Your Roll Number: ....

Department of Mathematics, University of Delhi
M.Sc. Mathematics Examinations, May 2025
Part II Semester IV

MMath18- 401(i): Advanced Group Theory (UPC 223502401)

Time: 3 hours | Maximum Marks: 70

Instructions: • Write your roll number on the space provided at the top of this page immediately on receipt of this question paper • **Q 1** is compulsoty • Answer **any four** questions from Q2 to Q7. • Each question carries 14 marks.

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(1) (a) Give an example of an infinite group having composition series. [$3\frac{1}{2}$ Marks]

(b) Show that the additive group of rational numbers is characteristically simple group. [$3\frac{1}{2}$ Marks]

(c) Show that every nilpotent group is sovable. [$3\frac{1}{2}$ Marks]

(d) Let p be a prime. Show that the group $\mathbb{Z}/p^2\mathbb{Z}$ is indecomposable. [$3\frac{1}{2}$ Marks]

(2) (a) Let G be a group such that $G/Z(G)$ is finite. Show that the commutator subgroup G' is finite. [9 Marks]

(b) Define lower centers $\gamma_n(G)$ of a group G, for all n in $\mathbb{N}$. Show that, for all n, $\gamma_n(G)$ is characteristic in G. [5 Marks]

(3) (a) Let G be a finite group. Show that the frattini subgroup is the set of nongenerators of G. [9 Marks]

(b) Let H, K and L be normal subgroups of a group G. Show that $[L, H, K] \leq [H, K, L][K, L, H]$. [5 Marks]

(4) (a) Let G be a simple group having both chain conditions, and let ϕ and ψ be normal nilpotent endomorphism of G. If $\phi + \psi$ is an endomorphism of G then prove that $(\phi + \psi)^k = 0$, for some positive integer k. [9 Marks]

(b) Define normalizer condition. Show that a nilpotent group satisfies normalizer condition. [5 Marks]

(5) (a) Show that every two normal series of an arbitrary group have refinements that are equivalent. [9 Marks]

(b) Prove that the normal closure of a subset X of a group G is equal to the subgroup generated by the conjugates of elements in X. [5 Marks]

(6) (a) Show that every finite group has maximal normal nilpotent subgroup. [9 Marks]

(b) Define semidirect product. Is the quaternion group Q_8 semidirect product of any of its two subgroups ? Justify. [5 Marks]

(7) (a) Determine the free group generated by one element. [4 Marks]

(b) Define presentation of a group. Give two presentations of a cyclic group of order 35. Justify. [5 Marks]

(c) Let N be normal subgroup of a finite group G. If P is Sylow subgroup of N, then show that $G = N_G(P)N$. [5 Marks]

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Your Roll No..

Department of Mathematics
University of Delhi, Delhi

M.A./M.Sc. Mathematics, Part-II, Semester-IV
Examination, May 2025

Paper: MMATH18 401(ii) Algebraic Number Theory
Unique Paper Code: 223502402

Maximum Marks: 70

Time: 3 Hour

Note: • Attempt five questions in all. • Question 1 is compulsory.

1. (a) Is $\mathbb{A}$, the field of all algebraic numbers, a finite extension of $\mathbb{Q}$? Justify. (2)

   (b) Let $f(t) \in \mathbb{Z}[t]$ be a monic polynomial and α an algebraic number. Show that if $f(\alpha)$ is an algebraic integer, then so is α. (3)

   (c) Is every principal ideal domain a Dedekind domain? Justify. (3)

   (d) Prove or disprove "*If $I \subsetneq J$ are ideals of the ring of algebraic integers of a number field K, then $N_K(I) > N_K(J)$.*" (3)

   (e) Find the class number of $\mathbb{Q}(\sqrt{13})$. (3)

2. (a) Let K be a number field of degree n and $\mathcal{O}_K$ its ring of algebraic integers. Prove that K has an integral basis and the additive group $(\mathcal{O}_K, +)$ is a free abelian group of rank n. (7+3)

   (b) Find the discriminant of $\mathbb{Q}(\theta)$, where $\theta^4 = 2$. (4)

3. (a) Give examples to show that for a fixed α, $N_K(\alpha)$ and $T_K(\alpha)$ depend on the number field K. (5)

   (b) Find the degree, integral basis and discriminant of the number field $K = \mathbb{Q}(\zeta)$, where ζ is a primitive pth root of unity for some odd prime p. (9)

4. (a) Prove that the ring of algebraic integers of only finitely many imaginary quadratic fields is a Euclidean domain. (8)

   (b) Solve $X^2 + 2 = Y^3$ for $X, Y \in \mathbb{Z}$. (6)

5. (a) Let R be an integral domain with quotient field K and I a non-zero fractional ideal of R. Prove that the set $I' = \{x \in K \mid xI \subseteq R\}$ is a fractional ideal of R. (4)

   (b) Let K be a number field and $\mathcal{O}_K$ its ring of algebraic integers. Prove that every non-zero fractional ideal of $\mathcal{O}_K$ is invertible. Deduce that every non-zero ideal of $\mathcal{O}_K$ can be uniquely written as a product of prime ideals. (6+4)

6. (a) State and prove Minkowski's Theorem. (1+5)

(b) Let K be a number field of degree n with ring of algebraic integers $\mathcal{O}_K$ and I a non-trivial ideal of $\mathcal{O}_K$. Prove that for the embedding $\sigma : K \to L^{st}$, $\sigma(I)$ is an n-dimensional lattice. (8)

7. (a) State and prove the Dedekind Theorem. (2+8)

(b) Write down the prime ideal factorization of the ideal $\langle 11 \rangle$ of $\mathbb{Z}[\sqrt{-5}]$. (4)

Roll Number:........

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics, End-semester Examination, May 2025
Part - II; Semester - IV
MMATH18-401(iii) : **Simplicial Homology Theory**
Unique paper code: 223502403

Time: 3 Hours | Max Marks: 70

Instructions: • All symbols have their usual meaning. • Question No.1 in **Part-A** is compulsory. • Answer any four questions from remaining six questions in **Part-B**.

Part-A, Compulsory

1. Choose appropriate options. Justifications are not required: (2 × 7)

(a) Which of the following statements are true:

A. Every geometrically independent set with $k+1$ elements generates a unique k-simplex.

B. Every subset of a geometrically independent set is geometrically independent.

C. Weak topology on $\mathbb{R}^\infty$ and subspace topology on $\mathbb{R}^\infty$ induced from the product topology on $\mathbb{R}^\omega$ are same.

D. If a set of three points is geometrically independent then they are co-linear.

(b) Let K and L be two simplicial complexes in $\mathbb{R}^m$, m sufficiently large. Then which of the following statements are true:

A. If $K = Cl(\sigma^n)$, $n < m$, then $|K|$ is homeomorphic to $\mathbb{S}^n$.

B. $|K|$ must be a Hausdorff subspace of $\mathbb{R}^m$.

C. Homology groups of K are independent of orientation upto an isomorphism.

D. If $H_p(K) \cong H_p(L)$ for all $p \geq 0$, then $|K|$ and $|L|$ are homeomorphic.

(c) Let $f, g : \mathbb{S}^1 \to \mathbb{S}^1$ be two continuous maps. Then which of the following statements are false:

A. If f and g are homotopic, then $\deg(f) = \deg(g)$.

B. If f is a homeomorphism, then $\deg(f) = \pm 1$.

C. If f is an antipodal map, then $\deg(f) = -1$.

D. If f is a constant map, then degree of f must be non-zero.

(d) Let K be a simplicial complex, $v \in K$ be a vertex and σ be a simplex in K. Then which of the following statements are true:

