $X^G = \{gxg^{-1} \mid g \in G, x \in X\}$. Prove that $< X^G > = < X >^G$. [4]

(6) (a) Let G be a finite p-group. Show that any two minimal generating sets have the same cardinality. Further show that any $x \notin \Phi(G)$ must belongs to some minimal generating set of G. [10]
(b) Let G be a group having both chain conditions and $\phi : G \to G$ be a normal endomorphism. Show that there exists invariant subgroups K, H under ϕ such that $\phi|_K$ is nilpotent, $\phi|_H$ is onto, and $G = K \times H$. [8]

(7) (a) State and prove the Krull-Schmidt theorem. [1+11]
(b) Let Q, K be groups and $\theta : Q \to Aut(K)$ be a homomorphism. Show that $G = K \rtimes_\theta Q$ is a semidirect product of K by Q that realizes θ. [6]

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examinations, December 2025
Part I Semester I
DSE-2(ii): (NONLINEAR OPTIMIZATION)
Unique Paper Code: 235137921104

Time: 3 Hours — Maximum Marks: 90

Instructions: • Attempt five questions in all. • Question 1 is compulsory. • All questions carry equal marks.

(1) (a) Define convex hull of a set in $\mathbb{R}^n$. If A is a nonempty set in $\mathbb{R}^n$ then prove that the convex hull of A is a convex set. [5 Marks]

(b) State and prove Weierstrass theorem for the existence of a global minimizer. [5 Marks]

(c) If U is an open set in $\mathbb{R}^n$, $f : U \to \mathbb{R}$ is Gâteaux differentiable at $x \in U$ and $d \in \mathbb{R}^n$ then prove that $f'(x, d) = \langle \nabla f(x), d \rangle$. [3 Marks]

(d) If C is a convex set in $\mathbb{R}^n$, $f : C \to \mathbb{R}$ is a strictly convex function then prove that there exists at the most one global minimizer of f on C. [3 Marks]

(e) State second-order sufficient optimality conditions for a feasible point of the following the problem [2 Marks]

$$\begin{aligned} &\text{minimize } f(x) \\ &\text{s.t. } g_i(x) \leq 0, i = 1, 2, ..., r, \\ &\quad h_j(x) = 0, j = 1, 2, ..., m, \end{aligned}$$

to be a strict local minimizer where f, g_i and h_j are real-valued functions defined on $\mathbb{R}^n$ having continuous second-order partial derivatives.

(2) (a) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined as [9 Marks]

$$f(x,y) = \begin{cases} \dfrac{x^3}{x^2 + y^2}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

Is the function f Gâteaux differentiable at $(0,0)$? Is it Fréchet differentiable at $(0,0)$? Justify.

(b) Let U be an open set in $\mathbb{R}^n$ and $f : U \to \mathbb{R}$ be a twice Gâteaux differentiable function on U. If $x^* \in U$ is a local minimizer then prove that the Hessian matrix $Hf(x^*)$ is positive semidefinite. Is the converse true? Justify. [9 Marks]

(3) (a) Find the critical points of the function $f(x, y, z) = xyze^{x+y+z}$ defined on $\mathbb{R}^3$ and determine their nature. [9 Marks]

(b) Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a function. Prove that the function f is convex if and only if $\text{epi}(f)$ is a convex set in $\mathbb{R}^{n+1}$. [9 Marks]

(4) (a) Let $C \subseteq \mathbb{R}^n$ be a nonempty closed convex set. Prove that the projection $\Pi_C(x)$ of x onto C is characterized by the projection inequality [9 Marks]

$$\langle x - \Pi_C(x), z - \Pi_C(x) \rangle \leq 0 \text{ for all } z \in C.$$

(b) If $C \subseteq \mathbb{R}^n$ is a nonempty convex set and $x \notin \text{int}C$ then there prove that there exists $c \in \mathbb{R}^n, c \neq 0$ such that [9 Marks]

$$\langle c, z \rangle \geq \langle c, \bar{x} \rangle \text{ for all } z \in C.$$

(5) (a) If the functions f, g_i and h_j in Q.1(e) are continuously differentiable and x^* is a local minimizer of the problem then prove that x^* satisfies Fritz John conditions. [12 Marks]

(b) If A is $m \times n$ matrix, $c \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$, determine the dual of the problem [6 Marks]

$$\begin{aligned} &\text{minimize } \langle c, x \rangle \\ &\text{s.t. } Ax = b. \end{aligned}$$

(6) (a) Find KKT points of the following problem [9 Marks]

$$\begin{aligned} &\text{Minimize } (x+1)^2 - y^2 \\ &\text{subject to } x + y \leq 0 \\ &x^2 + y^2 = 1. \end{aligned}$$

(b) Let $A \subseteq \mathbb{R}^n$, $B \subseteq \mathbb{R}^m$ and $L : A \times B \to \mathbb{R}$. Prove that the set of saddle points of L is of the form $A_0 \times B_0$ where $A_0 \subseteq A$ and $B_0 \subseteq B$. Also, prove A_0 and B_0 are convex sets if A and B are convex sets and L is a convex-concave function. [9 Marks]

(7) (a) Solve the following problem using Wolfe's method [12 Marks]

$$\begin{aligned} &\text{maximize } z = x_1 - 2x_2 - x_1^2 - x_2^2 \\ &\text{subject to } x_1 + x_2 \leq 2 \\ &x_1 \geq 0, x_2 \geq 0. \end{aligned}$$

(b) Solve the problem in part (a) geometrically. [6 Marks]

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