Your Roll Number: ....

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations, May 2023
Part II Semester IV

MMath18- 401(i): Advanced Group Theory (UPC 223502401)

Time: 3 hours | Maximum Marks: 70

Instructions: • Write your roll number on the space provided at the top of this page immediately on receipt of this question paper • **Q 1** is compulsoty • Answer **any four** questions from Q2 to Q7. • Each question carries 14 marks.

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(1) (a) Show that the additive group of rational numbers $\mathbb{Q}$ has neither ascending chain condition (acc) nor descending chain condition (dcc). [$3\frac{1}{2}$ Marks]

(b) Show that a free abelian group with two generators is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. [$3\frac{1}{2}$ Marks]

(c) Let H be a nontrivial normal subgroup of a p-group G. Show that $H \cap Z(G) \neq 1$. [$3\frac{1}{2}$ Marks]

(d) Let π be set of primes. Define π-group and give an example that, for $|\pi| \geq 2$, a π-subgroup of a group need not exist. [$3\frac{1}{2}$ Marks]

(2) (a) Let G be a finite group and $H \subseteq Z(G)$. Prove that $F(G/H) = F(G)/H$, where $F(G)$ denotes Fitting subgroup of G. [6 Marks]

(b) Let G be a finite group having p-complement for every prime divisor p of $|G|$. If G has a nontrivial proper normal subgroup then show that G is solvable. [8 Marks]

(3) (a) Prove that minimal normal subgroup of a group of odd order is elementary abelian. [8 Marks]

(b) Let G be a nilpotent group of class 2. Prove that for $a \in G$, the map $\phi : G \to G$, defined by $\phi(x) = [a, x]$, is a homomorphism. Deduce that $C_G(a) \lhd G$. [6 Marks]

(4) (a) Give three examples of indecomposable groups which are not simple. Justify. [6 Marks]

(b) Define presentation of a group. Give two presentations of a cyclic group of order 15. [8 Marks]

(5) (a) Let A, B and C be subgroups of a group G. If two of the commutator subgroups $[A, B, C]$, $[B, C, A]$ and $[C, A, B]$ are contained in a normal subgroup N of G then show that third commutator is also contained in N. [6 Marks]

(b) Let ϕ and ψ be two normal endomorphisms on a finite simple group G. If $\phi + \psi$ is an endomorphism of G then show that it is nilpotent. [8 Marks]

(6) (a) Prove that every finite group has a unique maximal normal nilpotent subgroup. [9 Marks]

(b) Let G be a finite group such that $G/\Phi(G)$ is nilpotent, where $\Phi(G)$ is the Frattini subgroup of G. Show that G is nilpotent. [5 Marks]

(7) (a) Let T be a group of order 12 which is generated by two elements x and y such that $x^6 = 1$ and $y^2 = x^3 = (xy)^2$. Show that T is a semidirect product of $\mathbb{Z}_3$ by $\mathbb{Z}_4$. [6 Marks]

(b) Show that every two normal series of an arbitrary group have refinements that are equivalent. [8 Marks]

Department of Mathematics
University of Delhi, Delhi

M.A./M.Sc. Mathematics, Part-II, Semester-IV
Examination, May 2023

Paper: **MMATH18 401(ii) Algebraic Number Theory**
Unique Paper Code: 223502402

Time: 3 Hour *Maximum Marks: 70*

Note: • Attempt five questions in all. • Question 1 is compulsory. • All questions carry equal marks.

1. (a) Let $K = \mathbb{Q}(\theta)$ be a number field of degree n and $m \in \mathbb{Z}$. Show that $\Delta[1, \theta, \ldots, \theta^{n-1}] = \Delta[1, \alpha, \ldots, \alpha^{n-1}]$, where $\alpha = m + \theta$. (3)

   (b) Let K be a number field of degree 2 such that all the elements of K have a non-negative norm. Prove that K must be imaginary. (2)

   (c) Prove that every non-zero prime ideal of $\mathcal{O}_K$, the ring of algebraic integers of a number field K, contains exactly one prime number. (3)

   (d) Is the set $L = \{m + en \mid n, m \in \mathbb{Z}\}$, where $e = \sum_{n=1}^{\infty} \frac{1}{n!}$, a lattice in $\mathbb{R}$? Justify. (3)

   (e) Give an example of a number field whose class number is different from 1. (3)

2. (a) Let K be a number field and $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ a $\mathbb{Q}$-basis of K consisting of algebraic integers. Show that the discriminant $\Delta(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a non-zero integer. (9)

   (b) Find all monomorphisms and conjugate fields of (i) $\mathbb{Q}(\sqrt[3]{p})$ and (ii) $\mathbb{Q}(\sqrt[4]{p})$, where p is a prime number. (5)

3. (a) Find an integral basis and the discriminant of the number field $K = \mathbb{Q}(\sqrt{d})$, where d is a square-free integer. (10)

   (b) For $K = \mathbb{Q}(\sqrt[4]{3})$ and $L = \mathbb{Q}(\sqrt{3})$ calculate $N_{K/L}(\sqrt{3})$ and $N_{K/\mathbb{Q}}(\sqrt{3})$. (4)

4. (a) Give some necessary and sufficient condition under which the ring of algebraic integers of a number field K is norm Euclidean. (7)

   (b) Show that 6 and $2(1 + \sqrt{-5})$ both have 2 and $1 + \sqrt{-5}$ as factors but they do not have a highest common factor in $\mathbb{Z}[\sqrt{-5}]$. Do they have a least common multiple? Justify. (7)