A. $\mathrm{st}(v)$ is an open subset of $|K|$.

B. The intersection of $\mathrm{st}(v)$ with the set of vertices of K is $\{v\}$.

C. If $x \in \mathrm{int}(\sigma)$, then its barycentric coordinates not necessarily be positive.

D. For any simplicial complex K, $C_p(K) \cong C_p(K^1)$, for all $p \geq 0$.

(e) Which of the following statements are false :

A. Tetrahedron is a simple regular polyhedran.

B. 1st Betti number of a standard cylinder is 1.

C. The Euler's characteristic of $\mathbb{S}^2$ is zero.

D. Geometric carrier of a 2-simplex is a 2-pseudomanifold.

(f) Let K be a simplicial complex in $\mathbb{R}^m$. Then which of the following statements are true:

A. The relation of connectedness in K is an equivalence relation.

B. K can have infinitely many combinatorial components.

C. $|K|$ is necessarily compact and connected subset of $\mathbb{R}^m$.

D. K contains finitely many vertices.

(g) Let $K = \{\langle 0\rangle, \langle \frac{1}{3}\rangle, \langle 1\rangle, \langle 0, \frac{1}{3}\rangle, \langle \frac{1}{3}, 1\rangle\}$ and $L = \{\langle 0\rangle, \langle \frac{2}{3}\rangle, \langle 1\rangle, \langle 0, \frac{2}{3}\rangle, \langle \frac{2}{3}, 1\rangle\}$ be two simplicial complexes and $f : |K| \to |L|$ be a continuous map given by $f(x) = x^2$. Then which of the following statements are true:

A. $C_p(K) \not\cong C_p(L)$, for all $p \geq 0$.

B. $H_p(K) \cong H_p(L)$, for all $p \geq 0$.

C. There exists a simplicial map $\phi : K^{(1)} \to L$, which is a simplicial approximation of f.

D. There exists an $n > 0$ such that $\phi : K^{(n)} \to L$ is a simplicial approximation of f.

Part-B, Attempt any FOUR questions from the following

2. (a) Prove that a subset $X = \{v_0, v_1, \ldots, v_k\}$ of $\mathbb{R}^n$ is geometrically independent if and only if $\{v_1 - v_0, v_2 - v_0, \ldots, v_k - v_0\}$ is linearly independent subset of $\mathbb{R}^n$. (9)

   (b) Prove that if $\sigma^k = \langle v_0, v_1, \ldots, v_k\rangle$ is a k-simplex in $\mathbb{R}^n$, then σ^k equals union of all line segments joining v_0 to the points of the simplex $\langle v_1, \ldots v_k\rangle$. (5)

3. (a) Let $K = \{\langle v_0\rangle, \langle v_1\rangle, \langle v_2\rangle, \langle v_3\rangle, \langle v_4\rangle, \langle v_0, v_1\rangle, \langle v_0, v_2\rangle, \langle v_1, v_2\rangle, \langle v_0, v_3\rangle, \langle v_0, v_4\rangle, \langle v_3, v_4\rangle\}$ be a simplicial complex with orientation $v_0 < v_1 < v_2 < v_3 < v_4$. Compute $H_1(K)$. (7)

   (b) Prove that the set of interiors of all simplexes of a simplicial complex K forms a partition of $|K|$. (7)

4. (a) Prove that a subset $\{v_0, v_1, \ldots, v_k\}$ of vertices in a simplicial complex K forms a simplex in K if $\bigcup_{i=0}^{k} \mathrm{st}(v_i) \neq \emptyset$. (5)

   (b) Carefully state and prove the simplicial approximation theorem. (9)

5. (a) Prove that if K is a connected simplicial complex, then $|K|$ is path connected. (6)

   (b) Carefully state and prove the Euler-Poincaré theorem. (8)

6. (a) State and prove the Brower's degree theorem. (7)

   (b) For distinct $m, n \in \mathbb{N}$, show that $\mathbb{S}^n$ is not homeomorphic to $\mathbb{S}^m$, and use it to prove that $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}^m$. (7)

7. (a) Let $\phi : K \to L$ be a simplicial map between two simplicial complexes K and L, and $\{\phi_p\}_{p\geq 0}$ be the corresponding chain mapping. Prove that it induces a homomorphism $\phi_p^* : H_p(K) \to H_p(L)$, for all $p \geq 0$. (6)

   (b) Carefully state and prove the Brower's fixed point theorem. (8)

Your Roll Number: ....

Department of Mathematics, University of Delhi
M.Sc. Mathematics Examinations, MAY - 2025
Part II Semester IV

MMATH18- 402(i): Abstract Harmonic Analysis, UPC: 223502405

Time: 3 hours — Maximum Marks: 70

Instructions: • Answer **five** questions • **Question 1** is compulsory. Attempt any **four** questions from **Q2** to **Q7** • Each question carries 14 marks. All the symbols used have their usual meaning.

(1) (a) State Peter-Weyl Theorem. (2)

(b) Let G be a compact abelian group. Prove that for $\chi \neq \eta \in \hat{G}$ (the dual group), $\int \chi(x)\overline{\eta(x)}\, dx = 0$. (3)

(c) Prove that a positive definite function is bounded. (3)

(d) Give example of an orthogonal basis on H_n, the space of homogenous polynomials of degree n of two variables. (3)

(e) Give example of an irreducible representation which is not one-dimesnional. (3)

(2) (a) Let G be a locally compact group. For $\mu, \nu, \gamma \in M(G)$, prove that (7)

(i) $(\mu * \nu)^* = \nu^* * \mu^*$.

(ii) $\mu * (\nu * \gamma) = (\mu * \nu) * \gamma$

(b) Let π be unitary representation of a locally compact group G on H and M be a closed invariant subspace for π. Prove that $\pi \cong \pi|_M \oplus \pi|_{M^\perp}$. Give example to show that the result may fail if π is not unitary. (7)

(3) (a) Let $\mu \in M(\hat{G})$ be a positive measure and $\phi_\mu : G \to \mathbb{C}$ be defined as

$$\phi_\mu(x) = \int_{\hat{G}} \xi(x)\, d\mu(\xi).$$

Prove that ϕ_μ is bounded, continuous and is of positive type. (7)

(b) Let π_1 and π_2 be isomorphic irreducible unitary representations of a locally compact group G. Prove that $\mathcal{C}(\pi_1, \pi_2)$ is one dimensional. (7)

(4) (a) Let G be a locally compact group with left Haar measure μ and $\pi_L : G \to B(L^2(G))$ be defined as $\pi_L(x)(f) = L_x f$. Prove that π_L is a unitary representation.

(b) Define the dual group of a locally compact abelian group. Prove that the dual group of the circle group S^1 is (group) isomorphic to $\mathbb{Z}$. (7)

(5) (a) Let π be a unitary representation of a locally compact group G. Prove that there exists a $*$-representation ρ of $L^1(G)$ on H such that (7)

$$\langle \rho(f)u, v\rangle = \int f(x)\langle \pi(x)u, v\rangle\, dx \quad \forall\, u, v \in H, f \in L^1(G).$$

(b) Prove that every irreducible unitary representation of a compact group is finite dimensional. (7)

(6) (a) Let μ be a left Haar measure on a locally compact group G. Prove that (7)
(i) $\mu(U) > 0$, for any non-empty open subset $U \subseteq G$.
(ii) $\int f \, d\mu > 0$, for any $f \in C_c^+(G)$.

