Your Roll Number: ....

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations, Dec 2024
Part I Semester I, Unique Paper Code : 223501101
MMATH18-101:Field Theory

Time: 3 hours | Maximum Marks: 70

Instructions: • Question **1** is compulsory and attempt any **Four** questions from the rest. All questions carry equal marks.

(1) (a) Is every algebraically closed field perfect? Justify your answer. [2 Marks]

(b) For a prime number p, find the normal closure of the extension $\mathbb{Q}(p^{1/4})/\mathbb{Q}$. [3 Marks]

(c) Let E/k be a field extension of degree p a prime number. Show that E has no proper subfield other than k. [3 Marks]

(d) The cyclotomic polynomial Φ_n over $\mathbb{F}_p$, where $p \nmid n$ may be reducible. Justify. [3 Marks]

(e) Show that the group of the equation $x^3 + x^2 + 3x + 1 = 0$ over $\mathbb{Q}$ is the symmetric group S_3. [3 Marks]

(2) (a) Define the prime subfield of any field. Prove that either it is isomorphic to the field of rationals or $\mathbb{Z}/p\mathbb{Z}$, for prime number p. [8 Marks]

(b) Show that a normal extension of a normal extension need not be normal. [6 Marks]

(3) (a) Let E be the minimal splitting field of the family of all polynomials over k. Show that E is an algebraically closed field. [6 Marks]

(b) Let F/k be a normal extension of degree n. Show that the group of all $k-$automorphisms on F is of order n if and only if F is generated by finitely many separable elements over k. [8 Marks]

(4) (a) State and prove the Fundamental Theorem of Galois Theory. [7 Marks]

(b) Does there exist a bijection between the subfields and the subgroups of a group of automorphisms when the extension is infinite? Justify. [7 Marks]

(5) (a) Find the Galois group of Cyclotomic extension $\mathbb{Q}(\zeta)/\mathbb{Q}$ where ζ is a primitive nth root of unity. [7 Marks]

(b) Prove the existence of a Galois extension whose Galois group is isomorphic to the additive group $\mathbb{Z}/n\mathbb{Z}$. [7 Marks]

(6) (a) Is every finite extension of a finite field Galois? Justify your answer. [3 Marks]

(b) Describe all the subextensions of $\mathbb{F}_q$, where $q = p^n$ for prime p. [2 Marks]

(c) State and prove Steinitz theorem. [2+7 Marks]

(7) (a) Prove that an extension F/k of degree n is radical if and only if F/k is cyclic when the ground field contains a primitive n-th root of unity. [7 Marks]

(b) Prove that there exists an irreducible equation of degree n for every integer $n \geq 1$, over $\mathbb{Q}$ whose group is the symmetric group of degree n. [7 Marks]

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examinations, December-2024
Part I Semester I

MMATH18-102: COMPLEX ANALYSIS
(UPC NO. 223501102)

Time: 3 Hours | Maximum Marks: 70

Instructions: • Attempt **FIVE** questions in all. • Question 1 is compulsory.
• All symbols have their usual meaning unless otherwise specified.

Q1. (a) Give an example with the justification of a branch of the logarithm on the open connected set $G = \mathbb{C} \setminus \{z \in \mathbb{R} \mid z \geq 0\}$. [3 Marks]

(b) Give an example to show that if f is analytic in an unbounded set G and continuous on $\overline{G}$, the equality [3 Marks]

$$\max\{|f(z)| \mid z \in \overline{G}\} = \max\{|f(z)| \mid z \in \partial G\}$$

may not hold.

(c) Let γ be a closed rectifiable curve in G such that $\gamma \sim 0$. Show that $n(\gamma; w) = 0$ for all $w \in \mathbb{C} \setminus G$. Does converse hold? Justify. [4 Marks]

(d) Let $G = \{z \mid \text{Re}(z) < 0\}$ and $\mathbb{D} = \{z \mid |z| < 1\}$. Use the orientation principle to find an analytic function $f : G \to \mathbb{C}$ such that $f(G) = \mathbb{D}$. [4 Marks]

Q2. (a) Let $f : G \to \mathbb{C}$ be an analytic function and $\mathbb{D} \subset G$. Show that $f(z) = \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \frac{f(w)}{w-z} dw$ for all $z \in \mathbb{D}$. Evaluate $\int_{\partial \mathbb{D}} \frac{e^z - e^{-z}}{z^n} dz$, where n is a positive integer. [5+3 Marks]

(b) Suppose all z_j and w_j, $j = 1, 2, 3$ are distinct points in $\mathbb{C}_\infty$. Show that there is a unique Mobius transformation S such that $Sz_j = w_j$, $j = 1, 2, 3$. [6 Marks]

Q3. (a) State and prove Liouville's Theorem. Use it to show that a non-constant polynomial has atleast one zero in the complex plane $\mathbb{C}$. [4+4 Marks]