5. (a) Let K be a number field and $\mathcal{O}_K$ its ring of algebraic integers. Prove that every non-zero ideal of $\mathcal{O}_K$ can be uniquely written as a product of finitely many prime ideals of $\mathcal{O}_K$. (10)

   (b) Let K be a number field and I an ideal of $\mathcal{O}_K$, the ring of algebraic integers of K. Prove that if m is the least positive integer in I, then $m \mid N(I)$. (4)

6. (a) Find all fractional ideals of $\mathbb{Z}$ and $\mathbb{Z}[\sqrt{-1}]$. (4)

   (b) State and prove Minkowski's Theorem. (6)

   (c) Show that the volume of a fundamental domain of a given lattice is independent of the set of generators chosen. (4)

7. (a) Let K be a number field and $\mathcal{O}_K$ its ring of algebraic integers. Prove that every non-zero ideal A of $\mathcal{O}_K$ is equivalent to an ideal whose norm is $\leq (2/\pi)^t\sqrt{|\Delta_K|}$ and use it to deduce that class number of K is finite. (8)

   (b) Let K be a number field of degree $n = s + 2t$, with $\mathcal{O}_K$ its ring of algebraic integers. Suppose that for every prime $p \in \mathbb{Z}$ with $p \leq (\frac{4}{\pi})^t \frac{n!}{n^n}\sqrt{|\Delta_K|}$ every prime ideal of $\mathcal{O}_K$ dividing $\langle p \rangle$ is principal. Prove that K has class number 1. Use it to show that the class number of $\mathbb{Q}(\zeta)$ is 1, where ζ is a primitive 5th root of unity. (6)

Your Roll Number:...........

Department of Mathematics, University of Delhi
M.A./ M.Sc. Mathematics, Semester Exam, May 2023
Part-II, Semester IV
MMATH18-401(iii) : Simplicial Homology Theory
Unique paper code: 223502403

Time: 3 hours — Max marks: 70

Instructions: • All the symbols have their usual meaning. • Question No.1 in **Part - A** is compulsory.
• Answer any four questions from remaining six questions in **Part - B**.

Part A, Coumpulsory:

1. Choose the appropriate options:

(a) Let $A = \{a_0, a_1, \ldots, a_k\}, k \geq 1$ be a geometrically independent subset of $\mathbb{R}^n$, n sufficiently large and $\mathbb{H}$ be a hyperplane generated by A. Then which of the following statements are correct:

A. $\mathbb{H}$ must be a subspace of $\mathbb{R}^n$.
B. Barycentric coordinates of elements of $\mathbb{H}$ must be unique.
C. Dimension of $\mathbb{H}$ can not be $k-1$.
D. The hyperplane generated by A may not be unique.

(b) Let K and L be two triangulations of a polyhedron X. Then which of the following statements are correct:

A. $C_p(K) \cong C_p(L)$, for all $p \geq 0$.
B. Homology groups of K are independent of orientation upto isomorphism.
C. $\text{mesh}(K) = \text{mesh}(L)$.
D. $H_p(K) \cong H_p(L)$, for all $p \geq 0$.

(c) Which of the following statements are correct:

A. The 1^{st} Betti number of torus is 2.
B. The Euler's character of torus is zero.
C. The 2^{nd} homology group of torus is trivial.
D. Torus is not a 2-pseudomanifold.

(d) Let $f : \mathbb{S}^n \to \mathbb{S}^n$, $n \geq 1$ be a continuous map. Then which of the following statements are correct:

A. If $\deg(f) = \pm 1$, then f has a fixed point.
B. $\mathbb{S}^n$ is not contractible .
C. If f is the antipodal map, then $\deg(f) = (-1)^{n+1}$.
D. The degree of f can not be zero.

(e) Which of the following statements are correct:

A. A 2-simplex is a simple polyhedron.
B. For a simplicial complex K, the homologous relation on $Z_p(K)$ is an equivalence relation.
C. If $\phi : K \to L$ is a simplicial approximation to a continuous map $f : |K| \to |L|$, then $f(\text{st}(v)) \subseteq \text{st}(\phi(v))$ for all vertex $v \in K$.
D. Homotopic spaces are homeomorphic.

(f) Let K and L be two simplicial complexes in $\mathbb{R}^n$. Which of the following statements are correct:

A. If $|K|$ is convex then the 0^{th} homology group of K is trivial.

B. If $f : |K| \to |L|$ is a homeomorphism then the induced map $f_p^* : H_p(K) \to H_p(L)$ is a group isomorphism for all $p \geq 0$.

C. If $|K|$ is convex, then $|K|$ has fixed point property.

D. $|L|$ is homeomorphic to $\mathbb{R}^n$.

(g) Choose the incorrect statements:

A. $\mathbb{D}^n, n \geq 1$ is a retract of $\mathbb{R}^n$.

B. $\mathbb{D}^3$ is a rectilinear polyhedron.

C. A simple regular polyhedron can have 8-vertices, 12-edges and 6-faces.

D. Every polyhedron is a rectilinear polyhedron.

(2 X 7 =14)

Part B, Attempt any FOUR questions from the following:

2. (a) Show that the interior of a simplex is an open set in the hyperplane spanned by the vertices of that simplex. (10)

   (b) Prove that $\lim_{n\to\infty} \text{mesh}(K^n) = 0$, where K^n is the n^{th} barycentric subdivision of K. (4)