(b) For $n \in \mathbb{N} \cup \{0\}$, consider the unitary representation $\pi_n : SU(2) \to B(H_n)$ defined as $\pi_n(A)(f(z)) = f(zA)$, $A \in SU(2)$, H_n being the space of homogenous polynomials of degree n of two variables. Prove that π_n is irreducible. (7)

(7) (a) Let G be a topological group and U be an open subset of G. Prove that for any $x \in G$, Ux is open. Further prove that if, in addition, U is a subgroup then U is closed. (7)

(b) For a compact group G, define $ZL^2(G)$. Prove that for any $f \in ZL^2(G)$, (7)

$$f = \sum_{[\pi]\in\hat{G}} < f, \chi_\pi > \chi_\pi.$$

No. of Printed Pages 2

M.A./M.Sc. Mathematics Examination (2025)
Part II, Semester IV
MMATH18-402(ii): Frames and Wavelets
Unique Paper Code-223502406

Maximum Marks: 70

Time: 3 hr.

Instructions: • Attempt **FIVE** questions in all. • Q. No. 1 is **COMPULSORY**. • All questions carry equal marks. • Symbols have their usual meaning.

(Question No. 1 is Compulsory)

(1) (a) Show that a minimal frame for an infinite-dimensional Hilbert space $\mathcal{H}$ is ω-independent. [2 Marks

(b) Define exact frame. Give an example, with justification, of a frame in $L^2(\mathbb{R})$ which is exact. [1+3= Marks

(c) Find two dual frames, with justification, of the frame $\{0, e_1, e_2, e_3, \dots\}$ for $\ell^2(\mathbb{N})$, where $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis of $\ell^2(\mathbb{N})$. [4 Marks

(d) Give an example, with justification, of a Bessel sequence in $\mathbb{C}^5$ which is not a frame for $\mathbb{C}^5$. [4 Marks

(Answer any FOUR questions from Q. No. 2 to Q. No. 7)

(2) (a) Let $\{f_k\}_{k=1}^{m}$ be a frame for a subspace $\mathcal{W}$ of $\mathbb{C}^n$. Find a formula for the orthogonal projection of $\mathbb{C}^n$ onto $\mathcal{W}$ in terms of the frame $\{f_k\}_{k=1}^{m}$. [4 Marks

(b) State and prove the frame algorithm. [10 Marks

(3) (a) Let $\{f_k\}_{k=1}^{\infty}$ be a Bessel sequence in an infinite-dimensional Hilbert space $\mathcal{H}$ with Bessel bound B. Show that [6 Marks

(i) $\|f_k\|^2 \leq B$ for all $j \in \mathbb{N}$.

(ii) $\|f_j\|^2 = B$ for some positive integer $j \implies f_k \perp f_j$ for all $k \in \mathbb{N} \setminus \{j\}$.

(b) Let $\{f_k\}_{k=1}^{\infty}$ be a sequence in an infinite-dimensional Hilbert space $\mathcal{H}$ and $B > 0$ such that $\sum\limits_{k=1}^{\infty} |\langle f_j, f_k \rangle| \leq B$ for all $j \in \mathbb{N}$. Show that $\{f_k\}_{k=1}^{\infty}$ is a Bessel sequence with Bessel bound B. [8 Marks

(4) (a) Define the pre-frame operator of a continuous frame for an infinite-dimensional Hilbert space $\mathcal{H}$. [2 Marks

(b) Let $\{f_k\}_{k=1}^{\infty}$ be a frame for an infinite-dimensional complex Hilbert space $\mathcal{H}$ with pre-frame operator T. Show that for every $\epsilon \in (0, 1)$, there exist three orthonormal bases $\{e_k^{(1)}\}_{k=1}^{\infty}$, $\{e_k^{(2)}\}_{k=1}^{\infty}$ and $\{e_k^{(3)}\}_{k=1}^{\infty}$ for $\mathcal{H}$ such that [12 Marks

$$f_k = \frac{\|T\|}{1-\epsilon}\left(e_k^{(1)} + e_k^{(2)} + e_k^{(3)}\right), \ \forall\, k \in \mathbb{N}.$$

(5) (a) Give an example, with justification, of a Riesz basis for $\ell^2(\mathbb{N})$ with optimal Riesz bounds equal to 1. [2 Marks

(b) Show that a sequence $\{f_k\}_{k=1}^{\infty}$ in an infinite-dimensional Hilbert space $\mathcal{H}$ is a Riesz basis for $\mathcal{H}$ if and only if $\{f_k\}_{k=1}^{\infty}$ is a complete Bessel sequence, and it has a complete biorthogonal sequence $\{g_k\}_{k=1}^{\infty}$ which is also a Bessel sequence. [12 Marks

(6) (a) Show that the coefficient functionals associated to a Schauder basis for a Banach space are continuous. [4 Marks

(b) Show that any two similar Parseval frames are unitarily equivalent. [4 Marks

(c) State and prove the Haar decomposition theorem. [6 Marks

(7) (a) Let ϕ and ψ denote the Haar scaling function and Haar wavelet, respectively. Show that if $V_0 = \text{span}\{\phi(x-k)\}_{k\in\mathbb{Z}}$ and $W_j = \text{span}\{\psi(2^j x - k)\}_{k\in\mathbb{Z}}$, $j \in \mathbb{N} \cup \{0\}$, then $L^2(\mathbb{R}) = V_0 \bigoplus_{j=0}^{\infty} W_j$. [6 Marks

(b) Define multiresolution analysis. Give an example, with justification, of a multiresolution analysis in $L^2(\mathbb{R})$. [3+5=8 Marks

Roll Number: ....

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI

M.A./M.Sc. Examinations, May-2025

MMATH18-402 (iv): THEORY OF UNBOUNDED OPERATORS

(UPC NO. 223502408)

Time: 3 Hours | Maximum Marks: 70

Instructions: • Attempt five questions in all. • Question 1 is compulsory. • All the symbols have their usual meaning unless otherwise specified.

Throughout this paper, H will be a complex Hilbert space and X will be a Banach space.

Q1. (a) If a linear operator T is defined everywhere on H, show that its Hilbert-adjoint operator T^* is bounded. [4 Marks]

(b) Let $S : \mathcal{D}(S) \to H$ and $T : \mathcal{D}(T) \to H$ be linear operators which are densely defined in H. If $S \subset T$, then show that $T^* \subset S^*$. [3 Marks]

(c) Let $T(t)$ and $S(t)$ be C_0 semigroups of bounded linear operators with infinitesimal generators A and B, respectively. If $A = B$, show that $T(t) = S(t)$ for $t \geq 0$. [3 Marks]

(d) Let $T(t)$ be a C_0 semigroup of bounded operators. If for every $t > 0$, $T(t)^{-1}$ exists and is a bounded operator then show that $S(t) = T(t)^{-1}$ is a C_0 semigroup of bounded operators. Also, find its infinitesimal generator. [4 Marks]

Q2. (a) Let $T : \mathcal{D}(T) \to H$ be a densely defined linear operator in H. Moreover, suppose that T is injective and its range $\mathcal{R}(T)$ is dense in H. Prove that T^* is injective, and $(T^*)^{-1} = (T^{-1})^*$. [8 Marks]

(b) Let $T : \mathcal{D}(T) \to H$ be a self-adjoint densely defined linear operator. Show that the spectrum $\sigma(T)$ is real and closed. [6 Marks]

Q3. (a) Let $T : \mathcal{D}(T) \to H$ be a self-adjoint densely defined linear operator. Show that $\lambda \in \rho(T)$ if there exists a $c > 0$ such that for every $x \in \mathcal{D}(T)$, $\|(T - \lambda I)x\| \geq c\|x\|$. [8 Marks]

(b) Show that the multiplication operator $T : \mathcal{D}(T) \to L^2(-\infty, +\infty)$, $x \mapsto tx$, where $\mathcal{D}(T) \subset L^2(-\infty, +\infty)$, is unbounded and self-adjoint. [6 Marks]

Q4. (a) Show that the point spectrum $\sigma_p(T)$ of a symmetric linear operator $T : \mathcal{D}(T) \to H$ is real. If H is separable, show that $\sigma_p(T)$ is countable. [8 Marks]

