Your Roll Number: ....

Department of Mathematics, University of Delhi
M.Sc. Mathematics Examinations, May 2024
Part II Semester IV

MMath18- 401(i): Advanced Group Theory (UPC 223502401)

Time: 3 hours | Maximum Marks: 70

Instructions: • Write your roll number on the space provided at the top of this page immediately on receipt of this question paper • **Q 1** is compulsoty • Answer **any four** questions from Q2 to Q7. • Each question carries 14 marks.

(1) (a) Show that the additive group of rational numbers has no composition series. [$3\frac{1}{2}$ Marks]

(b) Let P be a normal sylow subgroup of a finite group G and H/P char G/P. Prove that H char G. [$3\frac{1}{2}$ Marks]

(c) Prove that every p-group satisfies the normalizer condition. [$3\frac{1}{2}$ Marks]

(d) Let G be a solvable group of order mn, where $(m,n)=1$. Let G contains a normal subgroup K of order m. Show that G is semidirect product of K by Q, where $Q \leq G$ and $|Q| = n$. [$3\frac{1}{2}$ Marks]

(2) (a) Let H, K and L be subgroups of a group G. If two of the commutator subgroups $[H,K,L], [K,L,H]$ and $[L,H,K]$ are contained in a normal subgroup N of G then show that the third commutator is also contained in N. [7 Marks]

(b) Let G be an abelian group of prime exponent p. Prove that G is a vector space over $\mathbb{Z}_p$ and every homomorphism $\phi : G \to G$ is a linear transformation. Moreover, show that a finite abelian p-group is elementary iff it has exponent p. [7 Marks]

(3) (a) Prove that a finite characteristically simple group is either simple or a direct product of isomorphic simple groups. [9 Marks]

(b) Give two representations of a group of order 21. Justify. [5 Marks]

(4) (a) Let $A \triangleleft A^*$ and $B \triangleleft B^*$ be four subgroups of a group G. Show that $B(B^* \cap A) \triangleleft B(B^* \cap A^*)$, and $A(A^* \cap B) \triangleleft A(B^* \cap A^*)$ and there is an isomorphism $B(B^* \cap A^*)/B(B^* \cap A) \cong A(B^* \cap A^*)/A(A^* \cap B)$. [7 Marks]

(b) Let G be a nilpotent group and H be a nontrivial normal subgroup of G. Show that $H \cap Z(G) \neq 1$. [7 Marks]

(5) (a) Let G be a group having ascending chain condition (ACC) and ϕ be a injective normal endomorphism of G. Show that ϕ is an automorphism. [5 Marks]

(b) Are the additive groups $\mathbb{Z}$ and $\mathbb{Q}$ indecomposable? Justify. [4 Marks]

(c) For given groups Q and K and a homomorphism $\theta : Q \to \mathrm{Aut}(K)$, prove that the set $K \times Q$ under the operation $(a, x)(b, y) = (a\theta_x(b), xy)$ is a semidirect product of K by Q, where $\theta_x = \theta(x)$ for all $x \in Q$. [5 Marks]

(6) (a) Let G be a group having both chain conditions, and let ϕ and ψ be a normal endomorphisms of G. If $\phi + \psi$ is an endomorphism of G then show that it is nilpotent. [7 Marks]

(b) Let G be a nilpotent group of class n. Show that $G/Z(G)$ is nilpotent of class $n - 1$. [7 Marks]

(7) (a) Does there exist a finite free group ? Justify. [3 Marks]

(b) Let T be a group of order 12 which is generated by two elements a and b such that $a^6 = 1$ and $b^2 = a^3 = (ab)^2$. Show that T is a semidirect product of $\mathbb{Z}_3$ by $\mathbb{Z}_4$. [7 Marks]

(c) Let N be normal subgroup of a finite group G, and let P be sylow subgroup of N. Show that $G = N_G(P)N$. [4 Marks]

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Department of Mathematics
University of Delhi, Delhi

M.A./M.Sc. Mathematics, Part-II, Semester-IV
Examination, May-June 2024

Paper: **MMATH18 401(ii) Algebraic Number Theory**
Unique Paper Code: 223502402

Time: 3 Hour *Maximum Marks: 70*

Note: • Attempt five questions in all. • Question 1 is compulsory. • All questions carry equal marks.

1. (a) Prove an algebraic integer is a rational number if and only if it is an integer. (2)
   (b) Let $d \equiv 1 \pmod 4$ be a square-free integer different from 1. Show that $\mathbb{Z}[\sqrt{d}]$ is never a unique factorization domain. (3)
   (c) Let $\mathcal{P}$ be a prime ideal of $\mathcal{O}_K$ containing a prime number p. Prove that $\mathcal{P}$ contains an integer a if and only if p divides a. (3)
   (d) Prove that the volume of the fundamental domain of a lattice is independent of the choice of generators. (3)
   (e) Factorize the ideal $\langle 5 \rangle$ in $\mathbb{Z}[\zeta]$, where $\zeta = e^{2\pi i/5}$. (3)

2. (a) Prove that the discriminant of any basis of a number field $K = \mathbb{Q}(\theta)$ is a non-zero rational number. (8)
   (b) Find the discriminant of any basis of $K = \mathbb{Q}(\sqrt{3}, \sqrt{5})$. (6)

3. (a) For $K = \mathbb{Q}(\sqrt[4]{p})$ and $L = \mathbb{Q}(\sqrt{p})$ calculate $N_{K/L}(\sqrt{p})$ and $N_{K/\mathbb{Q}}(\sqrt{p})$, where p is a prime number. (5)
   (b) Find the integral basis and discriminant of the number field $K = \mathbb{Q}(\theta)$, where $\theta = \cos\frac{2\pi}{p} + i\sin\frac{2\pi}{p}$ and p is an odd prime number. (9)

4. (a) Find the group of units of $\mathcal{O}_K$, the ring of algebraic integers of the number field $K = \mathbb{Q}(\sqrt{d})$, where $d < 0$ is a square-free integer. (8)
   (b) Prove that factorization into irreducibles is possible in the ring of algebraic integers $\mathcal{O}_K$ of a number field K. (4)
   (c) State the Ramanujan-Nagell Theorem. (2)

5. (a) Prove that the ring of algebraic integers of the number field $\mathbb{Q}(\sqrt{d})$ is never a Euclidean domain for all square-free integers $d < -11$. (8)
   (b) Let $K = \mathbb{Q}(\sqrt{-5})$. Find a $\mathbb{Z}$-basis $\{\alpha_1, \alpha_2\}$ of the ideal $I = \langle 2, 1+\sqrt{-5} \rangle$ of $\mathcal{O}_K$ and verify the formula $N(I) = \sqrt{\left|\frac{\Delta(\alpha_1,\alpha_2)}{d_K}\right|}$. (6)

6. (a) Prove that every fractional ideal of a number field K is generated by at most two elements. (6)

(b) State and prove the Two Squares Theorem. (6)

(c) Let A, B and I be ideals of $\mathcal{O}_K$, the ring of algebraic integers of a number field K and such that A, B are coprime and $A|I, B|I$. Show that $AB|I$. (2)

7. (a) Let K be a number field and $\mathcal{O}_K$ its ring of algebraic integers. Define an embedding of K into $L^{s,t}$ and prove that, under this embedding, the non-zero ideals of $\mathcal{O}_K$ are mapped to lattices in $L^{s,t}$. (8)

(b) Let K be a number field of degree $n = s + 2t$, with $\mathcal{O}_K$ its ring of algebraic integers. Suppose that for every prime $p \in \mathbb{Z}$ with $p \leq (\frac{4}{\pi})^t \frac{n!}{n^n} \sqrt{|\Delta_K|}$ every prime ideal of $\mathcal{O}_K$ dividing $\langle p \rangle$ is principal. Prove that K has class number 1. Use it to show that the class number of $\mathbb{Q}(\zeta)$ is 1, where ζ is a primitive 5th root of unity. (6)

Roll Number:.........

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics, Semester exams, May - June, 2024
Part - II; Semester - IV
MMATH18-401(iii) : **Simplicial Homology Theory**; Unique paper code: 223502403

Time: 3 Hours | Max Marks: 70

Instructions: • All symbols have their usual meaning. • Question No.1 in **Part-A** is compulsory. • Answer any four questions from remaining six questions in **Part-B**.

