Department of Mathematics
University of Delhi, Delhi

M.A./M.Sc. Mathematics, Part-I, Semester-II
Examination, May-June 2024

Paper:MMATH18 201 Module Theory
Unique Paper Code: 223501201

Time: 3 Hours *Maximum Marks: 70*

Note: • Attempt five questions in all. • Question 1 is compulsory. • All questions carry equal marks.

1. (a) Prove that every abelian group is a $\mathbb{Z}$-module. (3)
   (b) Find the number of $\mathbb{Z}$-module homomorphisms between $\mathbb{Z}_2 \oplus \mathbb{Z}_3$ and $\mathbb{Z}_4 \oplus \mathbb{Z}_6$. (3)
   (c) Is every submodule of a free module free? Justify. (3)
   (d) Let G be a finite abelian group and $\mathbb{Q}$ the set of rational numbers. Show that $G \otimes_{\mathbb{Z}} \mathbb{Q} = \{0\}$. (2)
   (e) Prove that the homomorphic image of a divisible module is divisible. (3)
2. (a) State and prove the universal property of coproduct (direct sum) of a family of left R-modules. (8)
   (b) Prove that a short exact sequence $0 \longrightarrow M' \xrightarrow{\lambda} M \xrightarrow{\mu} M'' \longrightarrow 0$, of R-modules and homomorphisms, splits if and only if μ has left inverse. (6)
3. (a) Give an example of a contravariant functor that is left exact but not right. (4)
   (b) Prove that any two free modules with bases having the same cardinalities are always isomorphic. (4)
   (c) Let M be a finitely generated free module over a non-trivial commutative ring R. Prove that all the bases of M are finite and have the same number of elements. (6)
4. (a) Let R be a PID and $a = p_1^{e_1} \cdots p_n^{e_n}$, where $p_1, \ldots, p_n$ are distinct (non-associate) primes in R and $e_1, \ldots, e_n$ are positive integers. Prove that $R/\langle a \rangle \cong R/\langle p_1^{e_1} \rangle \oplus \cdots \oplus R/\langle p_n^{e_n} \rangle$. (6)
   (b) Let R be a PID. Prove that a direct sum of finitely many non-zero cyclic R-modules is cyclic if and only if the modules (summands) are bounded and their bounds are pairwise coprime. (8)
5. (a) State and prove the Baer's Criterion. (8)
   (b) Let $M = \sum_{i \in I} S_i$, where each S_i is a simple submodule the left R-module M. Prove that M is semi-simple. (6)

6. (a) Prove that the fully invariant submodules of a semi-simple module are a sum of type components. (5)

   (b) State and prove the universal property of the tensor product of modules over a commutative ring. (1+8)

7. (a) Let N_1, N_2 be submodules of an R-module M such that M/N_1 and M/N_2 are both Artinian. Prove that $M/(N_1 \cap N_2)$ is also Artinian. (3)

   (b) Let N be a proper submodule of an R-module M of finite length. Prove that the length of N is less than the length of M. (6)

   (c) Prove that a module has a composition series if and only if it is both Noetherian and Artinian. (5)

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DEPARTMENT OF MATHEMATICS

M.A./M.Sc. Part-I, Semester-II, May-June 2024

MMATH18-202 : Introduction to Topology

UPC : 223501202

Time: 3 hours. Maximum Marks: 70

Instruction: Attempt any ***Five*** questions. Question **No.1 is compulsory**. Each question carries 14 marks.

1. (i) Justify the statement :*Sierpinski space* is a T_0 space but not T_1. [2]
   (ii) What are the limits of the sequence $\left\{\frac{1}{n}\right\}$ in $\mathbb{R}_f$, the space of real numbers with the co-finite topology? [3]
   (iii) Determine two different bases for the Euclidean topology on the set of real numbers. [3]
   (iv) Show that a continuous function $f : [0,1] \longrightarrow [0,1]$ has a fixed point. [3]
   (v) Let $x_n \longrightarrow x$ in a space X. Show that the set $\{x_n : n \in \mathbb{N}\} \cup \{x\}$ is a compact subset of X. [3]

2. (a) Let $\mathcal{B}$ be a collection of subsets of the set X such that $X = \bigcup\{B : B \in \mathcal{B}\}$, and for every two members B_1 and B_2 of $\mathcal{B}$ and for each points $x \in B_1 \cap B_2$ there exists $B_3 \in \mathcal{B}$ with $x \in B_3 \subset B_1 \cap B_2$. Show that there is a topology on X for which $\mathcal{B}$ is a basis. [7]
   (b) Show that the *closed topologist sine curve* $\{(x, sin\frac{1}{x}) : 0 < x \leq 1\} \cup \{(0,y) : |y| \leq 1\}$ is connected but not a path connected subset of $\mathbb{R}^2$. [7]