(b) Give an example to show that the resolvent set of the infinitesimal generator of a C_0 semigroup of contractions need not contain more than the open right half-plane. [6 Marks]

Q5. (a) Let A be a linear operator for which $\rho(A) \supset (0, \infty)$. If $\|\lambda^n R(\lambda : A)^n\| \leq M$ for $n = 1, 2, \cdots$, $\lambda > 0$ then show that there exists a norm $|\cdot|$ on X which is equivalent to the original norm $\|\cdot\|$ on X and satisfies $\|x\| \leq |x| \leq M\|x\|$ for $x \in X$; and $|\lambda R(\lambda : A)x| \leq |x|$ for $x \in X$, $\lambda > 0$. [8 Marks]

(b) Let $T(t)$ and $S(t)$ be uniformly continuous semigroups of bounded linear operators. If

$$\lim_{t \downarrow 0} \frac{T(t) - I}{t} = A = \lim_{t \downarrow 0} \frac{S(t) - I}{t}$$

then show that $T(t) = S(t)$ for $t \geq 0$. [6 Marks]

Q6. (a) Let A be a linear operator with dense domain $\mathcal{D}(A)$ in X. If A is dissipative and there is a $\lambda_0 > 0$ such that $\mathcal{R}(\lambda_0 I - A) = X$, then prove that A is the infinitesimal generator of a C_0 semigroup of contractions on X. [8 Marks]

(b) Let A be dissipative with $\mathcal{R}(I - A) = X$. If X is reflexive, prove that $\mathcal{D}(A)$ is dense in X. [6 Marks]

Q7. (a) State and prove Stone's Theorem. [8 Marks]

(b) Let X be a reflexive Banach space, and let $T(t)$ be a C_0 semigroup on X with infinitesimal generator A. Show that the adjoint semigroup $T(t)^*$ of $T(t)$ is a C_0 semigroup on X^*, whose infinitesimal generator is A^* (the adjoint of A). [6 Marks]

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Your Roll No:......................

M.A/M.Sc. Mathematics, Part-II, Sem-IV (2025)
MMATH18-403(i), Calculus on $\mathbb{R}^n$
Unique Paper Code: 223502409

Time : 3 hours | Maximum Marks : 70

Question No. **1** is compulsory. Attempt any **four** questions from Question Nos. **2** to **7**. Unless otherwise mentioned, U will be an open subset of $\mathbb{R}^n$.

(1) (a) Define k-surface. [1]

(b) State the Stokes' theorem. [2]

(c) Define oriented affine k-simplex. [2]

(d) Let $f(x, y, z) = \frac{x^2+y^2}{e^z}$ be a function and $a = (0, 0, 0)$. Find the directional derivative of f at a with respect to vector $u = (1, 0, 1)$. [3]

(e) Let $f : U \to \mathbb{R}^m$ be a map. Define the derivative of f at a point $a \in U$ and show that the derivative at a is unique. [3]

(f) Let ω be an exact 1-form. Show that $\int_\gamma \omega = 0$ for any closed 1-surface γ. [3]

(2) (a) Let $f : U \to \mathbb{R}$ be a map of class C^2. Show that $D_iD_jf(x) = D_jD_if(a)$ for all $x \in U$ for all $i, j \in \{1, 2, \ldots, n\}$. [8]

(b) Let $f : U \to \mathbb{R}$ be a map of class C^k. State and prove Taylor's formula with Lagrange's remainder of order k in a neighborhood of $a \in U$. [6]

(3) (a) Let E be an open set in $\mathbb{R}^m$ and $T : E \to U$ be a C^2-map. Let ω and λ be k-form and ℓ-form in U, respectively. Show that

(i) $(\omega \wedge \lambda)_T = \omega_T \wedge \lambda_T$. [3]

(ii) $d(\omega_T) = (d\omega)_T$. [3]

(b) State and prove the Implicit Function Theorem. [8]

(4) (a) Let $f : U \to \mathbb{R}^n$ be a map of class C^1 and $Df(x)$ is invertible for all $x \in U$. If f is injective, show that f is an open map. [8]

(b) Let $\omega = (2x + yz)dx + (xz)dy + (xy)dz$. Prove that ω is exact. [6]

(5) (a) Suppose ω is a k-form in an open set $E \subseteq \mathbb{R}^n$, Φ a k-surface in E with parameter domain $D \subseteq \mathbb{R}^k$ and Δ is k-surface in $\mathbb{R}^k$ with parameter domain D defined by $\Delta(u) = u$ for all $u \in D$. Show that [7]

$$\int_\Phi \omega = \int_\Delta \omega_\Phi.$$

(b) Let I^k be a k-cell and $f : I^k \to \mathbb{R}$ be a continuous map. Show that the integration $\int_{I^k} f = \int\int \cdots \int f dx_1dx_2 \cdots dx_k$ makes sense, and it is independent of the order of integrations. [7]

(6) (a) Let $\gamma : [0,1] \to \mathbb{R}$ defined by $\gamma(t) = (\cos t,\ \sin t)$, and $\omega = xdy$ be a 2-form in $\mathbb{R}^2$. Let $T(x,y) = (x^2 - y^2,\ xy)$ be a map. Evaluate $\int_\gamma \omega_T$ and $\int_{T\circ\gamma} \omega$? [7]

(b) Let V be neighborhood of $\mathbf{0} \in \mathbb{R}^n$, $F : V \to \mathbb{R}^n$ be a C^1 map such that $F(\mathbf{0}) = \mathbf{0}$ and $DF(\mathbf{0})$ is invertible. Let $P_{m-1}F(x) = P_{m-1}(x)$ for all $x \in V$, for some $1 \le m \le n-1$. Show that there exist a neighborhood U of $\mathbf{0}$ and a C^1-map $F' : U \to \mathbb{R}^n$ such that $F'(\mathbf{0}) = \mathbf{0}$, $DF'(\mathbf{0})$ is invertible and $P_m F(x) = P_m(x)$ for all $x \in U$. Here $P_k(x_1, \ldots, x_n) = (x_1, \ldots, x_k, 0, \ldots, 0)$. [7]

(7) (a) Let $\{U_1,\ U_2\}$ be open subsets of $\mathbb{R}^n$ such that $U_1 = \{x \in \mathbb{R}^n \mid ||x|| < 2\}$ and $U_2 = \{x \in \mathbb{R}^n \mid ||x|| > 1\}$. Construct a partition of unity $\varphi_i : \mathbb{R}^n \to [0,1]$ for $i = 1,2$ subordinate to cover $\{U_1,\ U_2\}$. [7]

(b) Let $\sigma = [\mathbf{0}, e_1, e_2]$ be the standard affine oriented 2-simplex in $\mathbb{R}^2$. Use the Stokes' theorem to Evaluate the line integral [7]

$$\int_{\partial\sigma} (x^2y + y)dx \; + \; (xy^2 + x)dy$$

M.A./M.Sc. Mathematics, Part-II, Semester IV
Examination, May 2025

Paper: MMATH18-403(ii) Differential Geometry
Unique Paper Code 223502410

Time: 3 Hours Maximum Marks: 70

Instructions: Attempt **five** (5) questions in all. **Question no. 1 is compulsory**. All questions carry equal marks. Unless otherwise stated U denotes an open subset of R^{n+1}.