3. (a) Let K be an oriented simplicial complex and σ^{p-2} be a $p-2$ face of a simplex σ^p of K. Show that $\sum[\sigma^p, \sigma^{p-1}][\sigma^{p-1}, \sigma^{p-2}] = 0$, where summation is over all $p-1$ simplexes σ^{p-1} of K. (6)

   (b) Let $K = \{\langle a_0\rangle, \langle a_1\rangle, \langle a_2\rangle, \langle a_3\rangle, \langle a_0, a_1\rangle, \langle a_1, a_2\rangle, \langle a_0, a_2\rangle, \langle a_1, a_3\rangle, \langle a_2, a_3\rangle, \langle a_0, a_1, a_2\rangle\}$ be a simplicial complex with orientation $a_0 < a_1 < a_2 < a_3$. Compute all homology groups of K. (8)

4. (a) If $\phi_p, \psi_p : C_p(K) \to C_p(L)$ are chain homotopic chain mappings from $C(K)$ to $C(L)$ then show that the induced homomorphisms $\phi_p^*, \psi_p^* : H_p(K) \to H_p(L)$ are equal for all $p \geq 0$.(5)

   (b) Let K and L be two simplicial complexes and $f : |K| \to |L|$ be a continuous map. Define the induced sequence of homomorphism $f_p^* : H_p(K) \to H_p(L)$, $p \geq 0$. Prove that two homeomorphic spaces have same homology groups upto an isomorphism and conclude that $\mathbb{S}^n$ is not homeomorphic to $\mathbb{S}^m$ for $n \neq m$. (2+5+2)

5. Define a coherent orientation on an n-pseudomanifold K. Show that $\mathbb{S}^n, n \geq 1$ admits a coherent orientation. Compute all homology groups of $\mathbb{S}^n$. (2+4+8)

6. (a) State and prove Brower's degree theorem. (10)

   (b) Use the Brower's degree theorem to show that every polynomial of positive degree with complex coefficients has a root in $\mathbb{C}$. (4)

7. (a) Let S be a simple polyhedron with V vertices, E edges and F faces. Then show that $V - E + F = 2$. (7)

   (b) State Lefschetz fixed point theorem and from this deduce Brower's fixed point theorem. (2+5)

DEPARTMENT OF MATHEMATICS
M. A./M. SC. MATHEMATICS PART (II) -SEMESTER IV
FINAL EXAMINATION, MAY, 2023
MMATH18- 402(I) ABSTRACT HARMONIC ANALYSIS, UPC: 223502405

Time: 3 HOURS | Maximum Marks: 70

• Attempt **five** questions in all. • **Question No. 1 is compulsory.** Attempt any **four** from the remaining six questions. • All the questions carry equal marks. • All the symbols have their usual meanings.

(1) (a) Prove that the left regular representation of a locally compact group G on $L^2(G)$ is unitary. (3)
(b) For $\mu \in M(G)$, define μ^*, and prove that $\|\mu^*\| = \|\mu\|$. (3)
(c) For a locally compact abelian group G, prove that dual group $\hat{G}$ is an abelian group. (3)
(d) Prove that $(f * g) * h = f * (g * h)$, for all $f, g, h \in L^1(G)$. (3)
(e) Prove that the function $\phi : \mathbb{R} \to \mathbb{C}$ defined as $\phi(x) = e^{i\sqrt{2}x}$, $x \in \mathbb{R}$ is positive definite. (2)

(2) (a) Let G be a locally compact abelian group, $\mu \in M(\hat{G})$ be a positive measure and $\phi_\mu : G \to \mathbb{C}$ be defined as $\phi_\mu(x) = \int_{\hat{G}} \chi(x)\, d\mu(\chi)$, $x \in G$. Prove that ϕ_μ is continuous and is of positive type. (3+4)
(b) Let π_1 and π_2 be isomorphic irreducible unitary representations of G. Prove that $C(\pi_1, \pi_2)$ is one dimensional. (7)

(3) (a) Let G be a locally compact group and $x \in G$. Prove that there exists a positive real number $\Delta(x)$ such that for every left Haar measure μ on G, $\mu(Ex) = \Delta(x)\mu(E)$, E being a Borel set. Further prove that the map $\Delta : G \to \mathbb{R}^+$ is a group homomorphism. (4+3)
(b) For $n \in \mathbb{N} \cup \{0\}$, consider the unitary representation $\pi_n : SU(2) \to B(H_n)$ defined as $\pi_n(A)(f(z)) = f(zA)$, $A \in SU(2)$, H_n being the space of homogenous polynomials of degree n of two variables. Prove that π_n is irreducible. (7)

(4) (a) Let G be a compact group, $[\pi] \in \hat{G}$ and $f \in L^2(G)$. Define $\hat{f}$, the Fourier transform of f. Prove that (1+3+3)
(i) $\widehat{f * g}(\pi) = \hat{g}(\pi)\hat{f}(\pi)$, $g \in L^2(G)$
(ii) $\widehat{L_x f}(\pi) = \hat{f}(\pi)\pi(x^{-1})$, $x \in G$.
(b) Let G be a locally compact group. Prove that there exists a norm decreasing mapping from $L^1(G)$ into $M(G)$, the Banach algebra of complex regular Borel measures on G. (7)

(5) (a) Let ρ be a non-degenerate $*$-representation of $L^1(G)$ on a Hilbert space H, G being a locally compact group. Prove that ρ is an integrated representation of a unitary representation of G on H. (10)