Part-A, Compulsory:

1. Choose appropriate options (Justifications are not required):

(a) Which of the following statements are true:

A. Every linearly independent set is geometrically independent.

B. Every geometrically independent set is linearly independent.

C. If a set of three points is geometrically independent then they are co-planer.

D. If a set of three points is geometrically independent then they are co-linear.

(b) Let K and L be any two simplicial complexes in $\mathbb{R}^m$, m sufficiently large. Then which of the following statements are true:

A. If $H_p(K) \cong H_p(L)$ for all $p \geq 0$, then $|K|$ and $|L|$ are homeomorphic.

B. $|K|$ must be a compact subset of $\mathbb{R}^m$.

C. $|K|$ must be a connected subset of $\mathbb{R}^m$.

D. If K is an n-simplex, $n < m$, then $|K|$ is homeomorphic to $\mathbb{D}^n$.

(c) Let $f, g : \mathbb{S}^1 \to \mathbb{S}^1$ be two continuous maps. Then which of the following statements are true:

A. If f and g are homotopic, then $\deg(f) = \deg(g)$.

B. If f is a homeomorphism, then $\deg(f) = \pm 1$.

C. The induced homomorphisms $f_p^*, g_p^* : H_p(\mathbb{S}^1) \to H_p(\mathbb{S}^1)$ are always equal for all $p \geq 0$.

D. The degree of f can not be zero.

(d) Which of the following statements are true:

A. 0^{th} homology group of a compact convex subset in $\mathbb{R}^n$ is isomorphic to $\mathbb{Z}$.

B. The interior $\text{int}(\sigma)$ of a simplex σ is open in the hyperplane spanned by its vertices.

C. If $x \in \text{int}(\sigma)$, then its barycentric coordinates not necessarily be positive.

D. If $\dim(K)$ of a simplicial complex K is n, then $\dim(K^{(1)})$ is $n+1$.

(e) Which of the following statements are incorrect:

A. Tetrahedron is a simple regular polyhedron.

B. Let $f : |K| \to |K|$ be a continuous map, for some simplicial complex K and Lefschetz number $\lambda(f) \neq 0$, then f has fixed point.

C. The Euler's characteristic of $\mathbb{S}^2$ is zero.

D. Cylinder is a 2-pseudomanifold.

(f) Let K be a simplicial complex. Then which of the following statements are true:

A. The homologous relation on $Z_p(K)$ is an equivalence relation.

B. K can have infinitely many combinatorial components.

C. $|K|$ need not be a polyhedron.

D. K contains finitely many vertices.

(g) Let $K = \{\langle 0\rangle, \langle \frac{1}{3}\rangle, \langle 1\rangle, \langle 0, \frac{1}{3}\rangle, \langle \frac{1}{3}, 1\rangle\}$ and $L = \{\langle 0\rangle, \langle \frac{2}{3}\rangle, \langle 1\rangle, \langle 0, \frac{2}{3}\rangle, \langle \frac{2}{3}, 1\rangle\}$ be two simplicial complexes. Let $f : |K| \to |L|$ be a continuous map. Then which of the following statements are true:

A. There exists no simplicial map $\phi : K \to L$, which is a simplicial approximation of f.

B. There exists a simplicial map $\phi : K^{(1)} \to L$, which is a simplicial approximation of f.

C. There exists an $n > 0$ and a simplicial map $\phi : K^{(n)} \to L$, which is a simplicial approximation of f.

D. There does not exist any n such that $\phi : K^{(n)} \to L$ is a simplicial approximation of f.

(2 × 7)

Part-B, Attempt any FOUR questions from the following:

2. (a) Let K be a simplicial complex and $v \in K$ be a vertex. Prove that $\mathrm{st}(v)$ is open subset in $|K|$. (8)

   (b) Prove that if $A = \{a_0, a_1, \ldots a_k\}$ is a geometrically independent subset of $\mathbb{R}^n$, then there exists a unique hyperplane $\mathbb{H}^k$ passing through A. (6)

3. (a) Let $K = \{\langle a_0\rangle, \langle a_1\rangle, \langle a_2\rangle, \langle a_3\rangle, \langle a_0, a_1\rangle, \langle a_0, a_2\rangle, \langle a_1, a_2\rangle, \langle a_1, a_3\rangle, \langle a_2, a_3\rangle\}$ be a simplicial complex with orientation $a_0 < a_1 < a_2 < a_3$. Compute $H_1(K)$. (8)

   (b) Compute the homology group $H_2(\mathbb{S}^2)$, where $\mathbb{S}^2$ is the usual 2-sphere. (6)

4. (a) Let K and L be two simplicial complexes and $\phi : K \to L$ be a simplicial map. Prove that there exists a sequence of chain maps $\{\phi_p : C_p(K) \to C_p(L)\}_{p \geq 0}$ satisfying $\partial\phi_p = \phi_{p-1}\partial$, for all $p \geq 1$. (7)

   (b) Prove that the path components of the geometrical carrier of a simplicial complex K coincide with the geometrical carrier of combinatorial components of K. (7)

5. (a) Let K be a simplicial complex with r combinatorial components. Show that $H_0(K) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \ldots \oplus \mathbb{Z}$ (r-copies). (10)

   (b) Let X be a polyhedron and let K and L be two triangulations of X. Then $H_p(K) \cong H_p(L)$, for all $p \geq 0$. (4)

6. (a) Prove that $\mathbb{S}^{n-1}$ is not a retract of $\mathbb{D}^n$, $n \geq 1$. (6)

   (b) State and prove Brouwer's fixed point theorem. (8)

7. (a) Define simple regular polyhedron and find all five of them. (8)

   (b) Prove that $\mathbb{S}^n$, $n \geq 1$ admits a tangent vector field if and only if n is odd. (6)

Your Roll Number: ....

Department of Mathematics, University of Delhi
M.Sc. Mathematics Examinations, MAY - 2024
Part II Semester IV
MMATH18- 402(i): Abstract Harmonic Analysis,
UPC: 223502405

Time: 3 hours Maximum Marks: 70

Instructions: • Answer **five** questions • **Question 1** is compulsory. Attempt any **four** questions from **Q2** to **Q7** • Each question carries 14 marks.

(1) (a) Give example of a positive definite function. (3)
(b) Prove that the Banach algebra $M(G)$ is unital. (3)
(c) Let G be a locally compact abelian group and $\mu \in M(G)$ with the Fourier transform $\hat{\mu}$. Prove that for $\mu, \nu \in M(G)$, $(\mu * \nu)^{\wedge} = \hat{\mu}\hat{\nu}$. (3)
(d) Let G be a locally compact group with left Haar measure μ and $\pi_R : G \to B(L^2(G))$ be defined as $\pi_R(x)(f) = R_x f$. Is π_R a unitary representation? Justify your answer. (3)
(e) Prove that the character of an irreducible unitary representation of a compact group G is a central function. (2)

(2) (a) For a locally compact group G, prove that G is abelian if and only if $L^1(G)$ is abelian. (7)
(b) Prove that every unitary representation of a locally compact group is a direct sum of cyclic representations. (7)

(3) (a) State and prove Schur's Lemma. (2+5)
(b) Prove that the Banach algebra $M(G)$ is a Banach*-algebra. (7)

(4) (a) Let G be a locally compact group. Prove that $L^1(G)$ has a bounded approximate identity. (7)
(b) For $n \in \mathbb{N} \cup \{0\}$, consider the representation $\pi_n : SU(2) \to B(H_n)$ defined as $\pi_n(A)(f(z)) = f(zA)$, $A \in SU(2)$, H_n being the space of homogenous polynomials of degree n of two variables. Prove that π_n is unitary. (7)

(5) (a) Prove that the cardinality of the dual group $\hat{S}_3$ is same as the number of conjugacy classes in the symmetric group S_3. Further prove that $\sum_{[\pi]\in\hat{S}_3} d_\pi^2 = 6$. (4+3)
(b) Determine the dual group of $\mathbb{R}$. (7)

(6) (a) For $f, g \in L^1(G)$, define $f * g$ (with justification) and prove that $f * g \in L^1(G)$. Also, prove that $(f * g)^* = g^* * f^*$. (7)
(b) Let $\phi : G \to \mathbb{C}$ be a bounded continuous function on a locally compact group G. If ϕ is of positive type, prove that it is positive definite. (7)

(7) (a) Let ρ be a non-degenerate $*$-representation of $L^1(G)$ on H. Prove that it is an integrated representation of some unitary representation of G on H. (10)
(b) If $[\pi] = [\gamma] \in \hat{G}$, G being a compact group, prove that $\mathcal{E}_\pi = \mathcal{E}_\gamma$ where $\mathcal{E}_\pi = span\{\pi_{ij}; 1 \leq i, j \leq d_\pi\}$ (4)

M.A./M.Sc. Mathematics Examination (May-June 2024)
Part II, Semester IV
MMATH18-402(ii): Frames and Wavelets
Unique Paper Code-223502406

Time: 3 hr. Maximum Marks: 70

Instructions: • Attempt **FIVE** questions in all. • Q. No. 1 is **COMPULSORY**. • All questions carry equal marks. • Symbols have their usual meaning.

