Your Roll Number: ....

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations May-2025
Part I Semester II

MMATH18-201: Module Theory, UPC- 223501201

Time: 3 hours   Maximum Marks: 70

Instructions: • Question 1 is compulsory. • Attempt any four Questions from the rest. • All questions carry equal marks.

(1) (a) Show that the bound of a cyclic module over a PID is equal to the order of its generator. [3]

(b) Every torsion-free divisible abelian group is a direct sum of copies of rationals $\mathbb{Q}$. Justify [3]

(c) Give an example of a projective non-free module with justification. [3]

(d) Give an example of a module which is Noetherian but not Artinian, and also of one which is both. [3]

(e) For a finite abelian group G determine the $\mathbb{Z}$-module $G \otimes \mathbb{Q}$. [2]

(2) (a) Let M and N be left R-modules, $k \in \mathbb{N}$. Prove that [6]

$$Hom_R(M, N^k) \cong (Hom_R(M, N))^k.$$

(b) Define a split exact sequence. Prove that a short exact sequence $0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$ splits if and only if α has a right inverse. [2+6]

(3) (a) Describe what is Five-Lemma. Provide a detailed proof of the Lemma. [2+8]

(b) Show that a zero morphism in the category R^{Mod} corresponds to a zero morphism in the category S^{Mod} and zero module in R^{Mod} is mapped to the zero module of the category S^{Mod} via an additive functor F from R^{Mod} to S^{Mod}. [2+2]

(4) (a) Show that every injective left module over an integral domain is divisible. Does the converse hold? Justify. [3+3]

(b) State and prove the Dual basis lemma. [1+7]

(5) (a) Show that $R \otimes M \cong M$, for left module M over R. [4]

(b) State and prove the universal property of the tensor product. [2+8]

(6) (a) Let N be a proper submodule of a module M of finite length. Prove that the length of N is strictly less than the length of M. [7]

(b) Prove that every submodule of a semisimple module M is complemented in M. [7]

(7) (a) Let $0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$ be a short exact sequence of R-modules and R-homomorphisms. Prove that M is Noetherian if and only if both M' and M'' are Noetherian modules. [7]

(b) Let M be a torsion module over a principal ideal domain R. Prove that $M = \bigoplus_p M_p$, where M_p is p-primary submodule of M for prime p in R. [7]

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examinations, May 2025
Part I Semester II

MMATH18-202: INTRODUCTION TO TOPOLOGY
(Unique Paper Code 223501202)

Time: 3 hours

Maximum Marks: 70

Instructions: • Attempt five questions in all. • **Question 1 is compulsory.** • All questions carry equal marks. • The symbols used have their usual meanings.

(1) (a) Show that $\mathbb{Z}$ is a discrete subspace of $\mathbb{R}$ with usual topology. [2]

(b) Show that the collection
$$\mathcal{B} = \{(a, b) : a < b \text{ and } a, b \in \mathbb{Q}\},$$
is a basis for the usual topology on $\mathbb{R}$. [3]

(c) Prove that a sequence in a discrete space is convergent if and only if it is eventually constant. [3]

(d) Define a T_1-space. Show that a finite T_1-space is discrete. [3]

(e) Prove that the Sorgenfrey line $\mathbb{R}_l$ is neither connected nor compact. [3]

(2) (a) On the set $\mathbb{R}$ of real numbers, consider the collection
$$\mathcal{T}_f = \{\phi\} \cup \{U \subseteq \mathbb{R} : \mathbb{R} - U \text{ is finite}\}.$$
Prove that $\mathcal{T}_f$ is a topology on $\mathbb{R}$, which is not first countable. Is $(\mathbb{R}, \mathcal{T}_f)$ separable? Justify. [4+3+1]

(b) Let $\{A_i : i \in I\}$ be a locally finite family of subsets of a space X. Prove that $\bigcup_{i\in I} \overline{A_i}$ is closed in X. [6]

(3) (a) Let $X \times Y$ be the product space. Prove that a function $f : Z \to X \times Y$ is continuous if and only if the coordinate functions $p_X \circ f$ and $p_Y \circ f$ are continuous, where $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ are projection maps. Does this result hold for the arbitrary product with box topology? Justify your answer. [4+3]

(b) If $\{X_i : i = 1, 2, \ldots n\}$ is a finite collection of metrizable spaces, prove that $\prod_{i=1}^{n} X_i$ is metrizable. [7]

(4) (a) Prove that the subspaces $\mathbb{S}^{n-1}$ and $\mathbb{R}^n - \{0\}$ of the Euclidean space $\mathbb{R}^n$ are connected for $n \geq 2$. [5]

(b) Let X be a space and $A \subseteq X$. Prove that a point $x \in \overline{A}$ if and only if there is a net in A which converges to x. [5]

(c) Prove that a separable metric space is second countable. [4]

(5) (a) Let $\{X_\alpha : \alpha \in A\}$ be a family of spaces. Prove that the product space $\prod_{\alpha\in A} X_\alpha$ is path-connected if and only if each X_α is path-connected. [6]