(b) Let π be a unitary representation of a locally compact group G, $u \in H_\pi$. Prove that the function ϕ defined as $\phi(x) = \langle \pi(x)u, u \rangle, x \in G$ is of positive type. (4)

(6) (a) Prove that the dual group of $\mathbb{Z}$ is (group) isomorphic to the circle group S^1. (6)
(b) State and prove Schur's orthogonality relations. (2+6)

(7) (a) Prove that $\{\chi_\pi : [\pi] \in \hat{G}\}$ is an orthonormal basis of $ZL^2(G)$. (8)
(b) Let $\mathcal{E} = span\{\pi_{ij} : \pi$ is a finite dimensional unitary representation of $G, i, j = 1, \dots d_\pi\}$, G being a compact group. Prove that $\mathcal{E}$ is a subalgebra of $C(G)$. (6)

Your Roll Number: .... No. of Printed Pages **2**

M.A./M.Sc. Mathematics Examinations, May 2023
Part II Semester IV
MMATH18-402(ii): Frames and Wavelets

Time: 3 hours Maximum Marks: 70

Instructions: • Write your roll number on the space provided at the top of this page immediately on receipt of this question paper. • **Attempt FIVE questions in all.** • **Question No. 1 is Compulsory.** • Each question carries 14 marks. • $\mathcal{H}$ denote an infinite dimensional separable Hilbert space.
• Symbols have their usual meaning.

Question No. 1 is Compulsory

(1) (a) Give an example, with justification, of a sequence in $\ell^2(\mathbb{N})$ which is not Bessel but satisfies the lower frame condition. [3 Marks]

(b) Give an example, with justification, of a frame for an infinite dimensional Hilbert space which is ω-independent but minimal. [3 Marks]

(c) What is the relation between the optimal frame bounds of a frame $\{f_k\}_{k=1}^{m}$ for $\mathbb{C}^n$ and eigenvalues of the frame operator of $\{f_k\}_{k=1}^{m}$? Explain. [4 Marks]

(d) Define Gabor frame for $L^2(\mathbb{R})$. Give an example, with justification, of a Gabor frame for $L^2(\mathbb{R})$ which is not exact. [1+3=4 Marks]

Answer any Four questions from Q. No. 2 to Q. No. 7.

(2) (a) Show that if $\{f_k\}_{k=1}^{m}$ is a frame for $\mathbb{C}^n$, then there exist $n-1$ vectors $\{h_j\}_{k=2}^{n} \subset \mathbb{C}^n$ such that $\{f_k\}_{k=1}^{m} \bigcup \{h_j\}_{k=2}^{n}$ forms a tight frame for $\mathbb{C}^n$. [6 Marks]

(b) State and prove the frame algorithm. Also give an application, with justification, of the frame algorithm. [6 +2 =8 Marks]

(3) (a) Show that if $\{f_k\}_{k=1}^{\infty} \subset \mathcal{H}$ and B is a positive real number such that $\sum_{k=1}^{\infty} |\langle f_j, f_k\rangle| \leq B$ for each $j \in \mathbb{N}$, then $\{f_k\}_{k=1}^{\infty}$ is a Bessel sequence with Bessel bound B. [6 Marks]

(b) State and prove the frame minus one theorem. [2+6=8 Marks]

(4) (a) State and prove the Haar reconstruction formula. [6 Marks]

(b) Show that every frame for $\mathcal{H}$ is a multiple of a sum of three orthonormal bases for $\mathcal{H}$. [8 Marks]

(5) (a) Show that if $\{f_k\}_{k=1}^{\infty}$ is a frame for $\mathcal{H}$ with frame operator S, then $\{S^{-1/2}f_k\}_{k=1}^{\infty}$ is a Parseval frame for $\mathcal{H}$. [5 Marks]

(b) Give an example, with justification, of a frame for $\ell^2(\mathbb{N})$ which is not a Riesz basis for $\ell^2(\mathbb{N})$. What is the relation between Schauder bases and Riesz bases? Justify your answer. [3+6=9 Marks]

(6) (a) Show that the coefficient functionals associated with a Schauder basis for an infinite dimensional Banach X space are continuous. [4 Marks]

(b) Prove that if $\{f_k\}_{k=1}^{\infty}$ be a frame for $\mathcal{H}$ with frame operator S, then dual frames of $\{f_k\}_{k=1}^{\infty}$ are precisely the families [10 Marks]

$$\{g_k\}_{k=1}^{\infty} = \left\{ S^{-1}f_k + h_k - \sum_{j=1}^{\infty}\langle S^{-1}f_k, f_j\rangle h_j \right\}_{k=1}^{\infty},$$

where $\{h_k\}_{k=1}^{\infty}$ is a Bessel sequence in $\mathcal{H}$.

(7) (a) Show that if range of the synthesis operator associated with a sequence $\{f_k\}_{k=1}^{\infty} \subset \mathcal{H}$ is closed, then $\{f_k\}_{k=1}^{\infty}$ is a frame sequence. [6 Marks]

(b) Define multiresolution analysis (MRA). Give an example, with justification, of an MRA in $L^2(\mathbb{R})$. [2+6=8 Marks]

Your Roll Number: ....

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations, May 2023
Part II Semester IV
MMATH18-402 (iii): Operators on Hardy Hilbert Space
Unique Paper Code : 223502407

Time: Three hours Maximum Marks: 70

Instructions: • Write your roll number on the space provided at the top of this page immediately on receipt of this question paper • Question 1 is compulsory • Attempt 4 questions from the remaining six questions • All symbols have their usual meaning.