(b) Suppose f is analytic in a region $G \subseteq \mathbb{C}$, and there is a point $a \in G$ such that $f^{(n)}(a) = 0$ for each $n \geq 0$. Show that $f \equiv 0$. Can we drop the connectedness of G? Justify your assertion. [4+2 Marks]

Q4. (a) Suppose f is analytic in $B(a; R)$ and let $\alpha = f(a)$. If $f(z) - \alpha$ has a zero of order m at $z = a$, then show that there exists an $\epsilon > 0$ and $\delta > 0$ such that for $|\zeta - \alpha| < \delta$, the equation $f(z) = \zeta$ has exactly m simple roots in $B(a; \epsilon)$. Hence deduce the open mapping theorem. [8 Marks]

(b) Define a simply connected region. Show that in a simply connected region an analytic function has a primitive. [6 Marks]

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Q5. (a) State and prove the Rouche Theorem. Let f be analytic in a neighborhood of $\overline{\mathbb{D}}$. If $|f(z)| < 1$ for $|z| = 1$, show that there is a unique z with $|z| < 1$ and $f(z) = z$. [5+3 Marks]

(b) Suppose f has an isolated singularity at a. Show that $z = a$ is a removable singularity if $\lim_{z \to a}(z-a)f(z) = 0$. [6 Marks]

Q6. (a) Let $f(z) = \frac{z^2-4}{z^2+5z+4}$. Give the Laurent expansion of $f(z)$ in each of the following annuli: (i) $ann(0; 1, 4)$ (ii) $ann(0; 4, \infty)$. [4 Marks]

(b) Find the singularity of each of the following functions: [4 Marks]
(i) $z^{-2}\log(1+z)$ (ii) $z^n \sin(\frac{1}{z})$, $n \geq 1$.

(c) Let f be analytic on a region $G \subseteq \mathbb{C}$. Suppose there is a constant M such that $\limsup_{z \to a}|f(z)| \leq M$ for all $a \in \partial_\infty G$. Show that $|f(z)| \leq M$ for all z in G. [6 Marks]

Q7. (a) State the Residue Theorem. Use it to evaluate $\int_0^\infty \frac{\sin x}{x} dx$. [2+6 Marks]

(b) State and prove Schwarz's Lemma. [6 Marks]

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examinations, December 2024
Part I Semester I
MMATH18-103: MEASURE AND INTEGRATION
(Unique Paper Code 223501103)

Time: 3 hours Maximum Marks: 70

Instructions: • Attempt **five** questions in all. **Question No. 1 is compulsory**. All questions carry equal marks.• The symbols used have their usual meanings.

(1) (a) For $k > 0$ and $A \subset \mathbb{R}$, show that $m^*(kA) = km^*(A)$. [3]

(b) If f is a measurable function, show that $\{x : f(x) = \alpha\}$ is measurable for each extended real number α. [3]

(c) Give an example, with justification, of a function of bounded variation on $[0, 1]$ which is not continuous. [2]

(d) Let $[X, \mathcal{S}, \mu]$ be a measure space, $f \in L(X, \mu)$ and $E = \{x \in X : f(x) \neq 0\}$. Show that there is a sequence $\{E_n\}$ of measurable sets with $\mu(E_n) < \infty$ for each n such that $E = \cup_{n=1}^{\infty} E_n$. [3]

(e) If $\{f_n\}$ is a sequence of measurable functions such that $f_n \to f$ almost uniformly, then show that $f_n \to f$ a.e. [3]

(2) (a) Show that every interval is measurable. [8]

(b) Show that the characteristic function χ_E of a set $E \subset \mathbb{R}$ is measurable if and only if E is measurable. [3]

(c) Show that the collection $\mathcal{A} = \{A \in \mathbb{R} : \text{either } A \text{ is finite or } A^c \text{ is finite}\}$ is an algebra. [3]

(3) (a) Show that a countable subset of $\mathbb{R}$ has zero outer measure. Show also that the Cantor ternary set P is an uncountable set of zero measure. [2+3+2]

(b) Let f be a nonnegative measurable function on $\mathbb{R}$. Show that $\int f dx = 0$ if and only if $f = 0$ a.e. [7]

(4) (a) Let f be a bounded and measurable function on a finite interval $[a, b]$ and let $\varepsilon > 0$. Show that there exists a continuous function g vanishing outside a finite interval such that $\int_a^b |f - g| \, dx < \varepsilon$. [5]

(b) If

$$f_n(x) = \frac{n\sqrt{x}}{1 + n^2x^2}, \; x \in [0, 1]$$

and $n \in \mathbb{N}$, show that $\lim_{n\to\infty} \int_0^1 f_n dx = 0$. [5]

(c) Compute the four derivatives of f at every $x \in \mathbb{R}$ where [4]

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ -1, & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

(5) (a) Let $f : [a, b] \to \mathbb{R}$ be a monotone increasing function, where a and b are finite. Show that f' is measurable and that $\int_a^b f' dx \le f(b) - f(a)$. Also give an example to show that strict inequality may occur. [7+2]

