Your Roll Number: ....

Department of Mathematics, University of Delhi
M.Sc. Mathematics Examinations, January 2024
Part II Semester III

MMath18- 301(i) : Algebraic Topology (UPC 223502301)

Time: 3 hours

Maximum Marks: 70

Instructions: • Answer **five** questions. • **Question 1** is compulsory. Any **four** questions from **Q2** to **Q7**. • Each question carries 14 marks.

(1) (a) Show that any two continuous maps from the plane $\mathbb{R}^2$ into the 2-sphere $\mathbb{S}^2$ are homotopic. [3.5 Marks]

(b) Let A be strong deformation retract of X. Show that the inclusion map $i : A \hookrightarrow X$ induces an isomorphism $i_{\#} : \pi_1(A, x_0) \to \pi_1(X, x_0)$. [3.5 Marks]

(c) Let $p : \widetilde{X} \to X$ be a 7-sheeted covering map, where $\widetilde{X}$ is simply connected. Determine the fundamental group of X. [3.5 Marks]

(d) Determine the group of deck transformations for the quotient map $p : \mathbb{S}^n \to \mathbb{S}^n/\sim$, where $\sim$ denotes the equivalence relation identifying each point on the sphere with.its antipodal point. [3.5 Marks]

(2) (a) Give an example of a deformation retract which is not a strong deformation retract. [7 Marks]

(b) Suppose that a contimuous map $f : \mathbb{S}^1 \to \mathbb{S}^1$ is not homotopic to the identity map. Prove that there is a point $x \in \mathbb{S}^1$ such that $f(x) = -x$. [7 Marks]

(3) (a) Prove that the fundamental group of the circle is isomorphic to the infinite cyclic group. [8 Marks]

(b) Show that the exponential map $p : \mathbb{R} \to \mathbb{S}^1$ is a covering map. [6 Marks]

(4) (a) Let X and Y be spaces, and $x_0 \in X$ and $y_0 \in Y$. Show that $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \oplus \pi_1(Y, y_0))$. [7 Marks]

(b) Let X be contractible space and Y be any space. Show that $X \times Y \simeq Y$. [7 Marks]

(5) (a) Let $p : \widetilde{X} \to X$ be a covering map, and $f : I \to X$ be a path with origin x_0. If $\tilde{x}_0 \in p^{-1}(x_0)$ then show that there exists path $\tilde{f} : I \to \widetilde{X}$ with $p\tilde{f} = f$ and $\tilde{f}(0) = \tilde{x}_0$. [7 Marks]

(b) Show that any two universal covering spaces of a topological space are equivalent. [5 Marks]

(c) Define regular covering map and give an example. [2 Marks]

(6) (a) Let $p : \widetilde{X} \to X$ be a covering map such that $p(\tilde{x}_0) = x_0$, where $\widetilde{X}$ is path connected. Show that the multiplicity of p is index of $p_\sharp(\pi_1(\widetilde{X}, \tilde{x}_0))$ in $\pi_1(X, x_0))$. Hence, or Otherwise, find the fundamental group of $\mathbb{R}P^n$, $n > 1$. [7 Marks]

(b) Prove that there is no continuous map $f : \mathbb{S}^2 \to \mathbb{S}^1$ such that $f(-x) = -f(x)$ for all $x \in \mathbb{S}^2$. [7 Marks]

(7) (a) Let $p : \widetilde{X} \to X$ be a covering map and $\tilde{x} \in \widetilde{X}$. Show that the group of deck transformations $\Delta(p)$ is isomorphic to [7 Marks]

$$N(p_\# \pi_1(\widetilde{X}, \tilde{x}))/p_\# \pi_1(\widetilde{X}, \tilde{x})$$

where $N(p_\# \pi_1(\widetilde{X}, \tilde{x}))$ is the normaliser of $p_\# \pi_1(\widetilde{X}, \tilde{x}$ in $\pi_1(X, p(\tilde{x}))$.

(b) Determine the fundamental group of $\mathbb{S}^1 \vee \mathbb{S}^1$. [7 Marks]

Your Roll Number: ....

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examinations, December 10, 2024
Part II Semester III
MMATH18-301(ii): COMMUTATIVE ALGEBRA
(Unique Paper Code: 223502302)

Time: 3 Hours Maximum Marks: 70

Instructions: • Question 1 is compulsory. • Attempt any four questions from question 2 to question 7. • All questions carry equal marks. • Throughout the paper all the rings are assumed to be commutative ring with nonzero identity.

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(1) (a) Define extension and contraction of ideals. [2]

(b) Let S be a multiplicative closed set in R. Show that every ideal in $S^{-1}R$ is an extended ideal. [3]

(c) Define the minimal primary decomposition of a decomposable ideal. Also give an example of a decomposable ideal with a minimal primary decomposition. [3]

(d) Let R be a subring of a field K. If $R \subset K$ is an integral extension of rings, then show that R is a field. [3]

(e) Let R be a valuation ring associated with a valuation v on a field K. For any nonzero $x \in K$, show that $v(x) = 0$ if and only if x is unit in R. [3]

(2) (a) Show that the contraction of a maximal ideal $\mathfrak{m}$ of $R[[x]]$ is a maximal ideal of R and $\mathfrak{m}$ is generated by $\mathfrak{m}^c$ and x. [7]

(b) Show that $Hom(M \otimes N, P) \cong Hom(M, Hom(N, P))$, where M, N, P are R-modules. [7]

(3) (a) Let $\mathfrak{p} \in Spec(R)$. Show that the prime ideals of $R_{\mathfrak{p}}$ are in one to one correspondence with the prime ideals of R contained in $\mathfrak{p}$. [7]

(b) Let $f : A \to B$ be a ring homomorphism and let $\mathfrak{p} \in Spec(A)$. Show that $\mathfrak{p}$ is the contraction of a prime ideal of B if and only if $\mathfrak{p}^{ec} = \mathfrak{p}$. [7]

(4) (a) Let $\mathfrak{m} \in Max(R)$. Show that $\mathfrak{m}^5$ is a primary ideal. [4]

(b) State and prove the first uniqueness theorem of primary decomposition. [5]

(c) Let S be a multiplicative closed set in R. Show that there is one to one correspondence between the primary ideals of $S^{-1}R$ and the contracted primary ideals of R. [5]

(5) (a) State and prove the Going-up theorem. [1+5]

(b) Let C be the integral closure of the ring A in B and let $\mathfrak{a}^e$ denote the extension of $\mathfrak{a}$ in C. Show that the integral closure of $\mathfrak{a}$ in B is the radical of $\mathfrak{a}^e$. [5]

(c) Let B be a valuation ring of a field K. Show that B is a local ring. [3]

(6) (a) Let R be a ring in which the zero ideal is the product of finitely many maximal ideals. Show that R is Noetherian if and only if R is Artinian. [6]

(b) Show that in a Noetherian ring R, every ideal contains a power of its radical. [4]

(c) Show that in an Artinian ring R, the nilradical is nilpotent. [4]

(7) (a) Let R be an integral domain. Show that R is a Dedekind domain if and only if every nonzero fractional ideal of R is invertible. [7]

(b) Let R be a Noetherian local one dimensional domain in which there exists an element x such that every nonzero ideal is of the form $< x^n >, n \geq 0$. Show that R is a discrete valuation ring. [7]

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations, Dec 2024
Part II, Semester III, Unique Paper Code- 223502303

MMATH18-301(iii):Representation of Finite Groups

Time: 3 hours Maximum Marks: 70

Instructions: • Attempt five questions in all.• **Question1** is compulsory. • All questions carry equal marks • Throughout this question paper G will denote a finite group and F a scalar field.