(Question No. 1 is Compulsory)

(1) (a) Show that if ϕ and ψ denote the Haar scaling function and Haar wavelet, respectively, then $\phi(2^j x) = \frac{\psi(2^{j-1}x)+\phi(2^{j-1}x)}{2}$, $j \in \mathbb{Z}$, $x \in \mathbb{R}$. [3 Marks]

(b) Give an example, with justification, of a Schauder basis for an infinite dimensional Hilbert space $\mathcal{H}$ which is not a Riesz basis for $\mathcal{H}$. [3 Marks]

(c) Can a frame for $\mathbb{C}^3$ contain infinitely many vectors? Justify your answer. [4 Marks]

(d) Show that if $\{f_k\}_{k=1}^m$ is a tight frame for $\mathbb{C}^n$ with frame operator S and $\|f_k\| = 1$ for all $k = 1, 2, \ldots, m$, then $\|S\| \geq 1$. [4 Marks]

(Answer any FOUR questions from Q. No. 2 to Q. No. 7)

(2) (a) Give an example, with justification, of a sequence in $\ell^2(\mathbb{N})$ which is not minimal. [2 Marks]

(b) Show that a complete sequence of vectors $\{f_k\}_{k=1}^\infty$ in an infinite dimensional Hilbert space $\mathcal{H}$ is a frame for $\mathcal{H}$ with frame bounds A, B if and only if the pre-frame operator T associated with $\{f_k\}_{k=1}^\infty$ is well-defined on $\ell^2(\mathbb{N})$ and [12 Marks]

$$A\|\{c_k\}_{k=1}^\infty\|^2 \leq \|T(\{c_k\}_{k=1}^\infty)\|^2 \leq B\|\{c_k\}_{k=1}^\infty\|^2 \text{ for all } \{c_k\}_{k=1}^\infty \in \mathcal{N}_T^\perp,$$

where $\mathcal{N}_T$ denotes the null space of T.

(3) (a) Show that if $\{f_k\}_{k=1}^m$ is a Parseval frame for $\mathbb{C}^n$, then $\operatorname{tr}\Theta = \sum_{k=1}^m \langle \Theta f_k, f_k \rangle$ for any linear operator Θ on $\mathbb{C}^n$. [4 Marks]

(b) How frames are used in stable reconstruction of signals? Explain. [5 Marks]

(c) How do you obtain the optimal frame bounds of a frame $\{f_k\}_{k=1}^m$ for $\mathbb{C}^n$ in terms of eigen-values of the frame operator of $\{f_k\}_{k=1}^m$? Explain. [5 Marks]

(4) (a) Show that an exact frame for an infinite dimensional Hilbert space $\mathcal{H}$ is ω-independent. [3 Marks]

(b) State and prove the scaling relation of a multiresolution analysis. [5 Marks]

(c) Let $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ be Bessel sequences in an infinite dimensional Hilbert space $\mathcal{H}$ such that $f = \sum_{k=1}^{\infty}\langle f, f_k\rangle g_k$ for all $f \in \mathcal{H}$. Show that $\left(\{f_k\}_{k=1}^{\infty}, \{g_k\}_{k=1}^{\infty}\right)$ is a dual frame pair. [6 Marks]

(5) (a) Give an example, with justification, of a Parseval frame for the jth approximate space $V_j \subset L^2(\mathbb{R})$, $j \in \mathbb{Z}$. [4 Marks]

(b) Show that a sequence $\{f_k\}_{k=1}^{\infty}$ in an infinite dimensional Hilbert space $\mathcal{H}$ is a Riesz basis for $\mathcal{H}$ if and only if $\{f_k\}_{k=1}^{\infty}$ is complete in $\mathcal{H}$ and its Gram matrix defines a bounded, linear, and invertible operator on $\ell^2(\mathbb{N})$. [10 Marks]

(6) Prove that a complete sequence of non-zero vectors $\{f_k\}_{k=1}^{\infty}$ in a Banach space X is a Schauder basis for X if and only if there exists a constant C such that for all $m, n \in \mathbb{N}$ with $n \geq m$, $\left\|\sum_{k=1}^{m} c_k f_k\right\| \leq C\left\|\sum_{k=1}^{n} c_k f_k\right\|$ for all scalar-valued sequences $\{c_k\}_{k=1}^{\infty}$. [14 Marks]

(7) (a) Let $\{f_k\}_{k=1}^{\infty}$ be a frame for an infinite dimensional Hilbert space $\mathcal{H}$ with frame bounds A, B and $\{g_k\}_{k=1}^{\infty} \subset \mathcal{H}$ is such that $\{f_k - g_k\}_{k=1}^{\infty}$ is a Bessel sequence with Bessel bound K. Show that if $K < A$, then $\{g_k\}_{k=1}^{\infty}$ is a frame for $\mathcal{H}$. [4 Marks]

(b) Let $\{f_k\}_{k=1}^{\infty}$ and $\{\varphi_k\}_{k=1}^{\infty}$ be frames for an infinite dimensional Hilbert space $\mathcal{H}$. Show that the bi-infinite matrix $U = \left[\langle \varphi_\ell, S^{-1}f_k\rangle\right]_{\ell,k=1}^{\infty}$ defines a bounded operator on $\ell^2(\mathbb{N})$, where S is the frame operator of $\{f_k\}_{k=1}^{\infty}$. [5 Marks]

(c) How do you decompose $L^2(\mathbb{R})$ in terms of spaces associated with a Haar scaling function and Haar wavelet? Explain. [5 Marks]

Your Roll Number: ....

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations, May-June 2024
Part II Semester IV

MMATH18-402 (iii): Operators on Hardy Hilbert Space

Unique Paper Code : 223502407

Time: Three hours

Maximum Marks: 70

Instructions: • Write your roll number on the space provided at the top of this page immediately on receipt of this question paper • Question 1 is compulsory • Attempt 4 questions from the remaining six questions • All symbols have their usual meaning.

Section A

This section is compulsory.

(1) (a) Give a non-trivial example of a function which is in $H^2(\mathbb{D})$ but not in $H^\infty(\mathbb{D})$. (2)

(b) Let $T : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ be given by $(Tf)(z) = (z+1)f(z), z \in \mathbb{D}$. Check whether or not T is an isometry. (3)

(c) Find all the eigenvalues of the operator M_z acting on $H^2(\mathbb{D})$. (3)

(d) Show that if a function and its conjugate both lie in $\tilde{H}^2(S^1)$ then the function is a constant. (3)

(e) Find the matrix of the operator W_j where W is the bilateral shift, w.r.t the standard orthonormal basis of $L^2(S^1)$. (3)

Section B

Attempt four questions in all from this section.

(2) For $z_0 \in \mathbb{D}$ let the function k_{z_0} be given by $k_{z_0}(z) = \frac{1}{1-\bar{z}_0 z}, z \in \mathbb{D}$.