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(1) (a) Give a non-trivial example of a function which is in $H^\infty(\mathbb{D})$. Does it also lie in $H^2(\mathbb{D})$? (2)

(b) Let $T : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ be given by $(Tf)(z) = z^2 f(z), z \in \mathbb{D}$. Check whether or not T is an isometry. (3)

(c) Give an example of (i) an inner function which has exactly four distinct zeros in $\mathbb{D}$, (ii) an inner function which has a countably infinite number of zeros in $\mathbb{D}$, justifying your answers. (3)

(d) Let $T = 2I - 3U$, where U is the unilateral shift on $\tilde{H}^2(S^1)$. Show that T is a Toeplitz operator and find its symbol. (3)

(e) Find the matrix of the operator $W + J$, where W is the bilateral shift and J is the flip operator, w.r.t the standard orthonormal basis of $L^2(S^1)$. (3)

Section B

Attempt four questions in all from this section.

(2) (a) State Fatous' Theorem. Use it to show that if $f \in H^2(\mathbb{D})$ then $\lim_{r\to 1^-} f\left(re^{i\theta}\right) = \tilde{f}(e^{i\theta})$, for almost all θ, where $\tilde{f}$ is the boundary function of f. (5)

(b) Let $f(z) = \frac{1}{3}(1 + z^2), z \in \mathbb{D}$. Verify that $f \in H^2(\mathbb{D})$ and find $\tilde{f}$ the boundary function of f. Is f an inner function? (3)

(c) Show that any non-zero function in $H^2(\mathbb{D})$ may be written as a product of an inner and an outer function. (6)

(3) (a) Let $M_z : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ be given by $(M_z(f))(z) = zf(z), z \in \mathbb{D}$. Show that this operator is bounded and find its adjoint. (5)

(b) Prove that the bilateral shift W defined on $l^2(\mathbb{Z})$ is unitarily equivalent to the operator M on $L^2(S^1)$ where M is given by $(Mf)(e^{i\theta}) = e^{i\theta} f(e^{i\theta})$. Find the adjoint W^* of W acting on $l^2(\mathbb{Z})$. (7)

(c) Is the subspace $\tilde{H}^2(S^1)$ of $L^2(S^1)$ invariant under the adjoint of the bilateral shift? Justify. (2)

(4) For $\phi \in L^\infty(S^1)$ let M_ϕ be defined by $M_\phi f = \phi f$ for $f \in L^2(S^1)$.

(a) Prove that a bounded linear operator A on $L^2(S^1)$ is equal to M_ϕ if and only if the matrix of A with respect to the standard basis in $L^2(S^1)$ is a Toeplitz matrix. (4)

(b) Find the commutant of the unilateral shift acting on $\tilde{H}^2(S^1)$.(7)

(c) Show that $JPJ = M_{e_1}(I - J)M_{e_{-1}}$ where J is the flip operator and P the orthogonal projection of $L^2(S^1)$ onto $\tilde{H}^2(S^1)$. (3)

(5) (a) When is the product of two Toeplitz operators again a Toeplitz operator? Justify. (6)

(b) Find a $\psi \in L^\infty(S^1)$ such that $T_\phi T_\psi = T_\psi T_\phi$ when (i) $\phi(e^{i\theta}) = e^{3i\theta} - 2e^{i\theta}$(ii) $\phi(e^{i\theta}) = e^{-3i\theta} + e^{-i\theta}$ (iii) $\phi(e^{i\theta}) = e^{-i\theta} + 1 + e^{i\theta}$. (6)

(c) Find the adjoint of the Toeplitz operator T_ϕ where $\phi \in L^\infty(S^1)$. (2)

(6) (a) Prove a necessary and sufficient condition, (involving the unilateral shift) for an operator A to have a Hankel matrix with respect to the standard basis of $\tilde{H}^2(S^1)$. (4)

(b) Show that $k_{\bar{w}} \otimes k_w$ is a Hankel operator of rank one, where $w \in \mathbb{D}$. Find a symbol of such a Hankel operator. (6)

(c) Show that for every $\phi \in L^\infty(S^1)$, $H_{\phi^*} = H_\phi^*$. (4)

(7) (a) For $f \in L^2(S^1)$, let $\breve{f}(e^{i\theta}) = f(e^{-i\theta})$. For $\phi, \psi \in L^\infty$, suppose that there exist constants a, b such that $P\breve{\phi} = ak_{\bar{w}}$ and $P\breve{\bar{\psi}} = bk_w$. Show that then both H_ϕ and H_ψ are multiples of the operator $k_{\bar{w}} \otimes k_w$. (6)

(b) Prove that the product of two Hankel operators is a Toeplitz operator only if at least one of the Hankel operators is 0. (5)

(c) For $\phi, \psi \in L^\infty$, show that

$$H_{e_1\breve{\phi}} H_{e_1\psi} = T_{\phi\psi} - T_\phi T_\psi.$$

(3)

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Roll Number: ....

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI

M.A./M.Sc. Examinations, May-2023

MMATH18-402 (iv): THEORY OF UNBOUNDED OPERATORS

(UPC NO. 223502408)

Time: 3 hours | Maximum Marks: 70

Instructions: • Attempt **five** questions in all. • Question 1 is compulsory. • All the symbols have their usual meaning unless otherwise specified.

Throughout this paper, H will be a complex Hilbert space and X will be a Banach space.