(1) (a) If G is a non-abelian group of order 6, then find dimensions of all the irreducible $\mathbb{C}G$-modules. [3 Marks]

(b) Find all irreducible $\mathbb{C}G$-modules for $G = C_2 \times C_2$ [3 Marks]

(c) If x is a non identity element of the group G, then prove that $\chi(g) \neq \chi(1)$ for some irreducible character χ of G. [3 Marks]

(d) Define a regular representation and prove that the regular FG-module is faithful. [1+2 Marks]

(e) Let ω be the fifth root of unity. Is $\omega(\omega^2 + \omega^3)$ is an algebraic integer? Justify. [2 Marks]

(2) Prove that the representations of the form $g \mapsto [g]_\beta$, $g \in G$, for some basis β of an FG-module V, are equivalent to each other and also any two equivalent representations of G arise from some FG-moudule in this way. Illustrate this through an example when $G = C_3$. [7+7 Marks]

(3) (a) Let an FG-module V be a sum of finite number of irreducible FG-submodules of V, then show that V is a direct sum of some of the components. [7 Marks]

(b) Prove that if the regular $\mathbb{C}G$-module is a direct sum of irreducible $\mathbb{C}G$-submodules, then every irreducible $\mathbb{C}G$-module is isomorphic to one of the direct summands. [7 Marks]

(4) (a) Show that a group G is finite abelian iff every irreducible $\mathbb{C}G$-module has dimension one. [7 Marks]

(b) Prove that there is no faithful irreducible $\mathbb{C}G$-module when G is abelian but not cyclic. What is the conclusion when $G = C_2 \times C_2$. [7 Marks]

(5) (a) Lifting the irreducible characters of the quotient group S_n/S_n' find all the linear character of the group $S_n (n \geq 2)$ where S_n' is the derived subgroup of S_n. [3 Marks]

(b) Find the permutation character and all linear characters of S_5. Are they irreducible? [4 Marks]

(c) Compute the character table of $S_3 \times C_2$. [7 Marks]

(6) (a) Let χ be a faithful character which takes r distinct values over G and ψ any irreducible character. Prove that $\langle \chi^n, \psi \rangle \neq 0$ for some $n \in \{0, 1, \cdots, r-1\}$. Does the result still hold when χ is not faithful? Justify. [8+6 Marks]

(7) (a) Show that for every character and every element of a group, the image is an algebraic integer. [4 Marks]

(b) If the group G has a conjugacy class of order p^r , where p is a prime number and $r \geq 1$ an integer then show that G is not simple. [10 Marks]

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M.A./M.Sc. Mathematics Examinations, 2024
Part II, Semester III
MMATH18-302(i): Fourier Analysis
Unique Paper Code-223502304

Time: 3 hr

Maximum Marks: 70

Instructions: • Attempt Five questions in all • All questions carry equal marks • Question No. 1 is COMPULSORY • Symbols have their usual meaning.

(Section I: Question No. 1 is Compulsory)

(1) (a) Show that if $f \in L^1(\mathbb{T})$ and $f_{\tau_o}(t) = f(t - \tau_o)$, $\tau_o \in \mathbb{T}$, then $\lim_{\tau \to \tau_o} \|f_\tau - f_{\tau_o}\|_{L^1(\mathbb{T})} = 0$. [3 Marks]

(b) For a function $f \in L^1(\mathbb{R})$ and $\lambda > 0$, find the Fourier transform of the function $\varphi(x) = \lambda f(\lambda x)$, $x \in \mathbb{R}$. [3 Marks]

(c) Show that the modular function on a locally compact group is a homomorphism. [4 Marks]

(d) Find the characters on the additive group of real numbers $(\mathbb{R}, +)$. [4 Marks]

(Section II: Answer any Four questions from this section)

(2) (a) Define Poisson kernel in $L^1(\mathbb{T})$. Show that the Poisson kernel $P(r, t)$ can expressed as $P(r, t) = \frac{1-r^2}{1-2r\cos t + r^2}$, $r \in (0, 1)$, $t \in \mathbb{T}$. [1+5=6 Marks]

(b) Show that if $f \in L^1(\mathbb{T})$ and $\{k_n\}$ is a summability kernel in $L^1(\mathbb{T})$, then $f = \lim_{n \to \infty} k_n * f$ in the $L^1(\mathbb{T})$ norm. [8 Marks]

(3) (a) Show that if μ is a left Haar measure on a locally compact group G and $f \in C_c^+(G)$, then $\int_G f\, d\mu > 0$. [5 Marks]

(b) Prove that there exists a continuous function whose Fourier series diverges at a point. [9 Marks]

(4) (a) Show that if f is a continuous function with compact support on $\mathbb{R}$, then $\frac{1}{2\pi} \int |\hat{f}(\xi)|^2\, d\xi = \int |f(x)|^2\, dx$. [7 Marks]

(b) Show that if $f \in L^1(\mathbb{R})$ and $xf(x) \in L^1(\mathbb{R})$, then $\frac{d}{d\xi} \hat{f}(\xi) = \widehat{(-ixf)}(\xi)$, $\xi \in \widehat{\mathbb{R}}$. [7 Marks]

(5) (a) Show that if f is of bounded variation on $\mathbb{T}$ with total variation $\text{var}(f)$, then $|\hat{f}(n)| = \frac{\text{var}(f)}{2\pi|n|}$, n is a non-zero integer. [3 Marks]

(b) Let F be a compact subset of a topological group G and let e be the identity element of G. Show that if U is neighbourhood of e, then there exists a neighbourhood V of e such that $xVx^{-1} \subset U$ for all $x \in F$. [4 Marks]

(c) Let $f \in L^1(\mathbb{R})$ and $\varphi(t) = 2\pi \sum_{j=-\infty}^{\infty} f(t + 2\pi j)$, $t \in \mathbb{R}$. Show that $\|\varphi\|_{L^1(\mathbb{T})} \leq \|f\|_{L^1(\mathbb{R})}$ and $\widehat{\varphi}(n) = \widehat{f}(n)$, $n \in \mathbb{Z}$. [4+3=7 Marks]

(6) (a) Show that if a locally compact group G is discrete, then $L^1(G)$ has an identity. What happens if G is not discrete? Explain. [4+3=7 Marks]

(b) Show that if $f \in L^1(\mathbb{T})$ and $\int_{\mathbb{T}} \left|\frac{f(t)-f(t_o)}{t-t_o}\right| dt < \infty$, $t_o \in \mathbb{T}$, then $S_n(f, t_o) \to f(t_o)$; where $S_n(f, \cdot)$ denote the n^{th} partial sum of the Fourier series of f. [7 Marks]

(7) (a) Using the characteristic function $\chi_{[0,1]}$, find a summability kernel in $L^1(\mathbb{R})$. [4 Marks]

(b) Show that if λ and μ are left Haar measures on a locally compact group G, then there exists a positive real number c such that $\mu = c\lambda$. [10 Marks]

Your Roll Number: ....