(a) Show that $k_{z_0} \in H^2$. (2)

(b) Find $\|k_{z_0}\|$ and show that $f(z_0) = \langle f, k_{z_0}\rangle$, for all $f \in H^2$. Hence prove that the point evaluations $f \mapsto f(z_0)$ are bounded linear functionals on H^2. (5)

(c) Prove that if $\{f_n\} \to f$ in H^2 then $\{f_n\} \to f$ uniformly on compact subsets of $\mathbb{D}$. (4)

(d) Show that the functions k_{z_0} are eigenvectors for U^*, where U is the unilateral shift on H^2. (3)

(3) (a) Let $f(z) = \frac{1}{2}(i + z^3), g(z) = iz^2, z \in \mathbb{D}$. Verify that $f, g \in H^2(\mathbb{D})$ and find $\tilde{f}, \tilde{g}$ the boundary functions of f and g. Check if f, g are inner functions. (5)

(b) Show that any non-zero function in $H^2(\mathbb{D})$ may be written as a product of an inner and an outer function. Is this decomposition unique? Give details. (6+3)

(4) (a) Show that the bilateral shift W on $l^2(\mathbb{Z})$ is a unitary operator. Further, prove that W is unitarily equivalent to a multiplication operator on $L^2(S^1)$. (1+5)

(b) Find the commutant of W (regarded as an operator on $L^2(S^1)$. (8)

(5) (a) Suppose the matrix of a bounded linear operator A defined on $\tilde{H}^2$ w.r.t. the standard orthonormal basis of $\tilde{H}^2$ is of the form

$$\begin{pmatrix} -1 & -i & 0 & 0 & \\ 1 & -1 & -i & 0 & \ddots \\ 0 & 1 & -1 & -i & \ddots \\ 0 & 0 & 1 & -1 & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

Is A a Toeplitz operator? Is it analytic? In case A is a Toeplitz operator, find its symbol. Is this symbol unique? Justify. (6)

(b) Show that if $T_\psi T_\phi$ is a Toeplitz operator, then either T_ψ is co-analytic or T_ϕ is analytic. Hence deduce that the product of two Toeplitz operators is 0 if and only if one of the factors is 0. (5+3)

(6) (a) Show that $JPJ = M_{e_1}(I - J)M_{e_{-1}}$ where J is the flip operator and P the orthogonal projection of $L^2(S^1)$ onto $\tilde{H}^2(S^1)$. (3)

(b) Give a necessary and sufficient condition for $H_\phi = H_\psi$ for $\phi, \psi \in L^\infty(S^1)$ and prove your assertion. (5)

(c) State and prove a necessary and sufficient condition for a Toeplitz operator to be normal. (6)

(7) (a) Show that an operator A to has a Hankel matrix with respect to the standard basis of $\tilde{H}^2(S^1)$ if and only if $U^*A = AU$ where U is the unilateral shift. Use this to show that $k_{\bar{w}} \otimes k_w$ is a Hankel operator of rank one, where $w \in \mathbb{D}$. Find a symbol for this Hankel operator. (4+6)

(b) For $\phi, \psi \in L^\infty$, show that $H_{e_1\breve{\phi}} H_{e_1\psi} = T_{\phi\psi} - T_\phi T_\psi$. (4)

Roll Number: ....

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI

M.A./M.Sc. Semester Examinations, May/June-2024

MMATH18-402 (iv): THEORY OF UNBOUNDED OPERATORS

UPC-223502408

Time: 3 hours | Maximum Marks: 70

Instructions: • Attempt **five** questions in all. • Question 1 is compulsory. • All the symbols have their usual meaning unless otherwise specified.

Q1. (a) Let $T : l^2 \to l^2$ be given by $Tx = \left\{\frac{\xi_j}{j}\right\}_{j=1}^{\infty}$ where $x = \{\xi_j\}_{j=1}^{\infty} \in l^2$. Show that T has an unbounded self-adjoint inverse. [5 Marks]

(b) Show that if $T : D(T) \to H$ is closed, then for any $\lambda \in \mathbb{C}$, (i) $T_\lambda = T - \lambda I$ is closed and (ii) if T_λ^{-1} exists, then T_λ^{-1} is closed. [4 Marks]

(c) Show that for every C_0 semigroup $\{T(t)\}_{t\geq 0}$ on a Banach space X, there exists a $\delta > 0$ and a $K > 0$ such that $\|T(t)\| \leq K$ for all $t \in [0, \delta]$. [2 Marks]

(d) If A is a densely defined, closed operator on a Banach space X satisfying $(0, \infty) \subset \rho(A)$ and $\|R(\lambda, A)\| \leq \frac{1}{\lambda}$ for every $\lambda > 0$, show that $\lim_{\lambda\to\infty} \lambda R(\lambda, A)x = x \;\forall x \in X$. [3 Marks]

Q2. Let H be a Hilbert space and $S : D(S) \to H, T : D(T) \to H$ be linear operators with domains $D(S)$ and $D(T)$ respectively, and $D(S), D(T) \subset H$.

(a) If $D(S) = D(T) = H$, and $\langle Tx, y\rangle = \langle x, Sy\rangle$ for all $x, y \in H$, show that T is bounded and S is its Hilbert-adjoint. [6 Marks]

(b) Assuming T, S are densely defined, show that (i) $S \subset T$ implies $T^* \subset S^*$ and (ii) if T^* is densely defined, then $T^* \subset T^{**}$. [8 Marks]

Q3. Let $T : D(T) \to H$ be a linear operator, where $D(T) \subset H$ and H is a non trivial, complex Hilbert space.

(a) Show that if T is symmetric then $\overline{T}^* = T^*$. You may assume that $\overline{T}$ exists. [6 Marks]

(b) Define the Cayley transform of a self-adjoint operator and show that it always exists. Show further that it is a unitary operator. [8 Marks]

Q4. Let $D(T) = \{x \in X = L^2(-\infty, \infty) : \int_{-\infty}^{\infty} t^2|x(t)|^2\, dt < \infty\}$ and set $(Tx)(t) = tx(t), x \in D(T), t \in \mathbb{R}$.

(a) Specify a subspace $D \subset D(T)$ which is dense in $L^2(-\infty, \infty)$. Hence or [3 Marks] otherwise show that T is densely defined. [4 Marks]

(b) Show that $D(T) \neq X$. [7 Marks]

(c) Show that T is an unbounded, self adjoint operator.

Q5. Let A be the generator of a C_0 semigroup $(T(t))_{t\geq 0}$ on a Banach space X. Prove or disprove the following:

(a) For $x \in X$, $\int_0^t T(s)x\,ds \in D(A)$ and $A\left(\int_0^t T(s)x\,ds\right) = T(t)x - x$. [4 Marks]

(b) For $x \in D(A)$, $\frac{d}{dt}T(t)x = AT(t)x = T(t)Ax$. [4 Marks]

(c) A is necessarily closed and densely defined. [6 Marks]

Q6. Let A be a linear operator on a Banach space X.

(a) When is A said to be dissipative? State and prove a necessary and sufficient condition for A to be dissipative when X is a Hilbert space. [6 Marks]

(b) Show that if A is a dissipative operator and there is a $\lambda_0 > 0$ such that $range(\lambda_0 I - A) = X$, then A is the generator of a C_0 semigroup of contractions. [8 Marks]

Q7. (a) What is meant by a C_0 group? Show that a linear operator A is the generator of a C_0 group if and only if both A and $-A$ generate a C_0 semigroup. [6 Marks]

(b) Suppose that $\{T(t)\}t \geq 0$ is a C_0-semigroup and $0 \in \rho(T(t_0))$ for some $t_0 > 0$. Show that then $S(t) := T(t)^{-1}, t \geq 0$ defines a C_0 semigroup with generator $-A$. [8 Marks]

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Your Roll No:....................

M.A/M.Sc. Mathematics, Part-II, Sem-IV (2024)
MMATH18-403(i), Calculus on $\mathbb{R}^n$
Unique Paper Code: 223502409

Time : 3 hours

Maximum Marks : 70

Question No. **1** is compulsory. Attempt any **four** questions from Question Nos. **2** to **7**.
Unless otherwise mentioned, U will be an open subset of $\mathbb{R}^n$.