Q1. (a) For a densely defined linear operator $T : \mathcal{D}(T) \subset H \to H$, show that $\mathcal{N}(T^*) = \mathcal{R}(T)^{\perp}$. [3 Marks]

(b) Show that the Hilbert-adjoint operator T^* of a densely defined linear operator $T : \mathcal{D}(T) \subset H \to H$ is closed. [3 Marks]

(c) Prove that the infinitesimal generator of a uniformly continuous semigroup is bounded. [4 Marks]

(d) Let $T(t)$ and $S(t)$ be C_0 semigroups of bounded linear operators with infinitesimal generators A and B respectively. If $A = B$ then show that $T(t) = S(t)$ for $t \geq 0$. [4 Marks]

Q2. (a) Let $T : H \to H$ be a linear operator such that $\langle Tx, y\rangle = \langle x, Ty\rangle$ for all $x, y \in H$. Prove that T is bounded. [8 Marks]

(b) Let $T : \mathcal{D}(T) \to \ell^2$, where $\mathcal{D}(T) \subset \ell^2$ consists of all $x = (\xi_j)$ with only finitely many nonzero terms ξ_j and $y = (\eta_j) = Tx = (j\xi_j)$. Show that T is unbounded. Check whether T is closed or not. Justify your answer. [6 Marks]

Q3. (a) Let $T : \mathcal{D}(T) \subset H \to H$ be a self-adjoint densely defined linear operator. Show that a number λ belongs to the resolvent set $\rho(T)$ of T if there exists a $c > 0$ such that for every $x \in \mathcal{D}(T)$, $\|(T - \lambda I)x\| \geq c\|x\|$. [8 Marks]

(b) If $T : \mathcal{D}(T) \subset H \to H$ is a symmetric linear operator, show that its Cayley transform exists and is isometric. [6 Marks]

Q4. (a) Show that the multiplication operator $T : \mathcal{D}(T) \subset L^2(-\infty, +\infty) \to L^2(-\infty, +\infty)$, $x \mapsto tx$, has no eigenvalues and the spectrum $\sigma(T)$ of T is all of $\mathbb{R}$. [8 Marks]

(b) Give an example to show that the resolvent set of the infinitesimal generator of a C_0 semigroup of contractions need not contain more than the open right half-plane. [6 Marks]

Q5. State and prove the Lumer–Phillips Theorem. [14 Marks]

Q6. (a) Prove that a linear operator A is the infinitesimal generator of a C_0 semigroup $\{T(t) : t \geq 0\}$ satisfying $\|T(t)\| \leq M$ $(M \geq 1)$ if and only if [8 Marks]

(i) A is closed and $D(A)$ is dense in X.

(ii) The resolvent set $\rho(A)$ of A contains $\mathbb{R}^+$ and

$$\|R(\lambda : A)^n\| \leq \frac{M}{\lambda^n}; \text{ for } \lambda > 0,\ n = 1, 2, \ldots.$$

(b) Let $\{T(t) : t \geq 0\}$ be a C_0 semigroup of bounded operators. If $0 \in \rho(T(t_0))$ for some $t_0 > 0$ then show that $0 \in \rho(T(t))$ for all $t > 0$, and $T(t)$ can be embedded in a C_0 group. [6 Marks]

Q7. (a) Prove that a linear operator A is the infinitesimal generator of a C_0 group of unitary operators on a Hilbert space H if and only if iA is self-adjoint. [8 Marks]

(b) Let A be a linear densely defined operator in X. If $\lambda \in \rho(A)$, then prove that $\lambda \in \rho(A^*)$ and $R(\lambda : A^*) = R(\lambda : A)^*$. [6 Marks]

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Your Roll No:....................

M.A/M.Sc. Mathematics, Part-II, Sem-IV (May, 2023)
MMATH18-403(i), Calculus on $\mathbb{R}^n$
Unique Paper Code: 223502409

Time : 3 hours | Maximum Marks : 70

Question No. **1** is compulsory. Attempt any **four** questions from Question Nos. **2** to **7**.
Unless otherwise mentioned, U will be an open subset of $\mathbb{R}^n$.

(1) (a) Let $f(x,y,z) = x^2yz - xyz^3$ be a function and $a = (-2,-1,1)$. Find the directional derivative of f at a with respect to vector $u = (0,4,-3)$. [3]

(b) Consider the following function $f : \mathbb{R}^2 \to \mathbb{R}$ [3]

$$f(x,y) = \begin{cases} 1 & \text{if } xy = 0 \\ 0 & \text{if } xy \neq 0. \end{cases}$$

Find the points at which f is continuous and the points at which $\frac{\partial f}{\partial x}$ exists.

(c) State the implicit function theorem. [2]

(d) Suppose α and β are k- and m-forms, respectively, of class C^1 in U. If α is closed and β is exact, then prove that $\alpha \wedge \beta$ is also exact. [3]

(e) Suppose $k \geq 2$ and $\sigma = [P_0, P_1, \cdots, P_k]$ is an oriented affine k-simplex. Prove that $\partial(\partial\sigma) = 0$. [3]

(2) (a) Let $f : U \to \mathbb{R}^n$ be a one-one function of class C^r. Further, let $Df(x)$ be non-singular at each point $x \in U$ and let $V = f(U)$. Show that f^{-1} is continuous on V. [7]

(b) Prove that if U is also connected, and $f : U \to \mathbb{R}$ is differentiable such that $Df(x) = 0$ for all $x \in U$, then f is a constant. Is still f a constant if U is not connected? Justify your answer. [5]