Department of Mathematics, University of Delhi
M.Sc. Mathematics Examinations, December 2024
Part II Semester III
MMATH18- 302(ii): Matrix Analysis, UPC: 223502305

Time: 3 hours | Maximum Marks: 70

Instructions: • Answer **five** questions. • **Question 1** is compulsory. Attempt any **four** questions from **Q2** to **Q7**. • Each question carries 14 marks.

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(1) (a) Determine the spectral radius of a doubly stochastic matrix. (2)

(b) Determine the condition number for matrix inversion of a unitary matrix with respect to the spectral norm. (2)

(c) For $x, y \in \mathbb{R}^n$, prove that $x \prec y$, if $x \prec_w y$ and $-x \prec_w -y$. (3)

(d) For $A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$, prove that the series $\sum_{k=1}^{\infty} \frac{A^k}{k^2}$ is convergent. (3)

(e) Give example to show that e^{A+B} may not be equal to $e^A e^B$. (4)

(2) (a) Let $A \in M_n$ be a doubly stochastic matrix. Prove that there exists a permutation σ of $\{1, 2, \ldots, n\}$ such that $\Pi\, a_{i\sigma(i)} \neq 0$. (7)

(b) State Singular Value Decomposition for $A \in M_n$. Use it to derive the Polar Value Decomposition of A. (7)

(3) (a) If $(x_i) \in \mathbb{R}^n$ is such that $x_i \geq 0, \sum_{i=1}^{n} x_i = 1$, then prove that $\left(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}\right) \prec (x_1, x_2, \cdots, x_n)$. What can you say about (x_i) if $(x_1, x_2, \cdots, x_n) \prec \left(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}\right)$. (4+2)

(b) Prove that the Schur product of two positive definite matrices is positive definite. Is the result true for usual matrix multiplication? Justify your answer. (8)

(4) (a) For $A \in M_n$, prove that (7)

$$\rho(A) = \inf\{|||A||| : |||\cdot||| \text{ is an induced matrix norm}\}.$$

(b) Let $A, B \in M_n$ be positive definite matrices. Prove that there exists a non-singular matrix $C \in M_n$ such that C^*BC is a diagonal matrix and $C^*AC = I$. Use this to deduce that $A \succeq B$ if and only if $\rho(BA^{-1}) \leq 1$. (4+3)

(5) (a) For $x, y \in \mathbb{R}^n$, prove that $x \prec y$ if and only if there exist T-transforms $T_1, T_2, \ldots, T_m$ such that $x = yT_1T_2\ldots T_m$. (8)

(b) Define complex symplectic group $Sp(n, \mathbb{C})$. Is it compact? Give example of one non-trivial compact matrix Lie subgroup of this group. (6)

(6) (a) Prove that a matrix norm is minimal if and only if it is an induced norm. Further, let $|||\cdot|||_\alpha$ and $|||\cdot|||_\beta$ be matrix norms on M_n. Determine the conditions under which $\max\{|||\cdot|||_\alpha, |||\cdot|||_\beta\}$ is an induced norm. (6)

(b) For $A \in M_n$, let S be the union of k Gersgorin discs which forms a connected region disjoint from the remaining $(n-k)$ Gersgorin discs. Prove that exactly k eigenvalues of A lie in S. (5)

(c) Prove that if all the eigenvalues of a Hermitian matrix are positive, then it is positive definite. (3)

(7) (a) Let $\{v_1, v_2, \ldots, v_n\}$ be vectors in an inner product space V and G be the Gram matrix of these vectors. Prove that G is positive semidefinite. Further prove that G is positive definite if and only if $\{v_1, v_2, \ldots, v_n\}$ are linearly independent. (3 + 4)

(b) State and prove Minkowski's Determinant Inequality. (2+5)

Your Roll Number: ....

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations, December 2024
Part II Semester III
MMATH18-302(iii): Theory of Bounded Operators
Unique Paper Code 223502306

Time: 3 hours | Maximum Marks: 70

Instructions: • All notations used are standard • **Question no. 1 is compulsory** • Attempt any four questions from the remaining six questions.

Part A

(1) Classify each statement below as true or false, giving a brief justification in each case:

(a) $\sigma(T) = \emptyset$ when $T : l^2 \to l^2$ be defined by $T(x_1, x_2, \ldots) = (x_1, \frac{x_2}{2}, \frac{x_3}{3}, \ldots)$.

(b) If T is a positive bounded linear operator on a complex Hilbert space H then the approximate point spectrum of T coincides with the spectrum of T.

(c) If P_1, P_2 are projections on an infinite dimensional Hilbert space then so is P_1P_2.

(d) A compact linear operator on an infinite dimensional Hilbert space H can not be unitary.

(e) If (T_n) is a sequence of compact operators on a Banach space X converging in the strong operator topology to an operator T, then T is necessarily compact.

(f) Every projection is a partial isometry.

(g) A bounded linear operator T on a Hilbert space can be written uniquely as $T = UP$ where P is a positive operator on H and U is a unitary operator on H.

(2x7)

Part B

Attempt four questions in all.

(2) Let T be a bounded linear operator on a Hilbert space H. Show that

(a) if T is invertible, then $\sigma(T^{-1}) = \{\mu : \frac{1}{\mu} \in \sigma(T)\}$. (3)

(b) $\sigma(T^*) = \overline{\sigma(T)}$ where T is the Hilbert adjoint of T. (3)

(c) if $\{\lambda_n\} \subset \rho(T)$ and $\lambda_n \to \lambda$ with $\{R(\lambda_n, T)\}$ a bounded sequence in $\mathcal{B}(H)$ then $\lambda \in \rho(T)$. (3)

(d) the boundary of $\sigma(T)$ is contained in the approximate point spectrum of T. (5)

(3) Let T be a compact operator on a Banach space X and $\lambda \neq 0$.

(a) Show that there exists a smallest integer r such that

(i) $N(T_\lambda^r) = N(T_\lambda^m)$ for all $m = r+1, r+2, \ldots$ and

(ii) if $r > 0$ then $N(T_\lambda^m) \subsetneq N(T_\lambda^{m+1})$, $m = 0, 1 \ldots (r-1)$.

Here $N(S)$ denotes the null space of the operator S. (8)

(b) Let $X = l^2(\mathbb{N})$ and $T(x_1, x_2, x_3 \ldots) = (x_1, x_2, x_3, 0, 0 \ldots)$. Show that T is compact and illustrate (i) and (ii) in (a) for this T with $\lambda = 1$. (2+4)

(4) Let $T : l^2 \to l^2$ be defined by $T(\zeta_1, \zeta_2, \ldots) = (\zeta_2, \frac{\zeta_3}{2}, \frac{\zeta_4}{3}, \ldots)$.