1. (a) Compute $\nabla_{\mathbf{v}} f$ where $f(x_1, x_2) = x_1^2 - x_1 x_2$, $\mathbf{v} = (1, 2, cos\theta, sin\theta)$. [2]
   (b) Show that the set S of all unit vectors at all points of R^2 forms a 3-surface in R^4. [3]
   (c) Let $f : U \to R$ be a smooth function and let $\alpha : I \to U$ be an integral curve of the gradient field ∇f. Show that $\frac{d}{dt}(f \circ \alpha)(t) = \| \nabla f(\alpha(t)) \|^2$ [3]
   (d) Let S be an n-surface in R^{n+1}, $p \in S$ and $\mathbf{v} \in S_p$. Show that differential ∇ and the *covariant differential* $\mathbf{D}$ of tangent vector fields $\mathbf{X}$ and $\mathbf{Y}$ on S are related by: $\nabla_{\mathbf{v}}(\mathbf{X} \cdot \mathbf{Y}) = \mathbf{D_v X} \cdot \mathbf{Y}(p) + \mathbf{X}(p) \cdot \mathbf{D_v Y}$. [3]
   (e) Find the length of the parametrized curve $\alpha : I \longrightarrow R^3$ given by $\alpha(t) = (\sqrt{2}cos2t, sin2t, sin2t), I = [0, 2\pi]$ [3]

2. (a) Define local parametrization of a plane curve C. Show that local parametrization of a plane curve is unique up to reparametrization. [2+5]
   (b) Let S denotes the cylinder $x_1^2 + x_2^2 = r^2$ of radius $r > 0$ in R^3. Show that α is a geodesic of S if and only if it is of the form $\alpha(t) = (rcos(at + b) + rsin(at + b), ct + d)$ for some $a, b, c, d \in R$. [7]

3. (a) Let $f : U \to R$ be a smooth function. Let p be a regular point of f, and let $f(p) = c$. Show that the set of all vectors tangent to $f^{-1}(c)$ at p is exactly $[\nabla f(p)]^\perp$. [7]
   (b) Define the *Weingarten map* for an n-surface S in R^{n+1} oriented by the unit normal vector field $\mathbf{N}$. Show that the Weingarten map is self adjoint. [2+5]

4. (a) Let $S = f^{-1}(c)$ be an n-surface in R^{n+1} where $f : U \to R$ is such that $\nabla f(q) \neq 0$ for all $q \in S$. Suppose $g : U \to R$ is a smooth function and $p \in S$ is an extreme point of

g on S. Show that there is a real number λ such that $\nabla g(p) = \lambda \nabla f(p)$. Determine the extreme values of the function $g(x,y) = x^2 + y^2$ on the ellipse $ax^2 + 2bxy + cy^2 = 1 (ac - b^2 > 0)$. [5+3]

(b) Show that the product $f\omega$ of a 1-form ω and a function f on U is again a 1-form on U. Let $\omega = \sum_{i=1}^{n+1} f_i dx_i$ be a smooth 1-form on R^{n+1} and let $\alpha(t) = (\alpha_1(t), ..., \alpha_{n+1}(t))$ where $\alpha_i : [a,b] \longrightarrow R$ are smooth functions. Show that [2+4]

$$\int_\alpha \omega = \int_a^b \sum_{i=1}^{n+1} (f_i \circ \alpha) \frac{d\alpha_i}{dt} dt.$$

5. (a) Let V be a finite dimensional vector space with dot product and let $L : V \longrightarrow V$ be a self-adjoint linear transformation on V. Let $S = \{v \in V : v \cdot v = 1\}$ and define $f : S \longrightarrow R$ by $f(v) = L(v) \cdot v$. Show that f is stationary at $v_0 \in S$ if and only if v_0 is an eigenvector of L with eigenvalue $f(v_0)$. [9]

(b) Show that the 1-form $\omega = \dfrac{x}{x^2+y^2}dy - \dfrac{y}{x^2+y^2}dx$ on $R^2 \setminus \{0\}$ is not exact. [5]

6. (a) Let S be an n-plane $a_1x_1 + ... + a_{n+1}x_{n+1} = b$, let $p, q \in S$, and let $v = (p, v) \in S_p$. Show that if α is any parametrized curve in S from p to q then $P_\alpha(v) = (q, v)$. What can you say about parallel transport in an n-plane? [7]

(b) Let $a > b > 0$, define $\varphi : R^2 \rightarrow R^3$ by $\varphi(\theta, \phi) = ((a + b cos\phi)cos\theta, (a + b cos\phi)sin\theta, b sin\phi)$. Determine the principal curvatures and the Gaussian curvature of the parametrized surface φ. [7]

7. (a) Let S be an n-surface in R^{n+1} oriented by $\mathbf{N}$ and let $p \in S$. Let $\mathbf{Z}$ be any non-zero normal vector field on S such that $\mathbf{N} = \dfrac{\mathbf{Z}}{\| \mathbf{Z} \|}$ and let $\{\mathbf{v}_1, ..., \mathbf{v}_n\}$ be any basis for S_p. Show that the Gauss-Kronecker curvature $K(p)$ of S at p is given by

$$K(p) \| \mathbf{Z}(p) \|^n \det \begin{bmatrix} \mathbf{v}_1 \\ \cdot \\ \cdot \\ \mathbf{v}_n \\ \mathbf{Z}(p) \end{bmatrix} = (-1)^n \det \begin{bmatrix} \nabla_{\mathbf{v}_1}\mathbf{Z} \\ \cdot \\ \cdot \\ \nabla_{\mathbf{v}_n}\mathbf{Z} \\ \mathbf{Z}(p) \end{bmatrix}$$ [8]

(b) Write short notes on *Mobius band* and *circle of curvature*. [3+3]

Your Roll Number: ....

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations, May 2025
Part- II Semester- IV
MMATH18-403(iii): Topological Dynamics
(Unique Paper Code 223502411)

Time: **3 hours** Marks: **70**

Instructions: • All notations used are standard. • Question no. 1 is compulsory. • Attempt any four questions from the remaining six questions.

(1) Do as directed.

(a) Is $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 4x(1-x)$ topologically conjugate to $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^2 - 2$? Justify your claim. [3 Marks]

(b) Prove that if (X, d) is a compact metric space and $f : X \to X$ is an expansive homeomorphism then (X, f) has a generator. [2 Marks]

(c) If (X, d) is a compact metric space and $f^{-1} : X \to X$ is a homeomorphism having POTP then prove that f also has POTP. [3 Marks]

(d) Find the set of periodic points of the doubling map on the unit circle $\mathbb{S}^1$. Is it a closed subset of $\mathbb{S}^1$? [3 Marks]

(e) If a $k \times k$ matrix A with entries in $\{0, 1\}$ is irreducible then prove that its associated digraph is strongly connected. [3 Marks]

(2) (a) Let A be a $k \times k$ matrix with entries in $\{0, 1\}$ such that no row/column of A is full of zeros. If $A \vee (A * A) \vee \cdots \vee (\underbrace{A * \cdots * A}_{n-times}) = J$ for some $n \in \mathbb{N}$ then prove that $\sigma|_{X_A}$ is topologically transitive. [8 Marks]

(b) Prove that an irrational rotation on the unit circle $\mathbb{S}^1$ is a minimal homeomorphism. What can you say about a rational rotation on $\mathbb{S}^1$? Justify your claim. [6 Marks]

(3) (a) Prove Sarkovskii's theorem. Justify that the condition of continuity of the function in the hypothesis of the theorem is a necessary condition. [11 Marks]

(b) Discuss attracting and repelling fixed points of the logistic function $F_\mu : \mathbb{R} \to \mathbb{R}$ defined by $F_\mu(x) = \mu x(1-x), \mu \in \mathbb{R}, \mu > 0$. [3 Marks]

(4) (a) Let X be a compact metric space and $f : X \to X$ continuous. Prove that for every $x \in X, \omega(x)$ is a non-empty, closed f-invariant subset of X. [6 Marks]

(b) Do the graphical analysis of $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$. Draw the phase portrait of the orbit of $x = 2/3$. Find $W^s(0)$ and $W^u(\pm 1)$. [8 Marks]

P.T.O.