(b) Prove that a space X is Hausdorff if and only if the diagonal $\Delta = \{(x, x) : x \in X\}$ is closed in $X \times X$. Use this result to show that if $f : X \to Y$ is a continuous function and Y is Hausdorff, then the graph of f is closed in $X \times Y$. [5+3]

(6) (a) When do we say that a space X is locally connected? Prove that if $f : X \to Y$ is a continuous closed surjection and X is locally connected, then Y is also locally connected. [1+5]

(b) Let A be a compact subset of a Hausdorff space X and $x \in X - A$. Prove that there are disjoint open sets U and V such that $x \in U$ and $A \subseteq V$. Use this result to show that a compact subset of a Hausdorff space is closed. [5+3]

(7) (a) State and prove the Tube Lemma. Hence or otherwise, show that if X is a compact space, then the projection map $p : X \times Y \to Y$ defined by $p(x, y) = y$ is closed for all spaces Y. [6+4]

(b) State the Bolzano-Weierstrass property for a space X. Prove that it is satisfied by compact spaces. [1+3]

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examinations, May 2025
Part I Semester II
MMATH18-203: FUNCTIONAL ANALYSIS
(Unique Paper Code 223501203)

Time: 3 hours | Maximum Marks: 70

Instructions: • Attempt **five** questions in all. **Question 1 is compulsory**. All questions carry equal marks.• The symbols used have their usual meanings.

(1) (a) Show that for $0 < p < 1$, $\|x\|_p = (|\xi_1|^p + |\xi_2|^p)^{1/p}$, $x = (\xi_1, \xi_2) \in \mathbb{R}^2$ does not define a norm in $\mathbb{R}^2$. [3]

(b) Fix $a = (\alpha_n) \in l^2$, define $f(x) = \sum_{n=1}^{\infty} \alpha_n \xi_n$, $x = (\xi_n) \in l^2$. Find the norm of the functional f. [2]

(c) If (x_n) is a sequence in $C([a, b])$ which converges weakly to $x \in C([a, b])$, show that for every $t \in [a, b]$, $x_n(t) \to x(t)$. [3]

(d) If S and T are linear normal operators on a Hilbert space H satisfying $ST^* = T^*S$ and $TS^* = S^*T$, then show that their sum $S + T$ and product ST are normal. [3]

(e) Give an example, with justification, of a bounded linear operator on a normed space X which is injective but not surjective. [3]

(2) (a) Let X and Y be normed spaces, $T : \mathfrak{D}(T) \subseteq X \to Y$ be a linear operator. Show that T is continuous if and only if T is bounded. [5]

(b) Let $X = \mathbb{C}^n$ and $M = \{x = (\xi_j) \in X \mid \sum_{j=1}^{n} \xi_j = 1\}$. Show that M is convex and complete. [4]

(c) Define spectrum $\sigma(T)$ of a linear operator T on a complex normed space X and classify $\sigma(T)$ into point spectrum, continuous spectrum and residue spectrum. What can you say if dimension of X is finite? Justify. [5]

(3) (a) Let H_1, H_2 be Hilbert spaces and $T : H_1 \to H_2$ be a bounded linear operator. Show that there exists a unique bounded linear operator $T^* : H_2 \to H_1$ such that $\|T^*\| = \|T\|$ and for all $x \in H_1$ and $y \in H_2$, $\langle Tx, y\rangle = \langle x, T^*y\rangle$. [6]

(b) If the dual space X' of a normed space is separable, show that X itself is separable. [6]

(c) If (x_n) is a sequence in a normed space X such that $x_n \xrightarrow{w} x$ in X, show that the weak limit x of (x_n) is unique, [2]

(4) (a) Define Schauder basis for a normed space. If a normed space X has a Schauder basis, show that X is separable. [1+4]

(b) State Baire's category theorem. [2]

(c) Let X be a complex Banach space and $\Lambda \subseteq \mathbb{C}$ be open. For the operator function $S : \Lambda \to B(X, X)$ defined by $S(\lambda) = S_\lambda$, define the concepts of local holomorphicity and holomorphicity on Λ, and holomorphicity at a point $\lambda_0 \in \Lambda$. If $T : X \to X$ is a bounded linear operator, show that the resolvent $R_\lambda(T)$ is holomorphic at every point $\lambda_0 \in \rho(T)$ and hence is holomorphic on $\rho(T)$. [2+5]

(5) (a) State and prove Hahn-Banach theorem for normed spaces. [5]

(b) Let H be a Hilbert space, $T : H \to H$ be a isometric linear operator. Show that $T^*T = I$, the identity operator. [4]

(c) Show that the spectrum $\sigma(T)$ of a bounded linear operator T on a complex Banach space X is closed. [5]

(6) (a) State and prove closed graph theorem. [2+5]