(a) Show that T is a compact operator. (6)

(b) Show that $\sigma_p(T) = \{0\}$. (6)

(c) Hence or otherwise find $\sigma(T)$. (2)

(5) Let T be a bounded self-adjoint linear operator on a complex Hilbert space H.

(a) Prove that T is positive if and only if $\sigma(T) \subset [0, \infty)$. Hence or otherwise prove that $I + T^*T$ is invertible. (5)

(b) Show that every positive bounded self adjoint operator T with $\|T\| \leq 1$ has a positive square root. (7)

(c) If T is a projection on H find a square root of T. Is this square root unique? (2)

(6) Let $(P_n)_{n=1}^{\infty}$ be a monotone increasing sequence of projections P_n defined on a Hilbert space H.

(a) Show that (P_n) is strongly operator convergent to a projection P defined on H with range $P(H) = \overline{\cup_{n=1}^{\infty} P_n(H)}$. (9)

(b) Show by an example that in (a) above, even if $P_n(H)$ is finite dimensional for every $n \in \mathbb{N}$, the range $P(H)$ may not be finite dimensional. (5)

(7) (a) Let W be a partial isometry on a Hilbert space H. Show that then the range of W is closed. Further show that W^*W is a projection on H and find its range. (8)

(b) Let T be a trace class operator on a complex Hilbert space H. Define trT the trace of T and show that it is independent of the choice of the orthonormal basis of H. (6)

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M.A./M.Sc. Mathematics / Part II,Semester-III

Examination, December 2024

Paper: MMATH18 : 303(i) Advanced Complex Analysis.

(UPC 223502307)

Time: 3 hours.   Maximum Marks: 70

Instruction: Attempt any ***Five*** questions. Question No. **1 is compulsory**. Each question carries 14 marks.

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1. (a) Give an example, with justification of a convex function which is not logarithmically convex. [3]
   (b) Find the simplest meromorphic function with a pole at every integer n. [3]
   (c) Let $f \in H(G)$ and suppose $C_\infty \setminus G$ is connected. Use Runge's Theorem to show that there is a sequence of polynomials converging to f. [4]
   (d) Show that a harmonic function $u : G \longrightarrow R$ satisfies the Mean Value Property (MVP). [4]

2. (a) State *Phragmen-Lindelof theorem*. Let $a \geq \frac{1}{2}$ and $G = \left\{ z : |argz| < \frac{\pi}{2a} \right\}$. Suppose $f \in H(G)$ and $\limsup\limits_{z \to w} |f(z)| \leq M$ for all $w \in \partial G$. If there are positive constants P and $b < a$ such that $|f(z)| \leq Pexp(|z|^b)$ for all z with $|z|$ sufficiently large. Show that $|f(z)| \leq M$ for all $z \in G$. [2+6]
   (b) Let G be simply connected, $f \in H(G)$ and f does not assume the values 0 or 1. Show that there exists $g \in H(G)$ such that $f = -exp[\pi i cosh(2g)]$. [6]

3. State and prove the *Riemann mapping theorem*. [14]

4. (a) Let G be an open set, $\{a_k\}$ a sequence of distinct points in G without a limit point in G, and for each k let $S_k(z) = \sum_{j=1}^{m_k} \frac{A_{jk}}{(z - a_k)^j}$, $A_{jk} \in C$. Show that there is a meromorphic function f on G whose poles are exactly the points a_k and such that the singular part of f at a_k is $S_k(z)$.
   (b) Let $\mathcal{F}$ be a normal set in $C(G, \Omega)$. Show that for each $z \in G$ $\{f(z) : f \in \mathcal{F}\}$ has compact closure in Ω and $\mathcal{F}$ is equicontinuous at each point of G [3+4]

5. (a) Let D be the unit disk and $f : \partial D \longrightarrow R$ a continuous function. Show that there is a continuous function $u : \overline{D} \longrightarrow R$ such that: $u(z) = f(z)$ for $z \in \partial D$; u is harmonic in D; u is uniquely defined by the formula $u(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t) f(e^{it}) dt$ for $0 \leq r < 1, 0 \leq \theta \leq 2\pi$, where $P_r(.)$ is the *Poisson kernel*. [9]

(b) Let $\{f_n\}$ be a sequence in $H(G)$, each f_n a one-one function and $f_n \longrightarrow f$. Show that either f is one-one or f is a constant function [5]

6. (a) Define *elementary factor* $E_p(z)$ and show that $|1 - E_p(z)| \leq |z|^{p+1}$ for $p \geq 0$. Let $\{a_n\}$ be a sequence of non-zero complex numbers satisfying $\lim_{n\to\infty} |a_n| = \infty$ and $\{p_n\}$ is a sequence of integers such that $\sum_{n=1}^{\infty} \left(\frac{r}{|a_n|}\right)^{p_n+1} < \infty$ for all $r > 0$. Show that $\prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right)$ is an entire function having zeros precisely at the points a_n. [2+4+[illegible]]

(b) With brief explanation represent an open set G in the complex plane as countable union of compact sets. [4]

7. (a) Show that a sequence f_n in the space of continuous functions $(C(G, \Omega), \rho)$ converges to f if and only if it converges to f uniformly on all compact subsets of G. [6]

(b) Define *Landau's constant* L. If f is analytic on a region containing the closure of the unit disk D and $f(0) = 0,\ f'(0) = 1$, show that $f(D)$ contains a disk of radius L. [2+6]

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examinations, December 2024
Part II Semester III

MMATH18-303(ii): ADVANCED MEASURE THEORY
(Unique Paper Code 223502308)

Time: 3 hours Maximum Marks: 70

Instructions: • Attempt **five** questions in all. **Question No. 1 is compulsory**. All questions carry equal marks.• The symbols used have their usual meanings.

(1) (a) Give an example, with justification, to show that a signed measure may not satisfy monotonicity. [3]

(b) Let R be a measurable rectangle in $\mathbb{R}^n$, where we take Lebesgue measure on $(\mathbb{R}, \mathcal{M})$. For $y \in \mathbb{R}^n$ and $c > 0$, show that $R + y$ and cR are also measurable rectangles in $\mathbb{R}^n$ and that $m_n(R + y) = m_n(R)$ and $m_n(cR) = c^n m_n(R)$. [3]

(c) State Riesz representation theorem for bounded linear functionals on L^p spaces. [2]

(d) If $K = [0, 1/2] \cup [2/3, 1]$ and $U = (-1, 2)$, give an explicit example of a function $f \in C_c(\mathbb{R})$ such that $f = 1$ on K and $= 0$ on $\mathbb{R} \setminus U$. [4]

(e) When do you say a Borel measures on a locally compact Haüsdorff space X to be regular? [2]

(2) (a) Show that the Radon-Nikodym theorem for a finite measure μ implies the theorem for a σ-finite measure μ. [6]

(b) State Lebesgue decomposition theorem and prove only the uniqueness part. [2+6]

(3) (a) Let $(X, \mathcal{A}, \mu)$ be a finite measure space. Show that the class of measurable simple functions on X is dense in $L^p(\mu)$ for $1 \le p < \infty$. [7]