(1) (a) Let $f(x,y,z) = x^3yz^2 - xyz^2$ be a function and $a = (0,-2,1)$. Find the directional derivative of f at a with respect to vector $u = (1,0,-1)$. [3]

(b) Let $f : U \to \mathbb{R}^m$ be a map. Define the derivative of f at a point $a \in U$ and show that the derivative at a is unique. [3]

(c) State the Stone-Weierstrass theorem. [2]

(d) Let $f : \mathbb{R}^5 \to \mathbb{R}^3$ be of class C^1, $a = (2,1,-2,1,-2)$, $f(a) = \mathbf{0}$ and [3]

$$Df(a) = \begin{bmatrix} 1 & -1 & 0 & 2 & 0 \\ 2 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

(i) Show that there is a neighborhood V of $(2,1)$ in $\mathbb{R}^2$ and a C^1-map $\varphi : V \to \mathbb{R}^3$ such that $\varphi(2,1) = (-2,1,-2)$ and $f(x,\varphi(x)) = 0$, $\forall x \in V$.

(ii) Find $D\varphi(2,1)$.

(e) Suppose $k \geq 2$ and $\sigma = [P_0, P_1, \cdots, P_k]$ is an oriented affine k-simplex. Prove that $\partial(\partial\sigma) = 0$. [3]

(2) (a) Let $f : U \to \mathbb{R}^m$ be a map such that partial derivatives $D_i f_j(x)$ exist at each point $x \in U$ and also continuous on U, for $1 \leq i \leq n$ and $1 \leq j \leq m$. Show that f is differentiable at each point $x \in U$. [8]

(b) Let $f : U \to \mathbb{R}$ be a map of class C^k. State and prove the Taylor's formula with Lagrange's remainder of order k in a neighborhood of $a \in U$. [6]

(3) (a) Let ω be a k-form and λ be a m-form of class C^1 in U. Prove that [7]

$$d(\omega \wedge \lambda) = d\omega \wedge \lambda + (-1)^k \omega \wedge d\lambda.$$

(b) Let V be an open subset of $\mathbb{R}^m$, $f : U \to V$ be a map differentiable at a point $x \in U$, $g : V \to \mathbb{R}^k$ be a map differentiable at $f(x)$. Show that the $g \circ f$ is differentiable at x and $D(g \circ f)(x) = Dg(f(x)) \cdot Df(x)$. [7]

(4) (a) Let $f : U \to \mathbb{R}^n$ be a map of class C^1 and $Df(a)$ is invertible for some point $a \in U$. Show that f is injective in a neighborhood of a. [7]

(b) Let $T(s,t) = (x,y)$, where $x = s - st$ and $y = st$. Let [7]

$$S = \{(s,t) | 0 < s < 1,\ 0 < t < 1\}$$

be a square. Show that T is one-one on S and find $Q = T(S)$.
Let $f(x,y) = \frac{e^{x+y}}{x+y}$ be defined on Q, then find the integral of f on the region Q by using the change of variable with respect to T.

(5) (a) Let $V \subset \mathbb{R}^m$, $W \subset \mathbb{R}^p$ be open subsets and $T : U \to V$, $S : V \to W$ be two C^1-maps. Let ω be a k-form in W, then show that $(\omega_S)_T = \omega_{S \circ T}$. [7]

(b) Let I^k be a k-cell and $f : I^k \to \mathbb{R}$ be a continuous map. Show that the integration $\int_{I^k} f = \int \int \cdots \int f dx_1 dx_2 \cdots dx_k$ makes sense and it is independent of the order of integrations. [7]

(6) (a) Check the closeness and exactness of the following differential forms defined on $\mathbb{R}^2 \setminus \{(0,0)\}$: (1) $\omega = \frac{xdy - ydx}{x^2+y^2}$, (2) $\omega' = \frac{xdx + ydy}{x^2+y^2}$. [7]

(b) Define the functions ψ_n for $n = -2, -1, 0, 1, 2$, on $\mathbb{R}$ as follows:

$$\psi_0(x) = \begin{cases} 1 + 2x, & \text{if } -\frac{1}{2} \leq x < 0, \\ 1 - 2x, & \text{if } 0 \leq x < \frac{1}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

and $\psi_n = \psi_0(x - \frac{n}{2})$. Verify that the family $\{\psi_n ; n = -2, -1, 0, 1, 2\}$ defines a partition of unity for the compact set $[-1, 1]$ subordinate to the cover $\{(\frac{n}{2} - 1, \frac{n}{2} + 1) ; n = -2, -1, 0, 1, 2\}$. Sketch the graphs of ψ_n's. [7]

(7) (a) State and prove the Stoke's theorem. [10]

(b) Define $\Phi : [0,1]^2 \subset \mathbb{R}^2 \to \mathbb{R}^3$ as $\Phi(u,v) = (u^2+1, uv, v^2+1)$. Let $\alpha = xydx + yzdy + xzdz$ be a 1-form in $\mathbb{R}^3$. Calculate $\int_\Phi d\alpha$. [4]

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examinations, May 2024
Part II Semester IV
MMATH18-403(ii): DIFFERENTIAL GEOMETRY
(Unique Paper Code 223502410)

Time: 3 hours Maximum Marks: 70

Instructions: • Attempt **five** questions in all. **Question 1 is compulsory**. All questions carry equal marks.• The symbols used have their usual meanings.

(1) (a) Define tangent vector field on an n-surface in $\mathbb{R}^{n+1}$ and give an example with justification. [3]

(b) Show that a geodesic in an n-surface in $\mathbb{R}^{n+1}$ has constant speed. [2]

(c) If C is an oriented plane curve and $p \in C$, define a parametrization of a segment of C containing p. [3]

(d) Let S_1 be an n-surface in $\mathbb{R}^{n+1}$ and S_2 an m-surface in $\mathbb{R}^{m+1}$. If $\varphi : S_1 \to \mathbb{R}^{m+1}$ is a smooth map such that $\varphi(S_1) \subset S_2$, show that $d\varphi : T(S_1) \to T(S_2)$. [3]

(e) Let S be an oriented n-surface in $\mathbb{R}^{n+1}$ and $\widetilde{S}$ be the same surface with opposite orientation. Show that $\widetilde{K} = K$, where K and $\widetilde{K}$ are the Gauss-Kronecker curvatures of S and $\widetilde{S}$ respectively. [3]

(2) (a) If S is an $(n-1)$-surface in $\mathbb{R}^n$, define the cylinder over S and show that it is an n-surface in $\mathbb{R}^{n+1}$. Illustrate with a specific example. [7]

(b) Show by an example that if an n-surface S is not connected, then there can exist more than two smooth unit normal vector fields on S. Justify your claim. [7]

(3) (a) Let $\mathbf{X}$ be a smooth tangent vector field on an oriented n-surface S in $\mathbb{R}^{n+1}$, and $p \in S$. Show that there is an open interval I containing 0 and a parametrized curve $\alpha : I \to S$ such that [8]

(i) $\alpha(0) = p$

(ii) $\mathbf{X}(\alpha(t)) = \dot{\alpha}(t)$

(iii) If $\beta : \tilde{I} \to S$ is any other parametrized curve in S satisfying (i) and (ii), then $\tilde{I} \subset I$ and $\beta(t) = \alpha(t),\ \forall t \in \tilde{I}$.