(c) Let ω and λ be a k- and m- forms respectively. Show that $\omega \wedge \lambda = (-1)^{km}\lambda \wedge \omega$. [2]

(3) (a) Let ω be a k-form and λ be a m-form of class C^1 in U. Prove that [5]

$$d(\omega \wedge \lambda) = d\omega \wedge \lambda + (-1)^k \omega \wedge d\lambda.$$

(b) Suppose that $F : U \to \mathbb{R}^n$ is of class $C^1, 0 \in U, F(0) = 0$ and $F'(0)$ is invertible. For $1 \leq m \leq n$ show that there is a neighborhood V_m of $0 \in \mathbb{R}^n$ and a C^1-map $F_m : V_m \to \mathbb{R}^n$ such that $F_m(0) = 0$, $F'_m(0)$ is invertible and $P_{m-1}F_m(x) = P_{m-1}(x)$ for all $x \in V_m$, where P_m is the projection of first m-components. [5]

(c) Let $f : \mathbb{R}^5 \to \mathbb{R}^3$ be of class $C^1, a = (1,0,-1,2,-2), f(a) = \mathbf{0}$ and [4]

$$Df(a) = \begin{bmatrix} 2 & -1 & 0 & 1 & 0 \\ 2 & 0 & 1 & -1 & 0 \\ 0 & 1 & 2 & 1 & -1 \end{bmatrix}.$$

(i) Show that there is a neighborhood V of $(1,0)$ in $\mathbb{R}^2$ and a C^1-map $\varphi : V \to \mathbb{R}^3$ such that $\varphi(1,0) = (-1,2,-2)$ and $f(x,\varphi(x)) = 0, \forall x \in V$.

(ii) Find $D\varphi(1,0)$.

(4) (a) Let ω be an exact 1-form in U and γ be a closed smooth curve in U. Prove that $\omega(\gamma) = 0$. [3]

(b) Let $a \in U$. Show that $f : U \to \mathbb{R}^n$ is differentiable at a if and only if each component function f_i of f for $1 \le i \le n$ is differentiable at a. [4]

(c) Define $T(s,t) = (x,y)$, where $x = s - st$ and $y = st$. Let [7]

$$S = \{(s,t) | 0 < s < 1,\ 0 < t < 1\}$$

be a square. Show that T is one-one on S and find $Q = T(S)$.
Let $f(x,y) = \frac{e^{x+y}}{x+y}$ be defined on Q, then find the integration f on the region Q by using the change of variable with respect to T.

(5) (a) Let $\omega = \sum_I b_I(x) dx_I$ be the standard presentation of a k-form in U. If $\omega = 0$ in U, then prove that $b_I(y) = 0$ for every increasing k-index I and for every $y \in U$. [7]

(b) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by

$$(x,y,z) = T(u,v) = \frac{(2u, 2v, u^2 + v^2 - 1)}{1 + u^2 + v^2}$$

and $\omega = x dy \wedge dz + y dz \wedge dx + z dx \wedge dy$ be a 2-form in $\mathbb{R}^3$. Compute the pull back ω_T. [5]

(c) Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be such that $f(a) = (1,2)$ and [2]

$$Df(a) = \begin{bmatrix} -2 & 1 & 2 \\ -1 & 1 & 3 \end{bmatrix}.$$

Let $g : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $g(x,y) = (x^2 - y^2, 4xy, y^3)$. Find $D(g \circ f)(a)$.

(6) (a) Suppose ω is a k-form in U, ϕ is a k-surface in U with parameter domain $D \subset \mathbb{R}^k$ and Δ is the identity map on D, prove that [4]

$$\int_\phi \omega = \int_\Delta \omega_\phi.$$

(b) Let $T : \mathbb{Q}^n \to \mathbb{R}^n$ be one-one and C^2 map such that $\det J_T(y) > 0$ for all $y \in \text{int}(\mathbb{Q}^n)$, where $\mathbb{Q}^n$ is the standard n-simplex in $\mathbb{R}^n$. Let $E = T(\mathbb{Q}^n)$. Define the positive oriented boundary of the set E. Find the positive oriented boundary of the unit square $[0,1]^2 \subset \mathbb{R}^2$. [5]

(c) Define the functions ψ_n for $-2 \le n \le 2$, on $\mathbb{R}$ as follows

$$\psi_0(x) = \begin{cases} 1 + 2x, & \text{if } -\frac{1}{2} \le x < 0, \\ 1 - 2x, & \text{if } 0 \le x < \frac{1}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

and $\psi_n = \psi_0(x - \frac{n}{2})$. Verify that the family $\{\psi_n, -2 \le n \le 2\}$ defines a partition of unity for the compact set $[-1,1]$ subordinate to the cover $\{(\frac{n}{2} - 1, \frac{n}{2} + 1), -2 \le n \le 2\}$. Sketch the graphs of ψ_n's. [5]

(7) (a) Let $T : U \to \mathbb{R}^m$ be a C^2-map and ω be a k-form of class C^1 in $\mathbb{R}^m$. Then prove that $d(\omega_T) = (d\omega)_T$. [4]

(b) Let ω be a k-form of class C^2 in U. Prove that $d^2\omega = 0$. [3]

(c) State the Stoke's theorem. Define $\Phi : [0,1]^2 \subset \mathbb{R}^2 \to \mathbb{R}^3$ as $\Phi(u,v) = (u^2, uv, v^2)$. Let $\alpha = x dy + x dz + y dz$ be a 1-form in $\mathbb{R}^3$. Calculate $\int_\Phi d\alpha$ and $\int_{\partial\Phi} \alpha$ and verify Stoke's theorem. [7]

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examinations, May 2023
Part II Semester IV

MMATH18-403(ii): DIFFERENTIAL GEOMETRY
(Unique Paper Code 223502410)

Time: 3 hours Maximum Marks: 70

Instructions: • Attempt **five** questions in all. **Question 1 is compulsory**. All questions carry equal marks.• The symbols used have their usual meanings.