(b) Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be complete measure spaces and $E \subset X \times Y$ be a measurable set such that $(\mu \times \nu)(E) = 0$. Show that for a.e. $x \in X$, E_x is a measurable subset of Y and that $\nu(E_x) = 0$. Also give an example to show that $\{x \in X : E_x \text{ is not measurable}\}$ may be nonempty. [4+3]

(4) (a) If $f : \mathbb{R}^n \to \mathbb{R}^k$ is a continuous function, show that for every Borel set B in $\mathbb{R}^k$, $f^{-1}(B)$ is a Borel set in $\mathbb{R}^n$. Hence conclude that if E is a Borel set in $\mathbb{R}^n$ and $y \in \mathbb{R}^n$, then $E + y$ is also a Borel set in $\mathbb{R}^n$. [5+2]

(b) Let μ be a σ-finite measure on an algebra $\mathcal{A}$ of subsets of X and μ^* be the outer measure induced by μ. Then show that E is μ^*-measurable if and only if E is the proper difference $A \setminus B$ of a set $A \in \mathcal{A}_{\sigma\delta}$ and a set B with $\mu^*(B) = 0$. [7]

(5) (a) Show that the Hahn decomposition is unique except for null sets. [3]

(b) What do you mean by a complex measure on a measurable space $(X, \mathcal{A})$? Show that each complex measure ν may be expressed as $\nu = \mu_1 - \mu_2 + i\mu_3 - i\mu_4$, where μ_1, μ_2, μ_3 and μ_4 are finite measures. [1+5]

(c) Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be complete measure spaces and $E \subset X \times Y$ be a measurable set such that $(\mu \times \nu)(E) < \infty$. Prove Fubini's theorem for the function χ_E. [5]

(6) (a) Define the σ-algebra $\mathcal{B}a(X)$ of Baire sets in a locally compact Hausdorff space X and show that $\mathcal{B}a(X)$ is contained in the σ-algebra $\mathcal{A}$ generated by the compact G_δ sets of X. [1+6]

(b) Let F be a closed subset of a locally compact Hausdorff space X. Then show that F is also a locally compact Hausdorff space and the Borel sets of F are those Borel subsets of X which are contained in F. [7]

(7) (a) Let K be a compact subset of a locally compact Hausdorff space X and O an open subset of X such that $K \subset O$. Show that there is σ-compact open set U and a compact G_δ set H such that $K \subset U \subset H \subset O$. [5]

(b) Let X be a locally compact Hausdorff space, T be a positive linear functional on $C_c(X)$ and μ^* be its induced outer measure. Assuming that $\mu^*(K) < \infty$ for every compact set $K \subset X$, show that μ^* is additive on the collection of compact sets. [6]

(c) Prove that $(-1, 1) \subset \mathbb{R}$ is a σ-compact open set in $\mathbb{R}$ and it is the union of a countable collection of compact G_δ sets. [3]

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Your Roll Number: ....

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations, December 2024
Part- II Semester- III
MMATH18-303(iii): General Topology
(Unique Paper Code 223502309)

Time: **3 hours** Marks: **70**

Instructions: • All notations used are standard • **Question no. 1 is compulsory** • Attempt any **four** questions from the remaining six questions.

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(1) Do as directed.

(a) If $f : X \to Y$ is a proper surjection and X is Hausdorff then prove that Y is also Hausdorff. [3 Marks]

(b) Prove that the cone $C\mathbb{S}^n$ over the n-sphere $\mathbb{S}^n$ is homeomorphic to the closed disc $\mathbb{D}^{n+1}$. [3 Marks]

(c) Justify that $\mathbb{R}_l \times \mathbb{R}_l$ is not paracompact, where $\mathbb{R}_l$ is $\mathbb{R}$ with the lower limit topology. Is it completely regular? [2 Marks]

(d) Define '$\sim$' on $\mathbb{R}$ by $x \sim x'$ if $(x - x') \in \mathbb{Q}$. Describe the quotient topology τ on $\mathbb{R}/\sim$. Is $(\mathbb{R}/\sim, \tau)$ a normal space? Justify your claim. [3 Marks]

(e) Let X be a completely regular space and $F \subseteq X$ closed. Show that for each $x \in (X - F)$, there exists a continuous function $g_x : X \to [0,1]$ such that $g_x \equiv 0$ on F and $g_x \equiv 1$ on a nbd. of x. [3 Marks]

(2) (a) Let X be a locally compact Hausdorff space and K a compact subset of X. If U is an open subset of X with $K \subseteq U$, then prove that there exists an open subset V of X such that $K \subseteq V \subseteq \overline{V} \subseteq U$ and $\overline{V}$ is compact. [5 Marks]

(b) Prove that a subspace of a locally compact Hausdorff space is locally compact if and only if it is locally closed. [7 Marks]

(c) Find two spaces X and Y such that X is not homeomorphic to Y but X^* is homeomorphic to Y^*. [2 Marks]

(3) (a) Let $f : X \to Y$ be continuous. Then prove that f is proper if f is closed and $f^{-1}(y)$ is compact for every $y \in Y$. [6 Marks]

(b) Let $f : X \to Y$ be a continuous surjection. Prove that f is an identification map if and only if for any space Z, the continuity of a function $g : Y \to Z$ follows from that of $g \circ f : X \to Z$. [5 Marks]

(c) Justify that for $A \subseteq X$, the projection map $\pi : X \to X/A$ need not be an open map. [3 Marks]

P.T.O.

(4) (a) Let $\{X_\alpha\}_{\alpha\in\mathscr{A}}$ be a family of regular spaces then prove that the product space $\prod_{\alpha\in\mathscr{A}} X_\alpha$ is also regular. [4 Marks]

(b) Prove that for every completely regular space X, there exists a compact Hausdorff space $\beta(X)$ such that: [7 Marks]

(i) X is homeomorphic to a dense subspace X' of $\beta(X)$.

(ii) Every bounded real-valued continuous function on X' can be uniquely extended to a continuous function from $\beta(X)$ to $\mathbb{R}$.

(c) Prove that every locally compact Hausdorff space is completely regular. [3 Marks]

(5) (a) Let X be a normal space and A, B disjoint closed sets in X. Then prove that there exists a continuous function $f : X \to [0, 1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$. Justify that the condition of closedness of A and B is a necessary condition. [10 Marks]

(b) Let X be a normal space and A be a closed G_δ-subset of X. Then prove that there exists a continuous map $h : X \to \mathbb{R}$ such that $h^{-1}(\{0\}) = A$. [4 Marks]

(6) (a) State Nagata-Smirnov Metrization Theorem and prove Urysohn Metrization Theorem. [8 Marks]

(b) Let X be a T_3-space with the property that its every open covering has a countably locally finite open refinement then prove that X is a paracompact space. [6 Marks]

(7) (a) Prove that for every closed subset A of the real line $\mathbb{R}$ and continuous function $f : A \to [-1, 1]$, there exists a continuous function $g : \mathbb{R} \to [-1, 1]$ such that $g_{|A} = f$. [6 Marks]

(b) Prove that every open covering of $\mathbb{R}_l$ has a partition of unity subordinate to it. [8 Marks]

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Your Roll Number: ....