(5) (a) Prove that the space $X = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 4\}$ with the subspace topology of the Euclidean space $\mathbb{R}^2$ does not admit an expansive homeomorphism. [8 Marks]

(b) Let (X, d) and (Y, ρ) be two metric spaces, $g : X \to Y$ a homeomorphism such that $g^{-1} : Y \to X$ is uniformly continuous and $\phi : X \to X$ be an expansive homeomorphism. Prove that $g\phi g^{-1} : Y \to Y$ is an expansive homeomorphism. Justify that uniform continuity of g^{-1} is a necessary condition. [6 Marks]

(6) (a) Prove that every contraction map on $[0, 1]$ with the usual metric has POTP. Use it to prove $cos : [0, 1] \to [0, 1]$ has POTP. [9 Marks]

(b) Let (X, d) be a compact metric space and $f : X \to X$ an expansive homeomorphism. Prove that for any $\epsilon > 0$, a number less than an expansive constant for f, $W^s(x, d) = \cup_{n \geq 0} f^{-n}(W^s_\epsilon(f^n(x), d))$. [5 Marks]

(7) (a) Prove that a topologically Anosov self-homeomorphism on a compact metric X is topologically stable in the class of all self-homeomorphisms of X. [9 Marks]

(b) Let (X, d) be a compact metric space and $f : X \to X$ be a homeomorphism having POTP then prove that f has canonical coordinates. [5 Marks]

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examinations, May 2025
Part II Semester IV

MMATH18-404(i): ADVANCED FLUID DYNAMICS
UPC 223502412

Time: 3 Hours — Maximum Marks: 70

Instructions: • Attempt FIVE questions in all. • Q.No.1 is compulsory. • Attempt any FOUR questions from Q.No.2 to Q.No.7. Each question carries 14 marks. • All the symbols have their usual meaning.

(1) (a) Find dimension of medium porosity ϵ in the equation of continuity for the porous medium $\epsilon\frac{\partial\rho}{\partial t} = div(\rho\bar{v})$. [2 Marks]

(b) Show that under MHD approximation the electric field energy per unit mass is negligible in comparison with the magnetic field energy per unit mass. [2 Marks]

(c) Prove that Pre-Maxwell equation leads that a charge can not exist in a conductor at rest. [2 Marks]

(d) Show that the shearing stress τ_0 on a wall of length l and breadth b for a fluid flow of density ρ, viscosity μ and uniform stream speed U is proportional to $U^{\frac{3}{2}}$. [3 Marks]

(e) Define equation of state of a compressible substance. Write the equation of state for non-ideal gas and dusty gas. [2 Marks]

(f) Explain the difference between First and second law of thermodynamics. [3 Marks]

(2) (a) Write the total and kinetic energy equations for a compressible fluid flow. Derive the internal energy equation $\rho\frac{De}{Dt} = \frac{p}{\rho}\frac{D\rho}{Dt} - \nabla.\overrightarrow{q} + \overleftrightarrow{\tau} : (\nabla\overrightarrow{w})^t$. [8 Marks]

(b) Check whether the heat added Q and entropy S per unit mass of an ideal gas are function of state or not? Derive entropy equation for non-ideal gas with equation of state $p = \frac{RT}{(v-b)}$. [3+3 Marks]

(3) (a) Show that $c_p - c_v = -T(\frac{\partial v}{\partial T})_p^2(\frac{\partial p}{\partial v})_T$. Calculate $c_p - c_v$ for the non-ideal gas with equation of state $p(v-b) = RT$. [7 Marks]

(b) Investigate the maximum mass flow through a nozzle and show that for maximum isentropic mass flow, conditions at the exit plane are sonic. [7 Marks]

(4) (a) Use the method of characteristic to find the solution of problems $(i) u_t + cu_x = 0, x \in R, t > 0$, $(ii) u_t + uu_x = 0, x \in R, t > 0$; under the condition $u(x, 0) = 1, x < 0$, and $0, x > 0$. Draw the characteristic curves of each and comment on uniqueness of the two solutions. [7 Marks]

(b) Write the jump condition for the mass, momentum and energy equation across normal shock wave. Derive the $\frac{\rho_1}{\rho_2} = \frac{2+(\gamma-1)M^2}{(\gamma+1)M^2}$ where M is shock Mach number. [7 Marks]

(5) (a) Derive the magnetic field intensity equation for a conducting fluid in motion. Explain physically each term of the equation. [6 Marks]

(b) State and prove the Alfven's theorem. [5 Marks]

(c) Define the displacement and momentum thickness of a boundary layer and find the expression for each of them. [3 Marks]

(6) (a) Define Lorentz Force and derive an expression for the magnetic pressure. [7 Marks]

(b) Show that the MHD wave propagate with the speed $\sqrt{a^2 + V_A^2}$, where V_A is Alfven's velocity. [5 Marks]

(c) Define the concept of Boundary layer in viscous fluid flow. [2 Marks]

(7) (a) Derive the energy integral equation for steady, two dimensional boundary layer flow of incompressible fluid. [6 Marks]

(b) Derive the Prandtl's boundary layer equations along with the boundary conditions for two dimensional viscous incompressible fluid flow over a slender body. Write the corresponding equations for the steady flow. [8 Marks]

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Your Roll Number: ....

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations, May, 2025
Part II Semester IV

MMATH18-404(ii): COMPUTATIONAL METHODS FOR PDEs
(UPC: 223502413)

Time: 3 hours | Maximum Marks: 70

Instructions: • Section A is compulsory. • Answer any four questions from section B • All notations have usual meaning • Non-programmable scientific calculators are allowed

Section A

(1) (a) Find the range of values of c for which the Gauss-Siedel iteration scheme converges for the numerical solution of $AX = b$ where $A = \begin{pmatrix} 1 & c & 1 \\ c & 1 & c \\ -c^2 & c & 1 \end{pmatrix}$, c is a real non zero constant. [3]

(b) State true or false and justify the statement: Leap frog scheme for the numerical solution of one dimensional heat equation is unconditionally stable. [3]

(c) Find numerical domain of dependence at the point $(j\Delta x, k\Delta y, (n+1)\Delta t)$ of the scheme $u_{j,k}^{n+1} = a_2 u_{j,k-1}^n + a_3 u_{j,k}^n + a_4 u_{j+1,k}^n$. [4]

(d) State Gerschgorin Circle Theorem. Illustrate how it can be used to check the stability of a finite difference scheme. [4]

Section B

(2) (a) Consider the scheme [9]

$$\left(1 - \frac{\mu_x}{2}\delta_x^2\right) u_{j,k}^* = \left(1 + \frac{\mu_x}{2}\delta_x^2\right)\left(1 + \frac{\mu_y}{2}\delta_y^2\right) u_{j,k}^n$$

$$\left(1 - \frac{\mu_y}{2}\delta_y^2\right) u_{j,k}^{n+1} = u_{j,k}^*, \quad \text{where } \mu_x = \frac{\Delta t}{(\Delta x)^2},\ \mu_y = \frac{\Delta t}{(\Delta y)^2}$$

for the numerical solution of the problem $u_{xx} + u_{yy} = u_t$. Show that this scheme has truncation error of $O((\Delta x)^2 + (\Delta y)^2 + (\Delta t)^2)$.

(b) Examine the stability of the Dufort Frankel scheme [5]

$$U_k^{n+1} = \frac{2\mu}{1+2\mu}(U_{k+1}^n + U_{k-1}^n) + \frac{1-2\mu}{1+2\mu}U_k^{n-1}, \ \mu = \Delta t/(\Delta)x^2.$$

for the solution of the problem $u_t = u_{xx}$.