3. (a) Let $X_\alpha, \alpha \in A$, be a family of topological spaces, and $p_\beta : \prod X_\alpha \longrightarrow X_\beta$ be the projection onto the βth factor. Show that a net ϕ in $\prod X_\alpha$ converges to $x = (x_\alpha)$ if and only if $p_\alpha \circ \phi \longrightarrow x_\alpha$ for each $\alpha \in A$. [7]
   (b) Prove that a space X is locally connected if and only if the components of each open subsets of X are open. [7]

4. (a) Show that an infinite Hausdorff space has infintely many disjoint open sets. [7]
   (b) Show that a space X is compact iff each family of closed subsets of X with finite intersection property has non empty intersection. [7]

5. (a) Let X be a first countable space and Y a space. Show that a function $f : X \longrightarrow Y$ is continuous if and only if $x_n \longrightarrow x$ in X implies $f(x_n) \longrightarrow f(x)$ in Y. [7]
   (b) Show that a space X is Hausdorff iff each net in X converges to atmost one point. [7]

6. (a) Show that finite product of compact spaces is compact. [7]
   (b) Let X be a *compact Hausdorff space*. Show that a set $A \subset X$ is closed if and only if it is compact. [7]

7. (a) Show that a T_1 space X is *countably compact* if and only if it has the *Bolzano-Weierstrass* property. [7]

   (b) Show that every metric space is first countable. Give an example of a space which is not first countable. [4+3]

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI

M.A./M.Sc. Examinations, May/June-2024

MMATH18-203: FUNCTIONAL ANALYSIS (UPC : 223501203)

Time: 3 hours — Maximum Marks: 70

Instructions: • Attempt **five** questions in all. • Question 1 is compulsory. • All the symbols have their usual meaning unless otherwise specified.

Q1. Prove or disprove the following:

(a) The norm $\|\cdot\|$ on $\mathbb{R}^2$ defined by $\|x\| = |\xi_1| + |\xi_2|$, $x = (\xi_1, \xi_2) \in \mathbb{R}^2$ can be obtained from an inner product on $\mathbb{R}^2$. [2 Marks]

(b) The range of a bounded linear operator is closed. [3 Marks]

(c) The dual of a reflexive normed space is reflexive. [3 Marks]

(d) A weakly convergent sequence in a normed space is bounded. [3 Marks]

(e) Let X and Y be normed spaces and $T_n \in B(X,Y)$. If a sequence (T_n) is strongly operator convergent with limit T, then $T \in B(X,Y)$. [3 Marks]

Q2. (a) Show that a subset M of a normed space X is total in X if and only if every $f \in X'$ which is zero everywhere on M is zero everywhere on X. [6 Marks]

(b) Let H be a complex Hilbert space and $T \in B(H,H)$. Prove that T is normal if and only if $\|T^*x\| = \|Tx\|$ for all $x \in H$. Using this, show that for a normal linear operator T, $\|T^2\| = \|T\|^2$. [4 Marks]

(c) Consider $X = C^1[0,1]$ with the supremum norm. Is $f : X \to \mathbb{K}$ defined by $f(x) = x'(1)$, $x \in X$ a linear and continuous functional? Justify your assertion. [4 Marks]

Q3. (a) If $X \neq \{0\}$ is a complex Banach space and $T \in B(X,X)$, then prove that $\sigma(T) \neq \emptyset$. [6 Marks]

(b) Show that the dual space of ℓ^p is ℓ^q, where $\frac{1}{p} + \frac{1}{q} = 1$. [8 Marks]

Q4. (a) Let $\{x_1, \cdots, x_n\}$ be a linearly independent set of vectors in a normed space X (of any dimension). Show that there is a number $c > 0$ such that for every choice of scalars $\alpha_1, \cdots, \alpha_n$, we have [6 Marks]

$$\|\alpha_1 x_1 + \cdots + \alpha_n x_n\| \geq c(|\alpha_1| + \cdots + |\alpha_n|).$$

(b) State and prove the Hahn-Banach theorem for real vector spaces. [8 Marks]

Q5. (a) State and prove the Riesz representation theorem for bounded linear functionals. [6 Marks]

(b) Define the Hilbert-adjoint operator T^* of a bounded linear operator T. Let $T : \ell^2 \to \ell^2$ be defined by $Tx = (0, \xi_1, \xi_2, \cdots)$, $x = (\xi_j) \in \ell^2$. Show that T is an isometric operator but not unitary. Also, find T^* and $\|T^*\|$. [8 Marks]