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examinations, DECEMBER 2024
Part II Semester III,UPC 223502310

MMATH18-304(i): COMPUTATIONAL FLUID DYNAMICS

Time: 3 hours

Maximum Marks: 70

Instructions: • Attempt **five** questions in all • Question number 1 is compulsory and attempt any **four** from the remaining questions 2 to 7 • Each question from 1 to 7 carries 14 marks • Scientific non-programming calculator for doing numerical calculations is allowed for this examination • Notations and acronyms have their usual meaning.

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(1) (a) Define explicit, implicit, one-step and multistep finite difference schemes. Elucidate these schemes by giving examples for the equations $u_t + au_x = f(t,x)$, and $u_t = bu_{xx}$,where a is a constant and b is a positive constant. [3+4 Marks]

(b) Elucidate the drawbacks in a checker-board pressure field. What is the essence of a staggered grid in finding the solutions of pressure and velocity in steady flows. [3+4 Marks]

(2) (a) Find the order of accuracy, truncation error, consistency and stability of the Crank-Nicolson scheme for the equation $u_t + au_x = f(t,x)$, where a is a constant. [7 Marks]

(b) Deduce the Crank-Nicolson scheme from a finite volume discretised equation in familiar standard form for one-dimensional, unsteady heat conduction equation given by: [7 Marks]

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + S$$

where ρ is the density, c is the specific heat, k is the thermal conductivity, T is the temperature of the material and S is the source term.

(3) (a) Elucidate Alternating Direction Implicit (ADI) scheme. What are the D'yakonov split and First Lees ADI forms for the equation $u_{tt} = u_{xx} + u_{xx}$? [7 Marks]

(b) Use neat diagrams of the u and v control volumes to discretize the momentum equations and compute the convective flux/unit mass F and the diffusive conductance D at cell faces of the u and v control volume faces. [7 Marks]

(4) (a) Discuss most important fundamental properties of the finite volume discretisation scheme for one-dimensional convection-diffusion problems. [7 Marks]

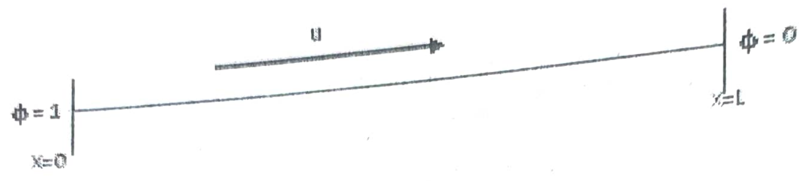
(b) Find the order of accuracy, truncation error, consistency and stability of the FTCS scheme for the pde $u_t = bu_{xx}$,where b is a positive constant. [7 Marks]

(5) (a) Solve the initial boundary value problem $u_{tt} = u_{xx}$; $u(0,x) = sin\pi x, u_t(0,x) = 0, 0 \le x \le 1$, $u(t,0) = 0, u(t,1) = 0, t \ge 0$ using the scheme $\delta_t^2 v_m^n = p^2 \delta_x^2 v_m^n$ with $h = \frac{1}{3}$ and $p = \frac{1}{2}$. Compute the solutions until two time levels. [7 Marks]

(b) Compute the solutions of the initial boundary value problem $u_t = u_{xx}$; $u(x,0) = cos(\frac{\pi x}{2}), -1 \le x \le 1, t = 0$; $u(-1,t) = u(1,t) = 0, t > 0$ by using the FTCS scheme at the third time level by choosing $h = \frac{1}{3}$, $\mu = \frac{1}{3}$. [7 Marks]

[7 Marks]

(6) (a) A property Φ is transported by means of convection and diffusion through one-dimensional domain as sketched below : Use the required



governing equations; the boundary conditions are $\Phi_0 = 1$ at $x = 0$ and $\Phi_L = 0$ at $x = L$. Divide the domain into 3 three control volumes and use the hybrid scheme for convection-diffusion, calculate the finite volume solution Φ at the given nodes for $u = 2.5m/s$. Observe that $F = F_e = F_w = 2.5$, $D = D_e = D_w = 0.5$, a Peclet number $Pe_w = Pe_e = \frac{\rho u \delta x}{\Gamma} = 5$.

[7 Marks]

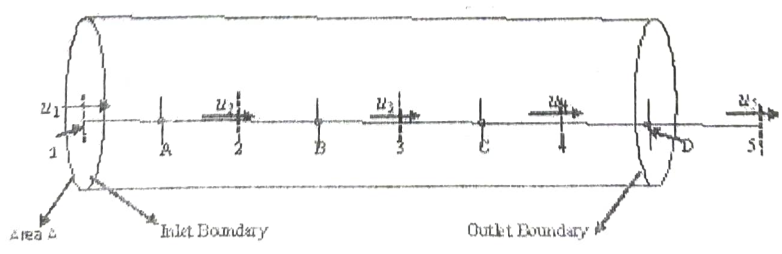
(b) Consider the problem of heat conduction that includes sources other than those arising from boundary conditions which is governed by the equation

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) + q = 0,$$

Draw a diagram of a large plate of thickness $L = 4\ cm$ with constant thermal conductivity k=0.5 W/m.K and uniform heat generation q=10000 kW/m^3. The two faces A and B are at temperatures of 100°C and 300°C respectively. Assuming that the dimensions in the y- and z-directions are so large that temperature gradients are significant in the x-direction only. Use the finite volume method to calculate the steady state temperature distribution in the whole domain which is divided into three control volumes.

[14 Marks]

(7) Consider the steady, one-dimensional flow of a constant-density fluid through a duct with constant cross-sectional area. Using staggered grid shown in the figure below, where the pressure p is evaluated at the main nodes $I = A, B, C$ and D, whilst velocity u is calculated at the backward staggered nodes i=1,2,3 and 4.



The problem data are as follows:

- Density $\rho = 1.0kg/m^3$ is constant.
- Duct area A is constant.
- Multiplier d defined by $u' = d(p'_I - p'_{I+1})$ is assumed to be constant; we take d=1.0.
- Boundary conditions: $u_1 = 10m/s$, $p_D = 0Pa$.
- Initial guessed velocity field: say $u_2^* = 8.0m/s$, $u_3^* = 11.0m/s$, $u_4^* = 7.0m/s$.

Use the SIMPLE algorithm and these problem data to calculate pressure corrections at nodes I=A to D and thereby obtain the corrected velocity fields at nodes i=2 to 4. Check whether the velocity is constant everywhere by continuity.

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Your Roll Number: ....