(3) (a) Solve the Laplace equation [9]

$$\nabla^2 u = x^2 + y^2, \quad (x, y) \in \mathbf{R}, u = (x^4 + y^4)/12 \text{ on } \partial\mathbf{R}$$

where $\mathbf{R}$ is the interior of the unit square $0 \le x, y \le 1$, $\Delta x = \Delta y = 1/4$ using five point difference scheme and one iteration of Gauss Seidel method for solving the resulting system of equations. Use initial approximation as $U_{i,j}^0 = 0, \forall\ i,\ j = 1(1)3$.

(b) Consider the system of equations [5]

$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = 0,$$

where $u = [u_1, u_2, u_3 \ldots, u_N]^T$ is an $N-$dimensional vector and A is a constant $N \times N$ matrix. Derive an implcit finite difference scheme for the solution of the above problem.

(4) (a) Show that for the problem $u_t(x,t) + a(x,t)u_x(x,t) = 0,\ x \in \mathbb{R},\ t > 0$ the Lax-Wendroff scheme is second order consistent in both space as well as time variable. [7]

(b) Consider the problem $u_{tt} = u_{xx} + 2$ together with initial and boundary conditions $u(x,0) = \sin\pi x,\ \frac{\partial u}{\partial t}(x,0) = 0,\ 0 \le x \le 1, u(0,t) = u(1,t) = 0,\ t \ge 0$. Find the numerical solution of the above problem using central time central space scheme ($\Delta x = 0.25,\ \Delta t = 0.1$) for two time levels ($t = 0.1,\ 0.2$). [7]

(5) (a) Derive Galerkin finite element approximation for the problem [7]

$$u_t - 3u_{xx} = 2,\ 0 < x < 1, u(x,0) = x + 1,\ u(0,t) = 1,\ u(1,t) = 2,$$

using Backward Euler scheme for time discretization and linear basis functions in space direction. Obtain the resulting system of algebraic equations for the first time level (take $\Delta x = 1/4$).

(b) For the linear advection equation $u_t + au_x + bu_y = 0$, where $a,\ b$ are positive constants, discuss CFL condition and stability results for the scheme [7]

$$U_{jk}^{n+1} + \frac{R_x}{2}\delta_{x0}U_{jk}^{n+1} + \frac{R_y}{2}\delta_{y0}U_{jk}^{n+1} = U_{j,k}^{n}.$$

(6) (a) Consider the following equation in cylindrical polar co-ordinates [7]

$$\frac{\partial^2 u}{\partial z^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2} = f(r,z,u),\ 0 \le r \le R,\ 0 < z \le c,$$

$$\frac{\partial u}{\partial r}(0,z) = 0, u(R,z) = g(z),\ t \ge 0\ u(r,0) = f(r) \quad 0 \le r \le R,\ u(r,c) = h(r).$$

Derive a finite difference scheme for the numerical solution of the above problem on a rectangular mesh.

(b) Consider $u_t = u_{xx}, 0 < x < 1, t > 0$, where $u(x,0) = \begin{cases} 2x, & 0 \le x \le 1/2 \\ 2(1-x) & 1/2 \le x \le 1, \end{cases}$ [7]

$u(0,t) = u(1,t) = 0\ \forall t > 0$. Solve the above problem using weighted theta method with $\theta = 1/2, \Delta x = 0.25, \Delta t = 0.1$. Find the solution at the first time level.

(7) (a) Consider the problem [10]

$$-(u_{xx} + u_{yy}) = 1, \quad 0 < x < 1,\ 0 < y < 1,$$
$$u(x,0) = u(x,1) = u(0,y) = u(1,y) = 0.$$

Solve the above problem using a finite element technique involving triangular elements and linear basis functions. Take $\Delta x = 1/2,\ \Delta y = 1/2$ and obtain the resulting system of algebraic equations in matrix form.

(b) Consider the problem $2u_{xx} - u_x + 2u_{yy} + 4u_y = x^2,\ 0 \le x, y \le 1$, with Dirichlet boundary conditions. Derive a second order scheme for the numerical solution of the above problem. [4]

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Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations, May-June 2025
Part II Semester IV
MMATH18-404(iii): Dynamical Systems (UPC-223502414)

Time: 3 hours | Maximum Marks: 70

Instructions: • Attempt any **FIVE** questions. • Question 1 is compulsory. • Each question carries 14 marks.

(1) (a) Solve the initial value problem $\dot{x} = Ax$ with [2 Marks]

$$A = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}.$$

(b) Define the flow ϕ_t of the non-linear system $\dot{x} = f(x)$. [2 Marks]

(c) Define the *Poincaré map*. [2 Marks]

(d) Determine the stability of the equilibrium points of the system $\dot{x} = f(x)$ with [2 Marks]

$$f(x) = \begin{bmatrix} x_1(3 - x_1 - 2x_2) \\ x_2(2 - x_1 - x_2) \end{bmatrix}.$$

(e) Compute the exponential of the matrix [2 Marks]

$$\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}.$$

(f) Define the following terms: stable focus and ω-limit point. [2 Marks]

(g) Find the maximal interval of existence (α, β) for the initial value problem [2 Marks]

$$\dot{x}_1 = \frac{1}{2x_1}, \quad x_1(0) = 1$$
$$\dot{x}_2 = x_1, \quad x_2(0) = 1$$

and if $\alpha > -\infty$ or $\beta < \infty$ then find the limit of the solution as $t \to \alpha^+$ or as $t \to \beta^-$ respectively.

(2) (a) Classify the equilibrium points of the Lorenz equation $\dot{x} = f(x)$ with [5 Marks]

$$f(x) = \begin{bmatrix} x_2 - x_1 \\ \mu x_1 - x_2 - x_1x_3 \\ x_1x_2 - x_3 \end{bmatrix}$$

for $\mu > 0$. At what value of the parameter μ do two new equilibrium points "bifurcate" from the equilibrium point at the origin?

(b) Sketch the phase portraits corresponding to [4 Marks]

$$\dot{x} = sin(x)$$

and determine the stability of all the fixed points.

(c) Use appropriate Liapunov function to determine the stability of the equilibrium points of the following system [5 Marks]

$$\dot{x}_1 = -x_2 - x_1x_2^2 + x_3^2 - x_1^3,$$
$$\dot{x}_2 = x_1 + x_3^3 - x_2^3,$$
$$\dot{x}_3 = -x_1x_3 - x_3x_1^2 - x_2x_3^2 - x_3^5.$$

(3) State and prove the Stable-Manifold Theorem. [14 Marks]

(4) (a) Define the term "Bifurcations". [2 Marks]

(b) Discuss the stability analysis and whole mechanism by making graphs for the following one-dimensional system [8 Marks]

$$\dot{x} = \mu x - x^3.$$

(c) Make bifurcation diagram for the pitchfork bifurcation. [4 Marks]

(5) (a) Using an appropriate theorem, show that the following system has a critical point with an elliptic domain at origin [5 Marks]

$$\dot{x} = y, \quad \dot{y} = -x^3 + 4xy.$$

(b) Solve the two-dimensional system [5 Marks]

$$\dot{y}_1 = -y_1, \quad \dot{y}_2 = -y_2 + z^2, \quad \dot{z} = z$$

and show that the successive approximations $\Phi_k \longrightarrow \Phi$ and $\Psi_k \longrightarrow \Psi$ as $k \longrightarrow \infty$ for all $(y_1, y_2, z) \in \mathbb{R}^3$.

(c) Find the stable, unstable and center subspaces E^s, E^u and E^c of the system $\dot{x} = Ax$ with the matrix [4 Marks]

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

(6) (a) Construct the phase portriat for the following undamped pendulum by converting it into Newtonian system [6 Marks]

$$\ddot{x} + \sin(x) = 0.$$

(b) Define the following systems: Hamiltonian system with n degrees of freedom on E, Gradient system on E and orthogonal system to a planar system. [8 Marks]
For the Hamiltonian function $H(x,y) = x^2 + 2y^2$, determine both Hamiltonian system and gradient system. Find the critical points for both the systems and determine the type and nature of equilibrium point.