(b) For the plane curve $C = f^{-1}(d)$, where $f(x_1, x_2) = x_1 - 3x_2^2$, oriented by $\frac{-\nabla f}{\|\nabla f\|}$, find global parametrization and the curvature κ. [6]

(4) (a) For $\theta \in \mathbb{R}$, let $\alpha_\theta : [0, \pi] \to \mathbb{S}^2$ be the parameterized curve from the north pole $p = (0,0,1)$ to the south pole $q = (0,0,-1)$ defined by

$$\alpha_\theta(t) = (\cos\theta \sin t, \sin\theta \sin t, \cos t),\ 0 \le t \le \pi.$$

If $\mathbf{N}$ is the outward orientation on $\mathbb{S}^2$ and $\mathbf{v} = (p, 1, 0, 0) \in S_p$, determine $P_{\alpha_\theta}(\mathbf{v})$. [8]

(b) Find the length of the connected oriented plane curve $f^{-1}(2)$ oriented by $\frac{\nabla f}{\|\nabla f\|}$, where [6]

$$f(x_1, x_2) = \frac{x_1^2}{2} + \frac{(x_2 - 2)^2}{2}.$$

(5) (a) Show that the Weingarten map at each point of an oriented n-surface in $\mathbb{R}^{n+1}$ is self adjoint. [7]

(b) Let $\alpha : I \to S$ be a geodesic in an n-surface S with $\dot{\alpha} \neq 0$, and $\beta = \alpha \circ h$ be a reparametrization of α (that is, there is an open interval J and a smooth map $h : J \to I$ with $h' > 0$ such that $\beta = \alpha \circ h$). Show that β is a geodesic in S if and only if there exist $a, b \in \mathbb{R}$ with $a > 0$ such that $h(t) = at + b \; \forall t \in I$. [7]

(6) (a) Let $U \subset \mathbb{R}^{n+1}$ be open and $f : U \to \mathbb{R}$ a smooth function. If p is a regular point of f and $c = f(p)$, show that the set of all vectors tangent to $f^{-1}(c)$ at p is equal to $[\nabla f]^{\perp}$. [7]

(b) Define parametrized torus in $\mathbb{R}^3$ and show that it is a parametrized 2-surface in $\mathbb{R}^3$. [7]

(7) (a) Let V be a finite dimensional vector space with dot product and $L : V \to V$ be a linear self adjoint transformation. Let $S = \{v \in V : v \cdot v = 1\}$ and define $f : S \to \mathbb{R}$ by $f(v) = L(v) \cdot v$. Show that f is stationary at $v_0 \in S$ if and only if $Lv_0 = f(v_0)v_0$. [9]

(b) Let $U \subset \mathbb{R}^{n+1}$ be open. Show that for each 1-form ω on U, there exist unique functions $f_i : U \to \mathbb{R}$, $i = 1, 2, \cdots, n$ such that [5]

$$\omega = \sum_{i=1}^{n+1} f_i dx_i.$$

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Your Roll Number: ....

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations, May/June 2024
Part- II Semester- IV

MMATH18-403(iii): Topological Dynamics
(Unique Paper Code 223502411)

Time: **3 hours** Marks: **70**

Instructions: • All notations used are standard. • Question no. 1 is compulsory. • Attempt any four questions from the remaining six questions.

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(1) Do as directed.

(a) Is every minimal homeomorphism on a compact metric space topologically Anosov also? Justify your claim. [3 Marks]

(b) For $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$, find $W^s(0)$ and $W^u(-1)$. [3 Marks]

(c) Can we say that for every non-zero $k \times k$ matrix A with entries in $\{0, 1\}$, X_A is non-empty? Justify your claim. [2 Marks]

(d) Prove that $\cos : [0,1] \to [0,1]$ defined by $x \to \cos(x)$ has POTP. [3 Marks]

(e) Prove that if f is an expansive homeomorphism on a compact metric space then the set of periodic points of f is a countable set. [3 Marks]

(2) (a) For $k \in \mathbb{N}$, let A be a $k \times k$ matrix with entries in $\{0, 1\}$. Prove that X_A is a closed, σ-invariant subset of Σ_k. Give an example of A for which $\sigma_{|A}$ is topologically transitive. Justify your claim. [9 Marks]

(b) Do the graphical analysis of $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = (-1/2)(x^3 + x)$ and draw the phase portrait of the orbit of $x = -1/2$. Find an attractive fixed point of f. [5 Marks]

(3) (a) Prove that the tent map $T : [0,1] \to [0,1]$ is topologically conjugate to logistic map $F_\mu : [0,1] \to [0,1]$ defined by $F_\mu(x) = \mu x(1-x)$ for $\mu = 4$. [5 Marks]

(b) Prove that if $f : \mathbb{R} \to \mathbb{R}$ has a periodic point of prime period 3 then it has periodic points of all prime periods $n \geq 1$. [9 Marks]

(4) (a) Prove that the unit circle $\mathbf{S}^1$ admits no expansive homeomorphism. [9 Marks]

(b) Let (X, d) be a metric space, $f : X \to X$ be a homeomorphism and $A \subseteq X$ such that $X \setminus A$ is finite. Prove that if f is expansive on A then f is expansive on X. [5 Marks]

(5) (a) Let (X, d) be a compact metric space and $f : X \to X$ be a homeomorphism. Prove that f is expansive if and only if f has a generator. [5 Marks]

P.T.O.

(b) Prove that the Dyadic Solenoid is a compact, connected metric space admitting an expansive homeomorphism. [9 Marks]

(6) (a) If (X, d) is a compact metric space then prove that the shift map $\sigma : X^{\mathbb{Z}} \to X^{\mathbb{Z}}$ has POTP. [9 Marks]

(b) Let (X, d) be a compact metric space and $f : X \to X$ be a homeomorphism having POTP then prove that f has canonical coordinates. [5 Marks]

(7) (a) Prove that a topologically Anosov homeomorphism f of a compact metric space (X, d) is topologically stable in the class of self-homeomorphisms of X. [9 Marks]

(b) Let (X, d) be a compact metric space and $f : X \to X$ be continuous. Prove that for any $x \in X$, $\omega(x) = \cap_{m \geq 0}(\overline{\cup_{n \geq m}\{f^n(x)\}})$. Conclude that $\omega(x)$ is a non-empty subset of X. [5 Marks]

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI

M.A./M.Sc. Mathematics Examinations, May- June 2024

Part II Semester IV

MMATH18-404(i): ADVANCED FLUID DYNAMICS

UPC 223502412

Time: 3 Hour Maximum Marks: 70

Instructions: • Attempt FIVE questions in all. • Q.No.1 is compulsory. • Attempt any FOUR questions from Q.No.2 to Q.No.7. Each question carries 14 marks. • All the symbols have their usual meaning.

(1) (a) Derive the Bernoulli'equation of motion for isentropic flow of an ideal gas in terms of critical and stagnation speed of sound. [3 Marks]

(b) Describe the second law of thermodynamics using its different variant. [2 Marks]

(c) Prove that Pre-Maxwell equation leads that a charge can not exist in a conductor at rest. [2 Marks]

(d) Derive an estimate of boundary layer thickness at a leading edge distance x for the laminar flow of a viscous fluid over a flat plate. [3 Marks]

(e) Define equation of state of a compressible substance. Write the van der Waal's equation of state for a non-ideal gas and derive its simplified form. [2 Marks]

(f) Compare the transport and diffusion phenomena in MHD flow using Magnetic-Reynolds number. [2 Marks]

(2) (a) Reduce the Navier Stokes equation of motion into non-dimensional form. Explain physically the obtained non-dimensional numbers. [7 Marks]

(b) Define function of state, enthalpy and free energy of a compressible substance. Derive the Maxwell's thermodynamic relations and explain them. [7 Marks]

(3) (a) State the integral form of conservation laws for one-dimensional flow of fluid with density $\rho(x,t)$ and flux $q(x,t)$. Under suitable condition deduce its differential form. Derive the jump condition across a discontinuity $x = s(t)$ for such a flow. Deduce the jump condition for the equation of motion of a gas. [8 Marks]

(b) Define shock-Mach number and shock strength. Prove that the shock wave is compressive in nature. [6 Marks]

(4) (a) Derive the equations of motion of a finite disturbance in one - dimensional unsteady flow of non-viscous gas in absence of external body force. Write its general solution and explain it geometrically. [6 Marks]

(b) Derive the kinetic energy equation of a gas and explain it physically. [3 Marks]

(c) State and prove the Alfven's theorem. [5 Marks]

(5) (a) Write the Maxwell's electromagnetic field equation for the conducting medium in rest and motion. Write the constitutive relations for [8 Marks]

a medium under electromagnetic field. Describe the electromagnetic force in conducting fluid.

(b) State the Magnetohydrodynamic approximations. Show that under these approximations displacement current density is negligible in Ampere's circuital law. [6 Marks]

(6) (a) Explain the Alfven's wave. Describe the Magnetohydrodynamic wave and derive its speed. [7 Marks]

(b) Define the mass, momentum and energy thickness of a boundary layer. Compute and compare the three thicknesses of a boundary layer over a flat plate for parabolic velocity distribution given by [7 Marks]

$$u(y) = \begin{cases} U_\infty \sqrt{\frac{y}{\delta}}, & 0 \leq y \leq \delta \\ U_\infty, & \delta \leq y < \infty, \end{cases}$$

where, U_∞ is uniform stream velocity.