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations, December 2024
Part II Semester III
MMATH18-304(ii) UPC: 223502311
COMPUTATIONAL METHODS FOR ODEs

Time: 3 hours Maximum Marks: 70

Instructions: • Section A is compulsory • Answer **any four** questions from Section B • Each question carries equal marks • Non-programmable scientific calculators are allowed. • Notations have their usual meaning

Section A

(1) (a) Derive a first order scheme for the numerical solution of $y'' + 4y' = 0,\ 0 < x < 1,\quad y'(0) = 0,\ y(1) = 1$. [4]

(b) State true or false and justify: The classical second order Runge-Kutta method is unconditionally stable for the solution of first order intial value problem. [3]

(c) Apply the Heun method with step length h to the problem $y' = y,\ y(0) = 1$. Show that $|y_j - y(t_j)| \le \frac{h^2}{6} t_j e^{t_j}$ where $y(t_j)$ is the exact solution. [4]

(d) State Routh Hurwitz criterion. Illustrate with an example. [3]

Section B

(2) (a) Let L be the linear difference operator associated with a linear multistep method for a first order initial value system. Show that the method has order p if and only if $L[x^r; h] \not\equiv 0,\ r = 0,\ 1, \dots,\ p$ and $L[x^{p+1}, h] \not\equiv 0$, and the error constant C_{p+1} satisfies $h^{p+1}(p+1)!C_{p+1} = L[x^{p+1}; h]$. [7]

(b) Give a second order scheme for the solution of the boundary value problem [7]

$$y'' = \text{sin}ty' + 1, \quad a < t < b, \quad y(a) = g_1,\ y'(b) + ky(b) = g_2.$$

(3) (a) Apply a fourth order method [7]

$$y_{j+1} - 2y_j + y_{j-1} = \frac{h^2}{12}[\bar{f}_{j-1} + 10\tilde{f}_j + \bar{f}_{j+1}],$$

where

$$\bar{y}'_j = (y_{j+1} - y_{j-1})/(2h),\ \bar{y}'_{j+1} = (3y_{j+1} - 4y_j + y_{j-1})/(2h),\ \tilde{y}'_j = \bar{y}'_j - [h(\bar{f}_{j+1} - \bar{f}_{j-1})/20],$$
$$\bar{y}'_{j-1} = (-y_{j+1} + 4y_j - 3y_{j-1})/(2h),\ \tilde{f}_j = f(x_j, y_j, \tilde{y}'_j),\ \bar{f}_{j\pm1} = f(x_{j\pm1}, y_{j\pm1}, \bar{y}'_{j\pm1})$$

to the boundary value problem $y'' = f(x, y, y')$, where

$$y'' - ky' = 0,\ k >> 1,\ y(0) = 1, y(1) = 0.$$

with step length h. Determine the explicit expression for y_j.

(b) Apply Nystrom method of second order for the numerical solution of the initial value problem $y' = y + y^2,\quad y(0) = 1$. Estimate $y(0.2)$ with $h = 0.1$ and use second order Taylor series method to get the starting value. [7]

(4) (a) Consider the non linear second order differential equation $y'' = f(x, y),\ a < x < b$ subject [7]

(b) Using the Hilbert-Schmidt method, solve $\phi(x) = \cosh x + \lambda \int_0^1 K(x,t)\phi(t)dt$, where [8 Marks]

$$K(x,t) = \begin{cases} \frac{\cosh x \cosh(t-1)}{\sinh 1}, & 0 \leq x \leq t, \\ \frac{\cosh t \cosh(x-1)}{\sinh 1}, & t \leq x \leq 1. \end{cases}$$

(5) (a) Define Green's function for the n^{th} order linear BVP with variable coefficient. Reduce the following BVP into integral equation $y'' + \lambda y = e^x, y(0) = y'(0), y(1) = y'(1)$. [3+5 Marks]

(b) Determine the Euler's equation for the functional $J[y] = \int_a^b F(x, y, y')dx$. Also, find the extremals of the functional $J[y] = \int_a^b (y'^2 + z'^2 + y'z')dx$. [3+3 Marks]

(6) (a) Define variation of a functional with a suitable example. [2 Marks]

(b) The thrust (P) of a propeller depends upon the diameter (D), speed (V), mass density (ρ), revolutions per minute (N) and coefficient of viscosity (μ). Show that [5 Marks]

$$P = \rho D^2 V^2 f\left(\frac{\mu}{\rho D V}, \frac{DN}{V}\right).$$

(c) Find the solution in terms of the sinh function for the WKB approximation to the IVP: $\epsilon^2 y'' - q(x)y = 0,\ y(0) = 0,\ y'(0) = 1;\ q(x) > 0$. [7 Marks]

(7) (a) Define the Abel's singular integral equation. Prove that the characteristic numbers of symmetric kernels are real. [2+5 Marks]

(b) Show that regular perturbation fails on the boundary value problem: $\epsilon y'' + y' + y = 0,\ 0 < t < 1,\ 0 < \epsilon << 1$ with $y(0) = 0$ and $y(1) = 1$. If $t = O(\epsilon)$, show that $\epsilon y''(t)$ is large; if $t = O(1)$, show that $\epsilon y''(t) = O(1)$. Find the exact solution and hence, find the inner and outer approximations. [7 Marks]

Your Roll Number: ....

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations, December 2024
Part II Semester III

MMATH18-304(iv): METHODS OF APPLIED MATHEMATICS
(UPC 223502313)

Time: 3 hours | Maximum Marks: 70

Instructions: • Question no. 1 is compulsory. • Answer any four questions from Question 2-7. • All notations have usual meaning.

(1) (a) Form an integral equation corresponding to the IVP: [2 Marks]
$y'' - 5y' + 6y = 0; y(0) = 0, y'(0) = 1.$

(b) State the Fredholm alternatives. [3 Marks]

(c) Find the resolvent kernel for Volterra-type integral equation with the following kernel $K(x,t) = \frac{cosh x}{cosh t}$. [3 Marks]

(d) Using Laplace transform, solve $\phi(x) = x - \int_0^x \sinh(x-t)\phi(t)dt.$ [3 Marks]

(e) If $\alpha(x)$ and $\beta(x)$ are continuous in $[a,b]$, and if $\int_a^b(\alpha(x)h(x) + \beta(x)h'(x))dx = 0$ for every function $h(x) \in \mathcal{D}_1(a,b)$ such that $h(a) = h(b) = 0$ then show that $\beta(x)$ is differentiable, and $\beta'(x) = \alpha(x)$ for all x in $[a,b]$. [3 Marks]

(2) (a) Solve the following integral equation [5 Marks]

$$\int_0^x \frac{\phi(t)dt}{\sqrt{x-t}} = \sin x.$$

(b) Solve the following integro-differential equation: [5 Marks]

$$\phi''(x) + 2\phi'(x) - 2\int_0^x \sin(x-t)\phi'(t)dt = \cos x; \phi(0) = \phi'(0) = 0.$$

(c) Using Fredholm determinants, find resolvent kernel of [4 Marks]

$$k(x,t) = x - sinh t, \qquad -1 \leq x \leq 1, -1 \leq t \leq 1.$$

(3) (a) Determine the relation between m^{th} and r^{th} iterarted kernels of a Fredholm integral equation. Find the iterated kernels of the kernel $k(x,t) = e^x \cos t$; with $a = 0$, $b = \pi$. [4+3 Marks]

(b) Solve the following intergral equation with degenerate kernel [7 Marks]

$$\phi(x) - \lambda\int_0^{2\pi} |\pi - t| sin x\, \phi(t)dt = x.$$

(4) (a) Find the characteristic numbers and eigenfunctions of the homogeneous integral equation if its kernel is given by $K(x,t) = e^{-|x-t|}, 0 \leq x \leq 1, 0 \leq t \leq 1.$ [6 Marks]