Q6. (a) Define the resolvent set $\rho(T)$ of $T \in B(X, X)$, where X is a complex Banach space, and show that it is an open subset of the complex plane $\mathbb{C}$. [6 Marks]

(b) Let T be a bounded linear operator from a Banach space X onto a Banach space Y. Prove that the image $T(B_0)$ of the open unit ball $B_0 = B(0; 1) \subset X$ contains an open ball about 0 in Y. [8 Marks]

Q7. (a) Let X and Y be normed spaces, $T : \mathcal{D}(T) \subset X \to Y$ be a linear operator. Show that T is closed if and only if $x_n \to x$ $[x_n \in \mathcal{D}(T)]$ and $Tx_n \to y$ together imply that $x \in \mathcal{D}(T)$ and $Tx = y$. [6 Marks]

(b) Let X be a normed space and Y be a Banach space. If $T : \mathcal{D}(T) \subset X \to Y$ is a bounded closed linear operator, then prove that $\mathcal{D}(T)$ is a closed subset of X. [4 Marks]

(c) Let X be a complex Banach space, $T : X \to X$ a linear operator, and $\lambda \in \rho(T)$. If T is closed, then show that $R_\lambda(T)$ is defined on the whole space X and is bounded. [4 Marks]

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.Sc. Mathematics Examinations, May-June 2024
Part I Semester II
MMATH18-204: FLUID DYNAMICS
UPC 223501204

Time: 3 hours | Maximum Marks: 70

Instructions: • Write your roll number on the space provided at the top of this page immediately on receipt of this question paper • Question No. 1 is compulsory and answer any **four** questions from the rest of questions no. 2 to 7. • All the symbols have their usual meaning unless otherwise mentioned.

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(1) (a) Define the acceleration of a moving fluid and reduce it for a flow of potiential kind. [3]

(b) Classify all type of forces acting on a fluid. [3]

(c) Doublets of strengths μ_1, μ_2 are situated at points A_1, A_2 whose cartesian coordinates are $(0, 0, c_1)$, $(0, 0, c_2)$, their axes being directed towards and away from the origin respectively. Find the condition that there is no transport of fluid over the surface of the sphere $x^2 + y^2 + z^2 = c_1c_2$ [3]

(d) Define Stokes stream function and explain its physical significance. [3]

(e) Write the tensor form of Navier-Stokes equation. [2]

(2) (a) Show that vortex lines and tubes cannot originate or terminate at internal points in a fluid. [3]

(b) For a fluid moving in a fine tube of variable section A, prove from first principles that the equation of continuity is

$$A\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial s}(A\rho v) = 0$$

where v is the speed at a point P of the fluid and s the length of the tube up to P. [4]

(c) Show that at any point P of a moving inviscid fluid the pressure p is same in all directions. [7]

(3) (a) If ϕ_P denotes the potential at any point P of the fluid, then prove that $\phi_P \to C$ as $P \to \infty$, where C is a constant. [7]

(b) State and prove Kelvin's circulation theorem. Also show that in a circuit $\mathscr{C}$ of fluid particles moving under the same conditions as in the Kelvin's circulation theorem, $\int_S$ [illegible] dS is constant, where S is an open surface with $\mathscr{C}$ as rim. [5+2]

(4) (a) State and prove Weiss's sphere theorem. [5]

(b) Determine the Stokes's stream function for uniform line source along $\overrightarrow{OZ}$. [5]

(c) Determine the condition for a surface to be a possible boundary for a moving fluid. [4]

(5) (a) Define the coefficient of viscosity and determine its dimension. [2]

(b) Define a line vortex. Also, obtain the complex velocity potential for a line doublet. [2+4]

(c) State and prove theorem of Blasius. [6]

(6) (a) Define the stress matrix. Show that the stress matrix is diagonally symmetric and contains only six unknowns. [2+5]

(b) Find relations between stress and rate of strain. Also, write the tensor form of stresses. [5+2]

(7) (a) Find the dimension of Stokes's stream function. [2]

(b) Assuming there is no slip between the fluid and either boundary and neglecting body forces, deteremine the mean velocity of viscous incomressible fluid under steady flow between parallel planes. [6]

(c) If $w = f(z)$ is the complex potential for a two-dimensional motion having no rigid boundaries and no singularities of flow within the circle $|z| = a$, show that on introducing the rigid boundary $|z| = a$ into the flow, the new complex potential is given by $w = f(z) + \bar{f}(a^2/z)$ for $|z| \geq a$. [6]