(7) (a) Define a boundary layer in fluid flow. Describe the boundary layer problem over a long flat plate and derive the Blasius's equation along with the boundary conditions. [2+6 Marks]

(b) Derive the momentum integral equation for steady, two dimensional boundary layer flow of incompressible fluid. [6 Marks]

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Your Roll Number: ....

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations, May -June 2024
Part II Semester IV

MMATH18-404(ii): COMPUTATIONAL METHODS FOR PDEs
(UPC: 223502413)

Time: 3 hours | Maximum Marks: 70

Instructions: • Section A is compulsory. • Answer any four questions from section B • All notations have usual meaning • Non-programmable scientific calculators are allowed

Section A

(1) (a) State true or false and justify the statement: Fully Implicit scheme for the numerical solution of one dimensional heat equation is unconditionally stable. [3]

(b) What is CFL condition? Explain with an example. [3]

(c) Find the necessary and sufficient condition for convergence of the SOR iterative scheme. [4]

(d) Consider the problem $-u_t+u_{xx}+u = f(x),\ x \in (0,1),\ u(x,0) = 1,\ u(0,t) = u(1,t) = 1$ and derive its equivalent weak form. [4]

Section B

(2) (a) State and prove Sherman-Morrison Formula. [4]

(b) State and prove Gerschgorin Circle Theorem. Use this theorem to derive the stability criteria for numerical solution of the problem $u_t = u_{xx},\ 0 < x < 1$ with condition $u_x(0,t) = h_1(u - v_1),\ u_x(1,t) = h_2(u - v_2),\ t \geq 0$ where h_1, h_2, v_1, v_2 are constants and $h_1 \geq 0,\ h_2 \geq 0$. Here boundary conditions are approximated by central difference and BTCS scheme is used. [5+5]

(3) (a) Analyze the stability of the Lax-Friedrich Scheme given by [7]

$$U_{jk}^{n+1} = \frac{1}{4}(U_{j+1k}^n + U_{j-1k}^n + U_{jk+1}^n + U_{jk-1}^n) - \frac{R_x}{2}\delta_{x0}U_{jk}^n - \frac{R_y}{2}\delta_{y0}U_{jk}^n$$

for approximating the solution to the problem $u_t + au_x + bu_y = 0$.

(b) Solve $u_t + u_x = 0,\ x \in [-1,3]$ with initial condition [7]

$$u(0,x) = \begin{cases} \cos^2(\pi x), & -1/2 \leq x \leq 1/2 \\ 0, & \text{otherwise} \end{cases}$$

using FTBS scheme and $\Delta x = 1, \Delta t/\Delta x = 0.8$. Find the solution at $t = 0.8$.

(4) (a) Consider the following equation in Cylindrical polar co-ordinates [7]

$$\frac{\partial^2 u}{\partial z^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2} = f(r, z, u),\ 0 \leq r \leq R,\ 0 < z \leq c$$

$$\frac{\partial u}{\partial r}(0, z) = 0,\ u(R, z) = g(z),\ t \geq 0,\ u(r, 0) = f(r) \quad 0 \leq r \leq R,\ u(r, c) = h(r).$$

Derive a second order finite difference scheme for solution of the above problem on a rectangular mesh.

(b) Consider the problem $u_{tt} = 4u_{xx}$ together with initial and boundary condi- [7]

tions

$$u(x,0) = \sin\pi x, \ \frac{\partial u}{\partial t}(x,0) = 0, \ 0 \le x \le 1, u(0,t) = u(1,t) = 0, \ t \ge 0.$$

Find the numerical solution of the above problem using second order finite difference scheme with $\Delta x = 0.25$, and $\Delta t = 0.1$ at $t = 0.1$. [7]

(5) (a) Derive a second order accurate numerical scheme for the solution of three dimensional Poisson's equation with Neumann boundary condition at one of the boundary . [7]

(b) Solve the problem

$$\nabla^2 u = 0, 0 \le x, y \le 1,$$
$$u(x,0) = 2x, \ u(x,1) = 2x - 1, \ 0 \le x \le 1,$$
$$(u_x + u)(0,y) = 2 - y, \ u(1,y) = 2 - y, \ 0 \le y \le 1$$

using five point finite difference scheme with $\Delta x = \Delta y = 1/3$. Use first order approximation for the derivative boundary conditions. [7]

(6) (a) Consider the problem

$$u_t - u_{xx} = f(x,t), \ 0 < x < 1, u(x,0) = u(0,t) = u(1,t) = 0.$$

Derive a finite element approximation for the above problem using Backward Euler scheme for time discretization and linear basis functions. Obtain the resulting system of algebraic equations. [7]

(b) Show that the truncation error of the fully implicit scheme for the numerical solution of the partial differential equation

$$u_t = \nu(u_{xx} + u_{yy} + u_{zz})$$

is of $O((\Delta t) + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2)$. [7]

(7) (a) Consider the problem

$$-(u_{xx} + u_{yy}) = 5, \quad 0 < x < 1, \ 0 < y < 1,$$
$$u(x,0) = u(x,1) = u(0,y) = u(1,y) = 0.$$

Solve the above problem using a finite element technique involving triangular elements with linear basis functions $\Delta x = 1/2$, $\Delta y = 1/3$ and write the resulting system of algebraic equations in matrix form. [9]

(b) Consider the problem

$$u_t = u_{xx} + u_{yy}, \quad (x,y) \in (0,1) \times (0,1), \ t > 0$$

with Dirichlet boundary conditions. Derive D'Yakanov ADI scheme for the solution of the above problem. [5]

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Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations, May-June 2024
Part II Semester IV
MMATH18-404(iii): Dynamical Systems (UPC 223502414)

Time: 3 hours

Maximum Marks: 70

Instructions: • Attempt **any FIVE** questions. • Question 1 is compulsory. • Each question carries 14 marks.

(1) (a) Solve the initial value problem $\dot{x} = Ax$ with [2 Marks]

$$A = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}.$$

(b) Define the flow ϕ_t of the non-linear system $\dot{x} = f(x)$. [2 Marks]

(c) Compute the exponential of the matrix [2 Marks]

$$\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}.$$

(d) Determine the stability of the equilibrium points of the system $\dot{x} = f(x)$ with [2 Marks]

$$f(x) = \begin{bmatrix} x_1(3 - x_1 - 2y_1) \\ y_1(2 - x_1 - y_1) \end{bmatrix}$$

(e) Explain the term "attracting set" with an example. [2 Marks]

(f) Define the terms: Separatrices, saddle-node and cusp. [2 Marks]

(g) Find the maximal interval of existence (α, β) for the following inital value problem: [2 Marks]

$$\dot{x} = x^2, \quad x(0) = x_0.$$

(2) (a) Define the term "Bifurcations". [2 Marks]

(b) Discuss the stability analysis and whole mechanism by making graphs for the following one-dimensional system [8 Marks]

$$\dot{x} = \mu x - x^3.$$

(c) Make bifurcation diagram for the pitchfork bifurcation. [4 Marks]

(3) (a) Construct the phase portriat for the following undamped pendulum by converting it into Newtonian system [5 Marks]

$$\ddot{x} + \sin(x) = 0.$$

(b) All sinks are asymptotically stable. However, not all asymptotically stable equilibrium points are sinks. Justify this statement with an appropriate example. [5 Marks]

(c) Find the stable, unstable and center subspaces E^s, E^u and E^c of the system $\dot{x} = Ax$ with the matrix [4 Marks]

$$A = \begin{bmatrix} -1 & -3 \\ 0 & 2 \end{bmatrix}.$$

(4) (a) Define the *Poincaré map*. Find the *Poincaré map* of the system [6 Marks]

$$\dot{x} = -y + x(1 - x^2 - y^2), \quad \dot{y} = x + y(1 - x^2 - y^2).$$

(b). Define the following systems: Hamiltonian system with n degrees of freedom on E, Gradient system on E and orthogonal system to a planar system. [8 Marks]
For the Hamiltonian function $H(x, y) = x^2 + 2y^2$, determine both Hamiltonian system and gradient system. Find the critical points for both the systems and determine the type and nature of equilibrium point.