(b) Using the Hilbert-Schmidt method, solve $\phi(x) = \cosh x + \lambda \int_0^1 K(x,t)\phi(t)dt$, where [8 Marks]

$$K(x,t) = \begin{cases} \frac{\cosh x \cosh(t-1)}{\sinh 1}, & 0 \leq x \leq t, \\ \frac{\cosh t \cosh(x-1)}{\sinh 1}, & t \leq x \leq 1. \end{cases}$$

(5) (a) Define Green's function for the n^{th} order linear BVP with variable coefficient. Reduce the following BVP into integral equation $y'' + \lambda y = e^x, y(0) = y'(0), y(1) = y'(1)$. [3+5 Marks]

(b) Determine the Euler's equation for the functional $J[y] = \int_a^b F(x, y, y')dx$. Also, find the extremals of the functional $J[y] = \int_a^b (y'^2 + z'^2 + y'z')dx$. [3+3 Marks]

(6) (a) Define variation of a functional with a suitable example. [2 Marks]

(b) The thrust (P) of a propeller depends upon the diameter (D), speed (V), mass density (ρ), revolutions per minute (N) and coefficient of viscosity (μ). Show that [5 Marks]

$$P = \rho D^2 V^2 f\left(\frac{\mu}{\rho D V}, \frac{DN}{V}\right).$$

(c) Find the solution in terms of the sinh function for the WKB approximation to the IVP: $\epsilon^2 y'' - q(x)y = 0,\ y(0) = 0,\ y'(0) = 1;\ q(x) > 0$. [7 Marks]

(7) (a) Define the Abel's singular integral equation. Prove that the characteristic numbers of symmetric kernels are real. [2+5 Marks]

(b) Show that regular perturbation fails on the boundary value problem: $\epsilon y'' + y' + y = 0,\ 0 < t < 1,\ 0 < \epsilon << 1$ with $y(0) = 0$ and $y(1) = 1$. If $t = O(\epsilon)$, show that $\epsilon y''(t)$ is large; if $t = O(1)$, show that $\epsilon y''(t) = O(1)$. Find the exact solution and hence, find the inner and outer approximations. [7 Marks]

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Your Roll No:.....................

M.A/M.Sc. Mathematics, Part-II, Sem-III (2024)
MMATH18-305(i), Coding Theory
Unique Paper Code: 223503301

Time : 1.5 hours

Maximum Marks : 35

Question No. 1 is compulsory. Attempt any **four** questions from Question Nos. **2** to **7**.
$\mathbb{F}_q$ denotes a finite field with q elements.

(1) (a) What is the dimension and distance of binary Hamming code $\text{Ham}(r, 2)$. [1]

(b) Define a maximum distance separable (MDS) code. [2]

(c) Write a generator matrix of binary Hamming code $\text{Ham}(3, 2)$. [2]

(d) Write a generator matrix of Reed-Muller code $\mathcal{R}(1, 3)$. [2]

(2) (a) Let C be a linear code over $\mathbb{F}_q$ with a parity check matrix H. Let $d(C)$ denote the distance of C. Show that $d(C) \geq d$ if and only if any $d-1$ columns of H is linearly independent. [5]

(b) Let C be a linear code over $\mathbb{F}_2$ with a parity-check matrix

$$H = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

Find the distance of code C. [2]

(3) Construct a syndrome-look-up table for a linear code C over $\mathbb{F}_2$ with a parity check matrix

$$H = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

Let $x = 100010$ be a received word. Using the above syndrome-look-up table decode x. [6+1]

(4) (a) Let C be an $[n, k, d]$-linear code over $\mathbb{F}_q$. Show that $C^{\perp}$ is an MDS code if C is an MDS code. [5]

(b) Let C be an $[n, k, d]$-linear code over $\mathbb{F}_q$. Show that $k + d \leq n + 1$. [2]

(5) (a) For an integer $q > 1$, and positive integers n, d such that $1 \leq d \leq n$. Prove that [5]

$$A_q(n, d) \geq \frac{q^n}{\sum_{i=0}^{d-1} \binom{n}{i} (q-1)^i}.$$

(b) Let $q \geq 2$ be a prime power. Show that $B_q(n, n) = A_q(n, n) = q$. [2]

(6) Let C_i be an $[n, k_i, d_i]$-linear code over $\mathbb{F}_q$ for $i = 1, 2$. Prove that $C = \{(u, u+v) \mid u \in C_1,\ v \in C_2\}$ is a $[2n, k_1 + k_2, \min\{2d_1, d_2\}]$-linear code over $\mathbb{F}_q$. [7]

(7) Let $\pi : \mathbb{F}_q^n \to \mathbb{F}_q[x]/(x^n - 1)$ be a defined as follows:

$$\pi(a_0, a_1, \ldots, a_{n-1}) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}.$$

Prove that a non empty subset C of $\mathbb{F}_q^n$ is a cyclic code if and only if $\pi(C)$ is an ideal of $\mathbb{F}_q[x]/(x^n - 1)$. [7]

Roll No. : ....

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations, December-2024
(Part II, Semester III)
MMATH18–305(ii): Stochastic Calculus for Finance
(UPC NO. 223503302)

Time: 2 hours | Maximum Marks: 35

Instructions: • Question 1 is compulsory. • Attempt **FIVE** questions in all.
• All symbols have their usual meaning unless otherwise specified.

(1) (a) Define a sigma algebra. [2 Marks]

(b) What is No Arbitrage Principle? [2 Marks]

(c) Show that a Brownian motion $\{W(t) : t \geq 0\}$ is a martingale. [3 Marks]

(2) Let $(\Omega, \mathcal{F}, P)$ be a probability space. Prove the following:

(a) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ where $A, B \in \mathcal{F}$. [2 Marks]

(b) If $\{A_n\}_{n=1}^{\infty}$ is an increasing sequence of sets in $\mathcal{F}$, then [5 Marks]

$$\lim_{n\to\infty} P(A_n) = P\Big(\bigcup_{n=1}^{\infty} A_n\Big).$$

(3) (a) Prove that the cumulative distribution function F_X of a random variable X defined on a probablity space $(\Omega, \mathcal{F}, P)$ is non-decreasing. [2 Marks]

(b) Let X be a random variable with density function given by [5 Marks]

$$f_X(x) = \begin{cases} 2e^{-2x}, & x > 0; \\ 0, & \text{otherwise.} \end{cases}$$

Find $E[X^2]$ and $\text{Var}(X)$.

(4) Write Ito-Doeblin formulae in the integral form for a Brownian motion. Compute the Ito integral $\int_0^t W(u)\, dW(u)$ using Ito-Doeblin formula. [7 Marks]

(5) (a) Define forward derivative. Write payoffs for long and short forward. [4 Marks]

(b) State Girsanov's theorem. [3 Marks]

(6) Derive the Black–Scholes formula for the price of a European call option for a non-dividend paying stock. [7 Marks]

(7) State and prove first fundamental theorem of asset pricing. [7 Marks]