(b) If f is absolutely continuous on $[a, b]$, where a and b are finite, show that $f \in BV[a, b]$. [5]

(6) (a) When do you say a measure μ on a ring $\mathcal{R}$ to be σ-finite? Show that the Lebesgue measure on the σ-algebra $\mathcal{M}$ of Lebesgue measurable sets is σ-finite, but the counting measure on the power set $\mathcal{P}(\mathbb{R})$ is not σ-finite. [1+2+2]

(b) Define convergence in measure and show that if $f_n \to f$ in measure and $g_n \to g$ in measure then $f_n + g_n \to f + g$ in measure. [1+5]

(c) Let $[X, \mathcal{S}, \mu]$ be a measure space and $p > 0$. Define $L^p(X, \mu)$ and show that if $f, g \in L^p(X, \mu)$ then $f + g \in L^p(X, \mu)$. [3]

(7) (a) Let $[X, \mathcal{S}, \mu]$ be a complete measure space, $\{f_n\}$ be a sequence in $L^\infty(\mu)$ such that $\|f_n - f_m\|_\infty \to 0$ as $n \to \infty$. Show that there is a function $f \in L^\infty(\mu)$ such that $f_n \to f$ a.e. and $\lim_{n\to\infty} \|f_n - f\|_\infty = 0$. [7]

(b) Let $\{f_n\}$ be a sequence of measurable functions. Show that if $f_n \to f$ uniformly a.e., then $f_n \to f$ almost uniformly, but the converse is not in general true. [1+6]

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Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics End Semester Examinations December 2024
Part I Semester I
MMATH18-104: Differential Equations
UPC: 223501104

Time: 3 Hours | Maximum Marks: 70

Instructions: • Attempt five questions in all. • Question 1 is compulsory. • All question carry equal marks. • All notations have usual meaning.

(1) (a) Define regular Sturm-Liouville equation. [2 Marks]

(b) Convert a three dimensional wave equation into space form and then write its elementary solution. [3 Marks]

(c) Obtain the neceassary condition for the solution of the Neumann problem for Laplace equation. [3 Marks]

(d) Consider the real valued function $\bar{f}(x,Y) = (y_2^2+2, x^2+y_1^2)$ defined on $R : |x| \leq 1, |Y| \leq 1$. Compute an upper bound of $\bar{f}$ and a Lipschitz constant on R,. [3 Marks]

(e) Define the Green's function for a harmonic function. [3 Marks]

(2) (a) State and prove the minimum principle for two-dimensional Laplace equation. [7 Marks]

(b) State and prove the uniqueness theorem for the solution of $y' = f(x,y), y(x_0) = y_0$. [7 Marks]

(3) (a) Define the Lyapunov function for a nonlinear autonomous system. [3 Marks]

(b) Suppose that $q(x) > 0$ and $q(x)$ is continuous in the interval $(0,\infty)$. Prove that every nontrivial solution of $y'' + q(x)y = 0$ has infinitely many zeros in $(0,\infty)$. [5 Marks]

(c) Solve by the method of eigenvalues : $y_1' = y_1 + 2y_2 + 2t$, $y_2' = 4y_1 + y_2 - \sin t$. [6 Marks]

(4) (a) Let $p(x), q(x)$ and $f(x)$ be continuous on $[a,b]$. Then prove that either the BVP $L[y] = f, U_1[y] = \alpha, U_2[y] = \beta$ has a unique solution for any given constants α and β, or else the associated homogeneous BVP $L[y] = 0$, $U_1[y] = 0, U_2[y] = 0$ has a nontrivial solution. [7 Marks]

(b) Determine the solution of Laplace equation for a sphere using the Green's function. [7 Marks]

(5) (a) Using the Green function, solve $y''+4y = e^x$, $y(0) = 0, y'(1) = 0$. [7 Marks]

(b) Convert a nonlinear autonomous system of two equations into equivalent perturbed system. If V is a Lyapunov function such that $-\nabla V(X).F(X)$ is positive definite in Ω, then show that the origin is asymptotically stable. [2+5 Marks]

(6) (a) State and prove the Helmholtz's second theorem for three-dimensional wave equation. [7 Marks]

(b) Derive the Lagrange identity for $L[y] = 0$ and hence, deduce for self-adjoint operator. [3 Marks]

(c) Find the eigenvalues and eigenfunctions of $y'' + \lambda y = 0, y(0) = y(2\pi), y'(0) = y'(2\pi)$. [4 Marks]

(7) (a) Write two elementary solutions of Diffusion equation. Determine the solution of space form of wave equation using Green's function, which is zero on the boudary of the domain. [2+5 Marks]

(b) Using the method of Fourier transform, determine the solution of two-dimensional Laplace equation under Neumann conditions for the upper half plane. [7 Marks]

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