(5) (a) State "The Local Center Manifold Theorem". [4 Marks]

(b) Use "The Local Center Manifold Thoerem" to find the approximation for the flow on local center manifold for the system [10 Marks]

$$\dot{x_1} = x_1 y - x_1 x_2^2, \quad \dot{x_2} = x_2 y - x_2 x_1^2, \quad \dot{y} = -y + x_1^2 + x_2^2,$$

Also, show that origin is a type of topological saddle that is unstable.

(6) Find the first three successive approximation $u^{(1)}(t, a)$, $u^{(2)}(t, a)$ and $u^{(3)}(t, a)$ for [14 Marks]

$$\dot{x_1} = -x_1$$

$$\dot{x_2} = x_2 + x_1^2$$

and use $u^{(3)}(t, a)$ to approximate S near the origin. Also approximate the unstable manifold U near the origin for this system. Find S and U for this problem.

(7) (a). Write the following system in polar coordinates and show that origin is stable focus for this nonlinear system [4 Marks]

$$\dot{x} = -y - x^3 - xy^2, \quad \dot{y} = x - y^3 - x^2 y.$$

(b) Using an appropriate theorem, show that the following system has a critical point with an elliptic domain at origin [4 Marks]

$$\dot{x} = y, \quad \dot{y} = -x^3 + 4xy.$$

(c) Consider the Lorenz system [6 Marks]

$$\dot{x} = \sigma(y - x)$$

$$\dot{y} = \rho x - y - xz$$

$$\dot{z} = xy - \beta z,$$

with $\sigma > 0$, $\rho > 0$ and $\beta > 0$. Show that this system is invariant under the transformation $(x, y, z, t) \to (-x, -y, z, t)$. Show that the $z-$axis is invariant under the flow of this system and that it consists of three trajectories. Also show a trajectory and the corresponding branched surface of the Lorenz system.

Your Roll Number: ....

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examinations, May 29, 2024
Part II Semester IV

MMATH18-405(i): CRYPTOGRAPHY

(Unique Paper Code: 223503401)

Time: 2 Hours Maximum Marks: 35

Instructions: • Attempt five questions in all. • Question 1 is compulsory. • All questions carry equal marks.

(1) Explain the following:

(a) Compute 2^{60} in modulo 181. [2]

(b) Define witness for compositeness of n. Also, give an example. [1+1]

(c) Explain Vigenere cipher. [1]

(d) Define an encryption scheme (cryptosystem). [1]

(e) Define block cipher. [1]

(2) Explain the following ciphers with examples:
CFB & OFB [3.5 + 3.5]

(3) In a long string of ciphertext which was encrypted by means of an affine map on single-letter message units in the 26-letter alphabet, you observe that the most frequently occurring letters are "Y" and "V", in that order. Assuming that those ciphertext message units are the encryption of "E" and "T", respectively, read the message "$QAOOYQQEVHEQV$". [7]

(4) You intercept the message "$ZRIXXYVBMNPO$", which you know resulted from a linear enciphering transformation of digraph-vectors in a 27-letter alphabet, in which $A - Z$ have numerical equivalents $0 - 25$, and blank $= 26$. You have found that the most frequently occurring ciphertext digraphs are "PK" and "RZ". You guess that they correspond to the most frequently occurring plaintext digraphs in the 27-letter alphabet , namely, "E " (E followed by blank) and "S " (S followed by blank). Find the deciphering matrix and read

the message. [7]

(5) Explain the following:

(i) The round key generations $K_1, K_2, \ldots, K_{16}$ in DES algorithm. [3]

(ii) The encryption and decryption in Feistel cipher in details. [4]

(6) Give an algorithm for finding discrete logs in finite fields. [7]

(7) Define Carmichael number with example. Show that an odd composite number $n \geq 3$ is a Carmichael number if and only if it is square-free and for each prime divisor p of n, $(p-1)$ divides $(n-1)$. [2 + 5]

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Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Final Examinations, 2024
Part II Semester IV
MMATH18-405(ii): Support Vector Machines
(Unique Paper Code: 223503402)

Time: 2 hours | Maximum Marks: 35

Instructions: • Attempt any five questions. • Question 1 is compulsory. • Answer any four questions from Q.2 to Q.7 • Each question carry equal marks. • All notations have standard meaning.

(1) (a) Give the KKT conditions for problem considered in Q(2). [2 Marks]

(b) State the weak duality theorem for the problem considered in Q(2). [1 Marks]

(c) Define graph kernel based on the path. [2 Marks]

(d) What do you mean by a support vector in linearly separable support vector classification? [2 Marks]

(2) Consider the problem [7 Marks]

$$\begin{aligned} \min \quad & f_0(x),\ x \in R^n, \\ s.t \quad & f_i(x) \le 0, \quad i = 1, \cdots, m, \\ & h_i(x) = a_i^T x - b_i = 0, \quad i = 1, \cdots, p, \end{aligned}$$

where $f_0(x)$ and $f_i(x)$, $i = 1, \cdots, m$ are continuous convex functions on R^n, and $h_i(x)$, $i = 1, \cdots, p$ are linear functions. If x^* is a local solution, then prove that x^* is a global solution.

(3) Consider the linearly seprable problem (P) [7 Marks]

$$\begin{aligned} \min \quad & \frac{1}{2}||w||^2, \\ s.t \quad & y_i((w \cdot x_i) + b) \ge 1, \quad i = 1, \cdots, l, \end{aligned}$$

corresponding to the training set $T = \{(x_1, y_1), (x_2, y_2), \ldots, (x_l, y_l)\}$. Prove that (P) has a solution (w^*, b^*) such that $w^* \neq 0$. Also, prove that there exists a $j \in \{i | y_i = 1\}$ and a $k \in \{i | y_i = -1\}$ such that $(w^* \cdot x_j) + b^* = 1$ and $(w^* \cdot x_k) + b^* = -1$, respectively.

(4) Show that the optimization problem [7 Marks]

$$\begin{aligned} \max_{\alpha, \beta} \quad & -\frac{1}{2}\sum_{i=1}^{l}\sum_{j=1}^{l} y_i y_j \alpha_i \alpha_j (\Phi(x_i) \cdot \Phi(x_j)) + \sum_{j=1}^{l} \alpha_j. \\ s.t. \quad & \sum_{i=1}^{l} y_i \alpha_i = 0, \\ & C - \alpha_i - \beta_i = 0;\ \alpha_i \ge 0;\ \beta_i \ge 0, \quad i = 1, \cdots, l, \end{aligned}$$

is the dual problem of the primal problem

$$\min_{w,b,\xi} \quad \frac{1}{2}||w||^2 + C\sum_{i=1}^{l} \xi_i,$$
$$s.t \quad y_i((w \cdot \Phi(x_i)) + b) \geq 1 - \xi_i;\ \xi_i \geq 0,\quad i = 1, \cdots, l.$$

(5) (a) Define $\bar{\epsilon}-$band of a hyperplane and hard $\bar{\epsilon}-$band for linear regression problem. [3 Marks]

(b) For the training set $T := \{(x_1, y_1), (x_2, y_2), \ldots, (x_l.y_l)\}$, where $x_i \in R^n, y_i \in Y = R$, $i = 1, \cdots, l$, and $\bar{\epsilon} > 0$. Show that a hyperplane $y = (w \cdot x) + b$ is a hard $\bar{\epsilon}-$band hyperplane if and only if the sets $D^+ = \{(x_i^T, y_i + \bar{\epsilon})^T, i = 1, \cdots, l\}$ and $D^- = \{(x_i^T, y_i - \bar{\epsilon})^T, i = 1, \cdots, l\}$, locate on both sides of this hyperplane respectively, and all of the points in D^+ and D^- do not touch this hyperplane. [4 Marks]

(6) State the algorithm for classification based on nonlinear sepearation. [7 Marks]

(7) (a) If $K_1(x, x')$ and $K_2(x, x')$ are kernals on $\mathbb{R}^n \times \mathbb{R}^n$ then prove that the product $K(x, x') = K_1(x, x') \cdot K_2(x, x')$ is also a kernal. [3 Marks]

(b) Prove that Gaussian radial basis function with parameter σ given by $K(x, x') = exp(-||x - x'||^2/\sigma^2)$ is a kernal. [4 Marks]