(7) (a) State "The Local Center Manifold Theorem". [4 Marks]

(b) Use "The Local Center Manifold Thoerem" to find the approximation for the flow on local center manifold for the system [10 Marks]

$$\dot{x}_1 = x_1 y - x_1 x_2^2, \quad \dot{x}_2 = x_2 y - x_2 x_1^2, \quad \dot{y} = -y + x_1^2 + x_2^2.$$

Also, show that origin is a type of topological saddle that is unstable.

Roll No. : ....

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations, May-2025
(Part II, Semester IV)

MMATH18–405(i): Cryptography

(UPC NO. 223503401)

Maximum Marks: 35

Time: 2 hours

Instructions: • Question 1 is compulsory. • Attempt **FIVE** questions in all.
• All symbols have their usual meaning unless otherwise specified.

(1) (a) In the DES algorithm, how many bits are given as input to each S-box and how many bits are produced as output from it? Is DES a symmetric or asymmetric encryption algorithm, and why? [2 Marks]

(b) What are the possible key sizes in the AES algorithm? Which AES key size requires 14 rounds of processing? [2 Marks]

(c) What is the Vernam One-Time Pad cryptosystem and how does it ensure perfect secrecy? [3 Marks]

(2) Explain the RSA encryption scheme. Prove that $m^{ed} \equiv m \bmod n$ where (n, e) is the public key and d is the corresponding private key. [7 Marks]

(3) Encrypt the plaintext 1011000101001010 using ECB mode and CBC mode explaining each step. Use the permutation cipher with block length 4 and the key $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$. For CBC, take Initialization Vector IV= 1010. [3+4 Marks]

(4) Describe the structure of a Feistel cipher and explain the encryption and decryption processes. [7 Marks]

(5) (a) What is the Blum Blum Shub generator and how does it work? [4 Marks]

(b) Starting with the key $(1, 0, 0, 0)$, determine the keystream and its period, given that the keystream is generated by the recurrence relation $z_{i+4} = (z_i + z_{i+1}) \bmod 2,\ i \geq 1$. [3 Marks]

(6) (a) Explain the round key generations $K_1, K_2, \ldots, K_{16}$ in the DES algorithm without explicitly writing PC1 and PC2. [5 Marks]

(b) In the AES algorithm, the state is a 4×4 matrix of bytes. How does the ShiftRows operation change the state? [2 Marks]

(7) (a) What is unconditional security of a cryptographic protocol? [2 Marks]

(b) What is Common Modulus Attack in RSA cipher and how does it allow message recovery? [5 Marks]

Your Roll Number: ....

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics End Semester Examinations, May 2025
Part II Semester IV

MMATH18 405(ii): SUPPORT VECTOR MACHINES
(UPC 223503402)

Time: 2 hours | Maximum Marks: 35

Instructions: • Question no. 1 is compulsory. • Answer any four questions from Q. 2 to Q. 7. • All notations have usual meaning.

(1) (a) What is the Slater's condition for convex programming prroblem? [2 Marks]

(b) Define ε-band hyperplane for a linear regression problem. [2 Marks]

(c) Illustrate adjacent matrix of the graph with suitable example. [3 Marks]

(2) (a) State the strong duality theorem. [2 Marks]

(b) Write the KKT conditions for the convex optimization problem: [5 Marks]

$$\begin{aligned} \min \quad & f_0(x), \quad x \in \mathbb{R}^n \\ \text{s.t.} \quad & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & h_i(x) = a_i^T x - b_i = 0, \quad i = 1, \ldots, p. \end{aligned}$$

Further, prove that for the above problem satisfying Slater's condition, it is necessary and sufficient for its solution x^* to satisfy KKT conditions.

(3) Consider the linearly separable problem (P): [7 Marks]

$$\begin{aligned} & \text{Minimize } \frac{1}{2}\|w\|^2 \\ & \text{subject to } y_i(w \cdot x_i + b) \geq 1, \quad i = 1, 2, \ldots, l, \end{aligned}$$

corresponding to the training set $T = \{(x_1, y_1), (x_2, y_2), \ldots, (x_l, y_l)\}$. Prove that (P) has a solution (w^*, b^*) such that $w^* \neq 0$. Also, prove that there exists a $j \in \{i \mid y_i = 1\}$ and a $k \in \{i \mid y_i = -1\}$ such that $(w^* \cdot x_j) + b^* = 1$ and $(w^* \cdot x_k) + b^* = -1$, respectively.

(4) Consider the linearly separable problem (P). Prove that for any solution $\alpha^* = (\alpha_1^*, \cdots, \alpha_l^*)^\top$ to the dual problem [7 Marks]

$$\begin{aligned} \min_{\alpha} \quad & \frac{1}{2}\sum_{i=1}^{l}\sum_{j=1}^{l} y_i y_j (x_i \cdot x_j)\alpha_i\alpha_j - \sum_{j=1}^{l}\alpha_j, \\ \text{s.t.} \quad & \sum_{i=1}^{l} y_i\alpha_i = 0, \alpha_i \geq 0, \quad i = 1, \cdots, l, \end{aligned}$$

there must be a nonzero component α_j^*. Furthermore, prove for any nonzero component α_j^* of α^*, the unique solution to the primal problem can be obtained as $w^* = \sum_{i=1}^{l} \alpha_i^* y_i x_i$, $b^* = y_j - \sum_{i=1}^{l} \alpha_i^* y_i (x_i \cdot x_j)$.

(5) (a) For the training set $T := \{(x_1, y_1), (x_2, y_2), \ldots, (x_l, y_l)\}$, where $x_i \in \mathbb{R}^n$, $y_i \in \mathcal{Y} = \mathbb{R}$, $i = 1, \ldots, l$, and $\bar{\varepsilon} > 0$. Show that a hyperplane $y = (w \cdot x) + b$ is a hard $\bar{\varepsilon}$-band hyperplane if and only if the sets $D^+ = \{(x_i^T, y_i + \bar{\varepsilon})^T, i = 1, \ldots . l\}$ and $D^- = \{(x_i^T, y_i - \bar{\varepsilon})^T, i = 1, \ldots, l\}$ lie on opposite sides of this hyperplane respectively, and all of the points in D^+ and D^- do not touch this hyperplane. [4 Marks]

(b) Write an algorithm for linear $\epsilon -$ *band* support vector regression. [3 Marks]

(6) (a) Find the dual of primal problem: [4 Marks]

$$\min_{w,b,\zeta} \quad \frac{1}{2}\|w\|^2 + C\sum_{i=1}^{l}\zeta_i$$
$$\text{s.t.} \quad y_i\left((w \cdot \Phi(x_i)) + b\right) \geq 1 - \zeta_i, \quad i = 1, \ldots, l$$
$$\zeta_i \geq 0, \quad i = 1, \ldots, l$$

where $x_i \in \mathbb{R}^n$, $y_i \in \mathcal{Y} = \{-1, 1\}$, and $\Phi : \mathbb{R}^n \to \mathcal{H}$ is defined as $\Phi(x) = x$.

(b) Write an algorithm for classification machine based on nonlinear separation. [3 Marks]

(7) (a) If $f(\cdot)$ is a real-valued function defined on $\mathbb{R}^n$, then prove that $K(x, x') = f(x)f(x')$ is a kernel. Particularly, the function $K(x, x') \equiv a$ where a is a nonnegative scalar, is a kernel. [3 Marks]

(b) Prove that the Gaussian radial basis function with parameter σ given by $K(x, x') = \exp\left(-\frac{\|x-x'\|^2}{\sigma^2}\right)$ is a kernel. [4 Marks]