(b) Let H be a Hilbert space. With every $z \in H$ we can associate $f_z \in H'$ defined by $f_z(x) = \langle x, z\rangle$. Show that $T : H \to H'$, defined by $T(z) = f_z$, is an isometric bijection which is conjugate linear. Hence conclude that the dual space H' is a Hilbert space with the inner product $\langle \cdot, \cdot\rangle_1$, where $\langle f_z, f_w\rangle_1 = \langle w, z\rangle$. [7]

(7) (a) State uniform boundedness theorem. Use it to prove that the normed space X of all polynomials x with the norm $\|x\| = \max_j |\alpha_j|$, where $\alpha_1, \alpha_2, \ldots$ are the coefficients of x, is not complete. [2+5]

(b) Let $T : \mathfrak{D}(T) \to Y$ be a bounded linear operator, where $\mathfrak{D}(T)$ lies in a normed space X and Y is a Banach space. Show that T has an extension $\tilde{T} : \overline{\mathfrak{D}(T)} \to Y$, where $\tilde{T}$ is a bounded linear operator of norm $\|\tilde{T}\| = \|T\|$. [7]

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examinations, May 2025
Part I Semester II,UPC 223501204

MMATH18-204: FLUID DYNAMICS

Time: 3 hours Maximum Marks: 70

Instructions: • Attempt **Five** questions in all • Question No.1 is compulsory and attempt any **Four** questions from question Nos. 2 to 7 • Each question carries 14 marks • Notations, names and words have their usual meaning.

(1) (a) State the conservation of mass and momentum using the control volume approach and write their mathematical formulations. [2+2 Marks]

(b) What is the Magnus effect? Elaborate on it. [1+3 Marks]

(c) State whether the statement "The value of Stokes stream function Ψ at any point in an incompressible, axi-symmetric flow depends solely on the position of the point and not on the path" is true or false and justify your assertion. [1+2 Marks]

(d) Write the expressions for the six distinct components of the stress matrix in terms of the principal stresses. [3 Marks]

(2) (a) Define steady and unsteady flow of a fluid. State whether the statement "In steady flow, the streamlines, pathlines and streaklines will coincide" is true or false and justify your assertion. What boundary conditions must be satisfied at a rigid boundary (stationary or moving) in contact with an inviscid or viscous fluid? [7 Marks]

(b) State and prove a necessary condition for a surface to be a boundary surface. Show that $\frac{x^2}{a^2}tan^2t + \frac{y^2}{b^2}cot^2t = 1$ is a possible form of the boundary surface and find an expression for the normal velocity. [3+4 Marks]

(3) (a) State and prove Kelvin's minimum energy theorem. A large mass of incompressible non-viscous fluid contains a spherical air bubble, the air inside the bubble obeying Boyle's law, $pv =$ constant. At a great distance from the bubble the pressure is zero. Neglecting the inertia of the system, Determine the equation to be satisfied by the radius R of the bubble at time t. [4+3 Marks]

(b) Write the Euler's equation of motion in vector and tensor forms. Elucidate the role of the Bernoulli's equation in the working of the Pitot tube which is used for measuring the fluid velocity. [4+3 Marks]

(4) (a) State and prove the theorem of Blasius. Use this theorem to determine the force exerted on a circular cylinder $|z| = a$ in an irrotational flow produced by a line source of strength m at $z = 3a$. [5+3 Marks]

(b) Show that the steam function $\psi(x, y)$ for a two-dimensional motion exist whether the motion is irrotational or not but the velocity potential $\varphi(x, y)$ does exist only when the motion is irrotational. In a two-dimensional motion, a source of strength m is placed at each of the points $(-2, 0)$ and $(2, 0)$. and a sink of strength $4m$ is placed at origin. Determine the equation of the streamlines. [6 Marks]

(5) (a) Find the complex potential, complex velocity, stagnation points and the streamlines of a combined flow of a source of strength m at the origin and a uniform stream parellel to the $x-$axis. Determine the velocity components at a point P of a uniform flow past a fixed infinite circular cylinder. [3+3 Marks]

(b) Derive an expression for the velocity potential due to a three-dimensional doublet and hence find the components of velocity and the equation of the the stream lines. State and prove Butler's sphere theorem. [4+4 Marks]

(6) (a) Find the Stokes Stream function Ψ of a three-dimensional simple source of strength m at the origin. Determine the stream surfaces of a doublet in a uniform stream whose undisturbed velocity is $-Uk$. [3+4 Marks]

(b) State and prove Weiss's sphere theorems. Determine the image of a source in a solid sphere. [5+2 Marks]

(7) (a) Define stress matrix, rate of dilatation, laminar flow, no-slip condition and kinematic coefficient of viscosity for real fluids. Write the tensor form of the Navier-Stokes equations of motion. [5+2 Marks]

(b) Determine the fluid velocity profile from the expression of $w(R)$ and also the volume of fluid discharged over any section per unit time (Q) for the problem of a steady flow through tube of uniform circular cross-section. [7 Marks]

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