(1) (a) Let S be an n-surface in $\mathbb{R}^{n+1}$ and α be a parameterized curve in S. Show that velocity vector field along α is parallel if and only if α is a geodesic. [3]

(b) Show that the 1-form $\eta = \frac{-x_2}{x_1^2+x_2^2}dx_1 + \frac{x_1}{x_1^2+x_2^2}dx_2$ in $\mathbb{R}^2 \setminus \{0\}$ is not exact. [4]

(c) Let S be an oriented n-surface in $\mathbb{R}^{n+1}$ and $\tilde{S}$ be the same surface with opposite orientation. Let $p \in S$. If L_p and $\tilde{L}_p$ are the Wiengarten maps at p of S and $\tilde{S}$ respectively, then what is the relation between L_p and $\tilde{L}_p$? Justfy your answer. [2]

(d) If $U \subseteq \mathbb{R}^n$ is open and $\varphi : U \to \mathbb{R}^m$ is a smooth map, define the differential $d\varphi$ of φ. Compute the differential $d\psi$ for the function $\psi : U \to \mathbb{R}^{n+1}$ defined by $\psi(p) = (p, f(p))$, where $f : U \to \mathbb{R}$ is a smooth map. [1+2]

(e) Let S be an oriented n-surface in $\mathbb{R}^{n+1}$, let $p \in S$ and $\{k_1(p), k_2(p), \ldots, k_n(p)\}$ be the principal curvatures of S at p with the corresponding principal curvature directions $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$. Show that the normal curvature $k(\mathbf{v})$ in the direction $\mathbf{v} \in S_p$ is given by $k(\mathbf{v}) = \sum_{i=1}^{n} k_i(p)(\mathbf{v}.\mathbf{v}_i)^2$. [2]

(2) (a) Let S be an n-surface in $\mathbb{R}^{n+1}$, let $p, q \in S$ and let $\alpha : [a, b] \to S$ be a piecewise smooth curve from p to q. Show that the parallel transport $P_\alpha : S_p \to S_q$ is a vector space isomorphism that preserves dot product. [7]

(b) Show that the Weingarten map at each point of a parameterized n-surface in $\mathbb{R}^{n+1}$ is self adjoint. [7]

(3) (a) Define derivative of a smooth vector field $\mathbf{X}$ on an n-surface S in $\mathbb{R}^{n+1}$ with respect to $\mathbf{v} \in S_p$. If $\mathbf{X}$ is a unit vector field, show that $\nabla_\mathbf{v}\mathbf{X} \perp \mathbf{X}(p)$ for all $\mathbf{v} \in S_p$, $p \in S$. If $\mathbf{X}$ is also a tangent vector field on S, then show that for any tangent vector field $\mathbf{Y}$ on S and for all $\mathbf{v} \in S_p, p \in S$, $\nabla_\mathbf{v}(\mathbf{X}.\mathbf{Y}) = (D_\mathbf{v}\mathbf{X}).\mathbf{Y}(p) + \mathbf{X}(p).(D_\mathbf{v}\mathbf{Y})$, and hence conclude that $D_\mathbf{v}\mathbf{X} \perp \mathbf{X}(p)$. [2+5]

(b) Prove that on a compact oriented n-surface S in $\mathbb{R}^{n+1}$ there exists a point p such that the second fundamental form at p is definite. [7]

(4) (a) If S is the ellipsoid $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = 1$, $a, b, c > 0$, oriented by its inward normal, determine its Gaussian curvature $K(p)$. [7]

(b) Let $S = f^{-1}(c)$ be an n-surface in $\mathbb{R}^{n+1}$, where $f : U \to \mathbb{R}$ is such that $\nabla f(q) \neq 0$, for all $q \in S$. Suppose $g : U \to \mathbb{R}$ is a smooth function and $p \in S$

is an extreme point of g on S. Show that there is a real number λ such that $\nabla g(p) = \lambda \nabla f(p)$. [3]

(c) Let $a, b, c \in \mathbb{R}$ be such that $ac - b^2 > 0$. Show that the maximum and minimum values of the function $g(x_1, x_2) = x_1^2 + x_2^2$ on the ellipse $ax_1^2 + 2bx_1x_2 + cx_2^2 = 1$ are of the form $1/\lambda_1$ and $1/\lambda_2$, where λ_1 and λ_2 are the eigenvalues of the matrix [4]

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}.$$

(5) (a) Let S be an oriented n-surface in $\mathbb{R}^{n+1}$ and $L_p : S_p \to S_p$ be the Weingarten map. Show that there is an orthonormal basis for S_p consisting of eigenvectors of L_p. [6]

(b) For $\theta \in \mathbb{R}$, let $\alpha_\theta : [0, \pi] \to \mathbb{S}^2$ be the parameterized curve from the north pole $p = (0, 0, 1)$ to the south pole $q = (0, 0, -1)$ defined by

$$\alpha_\theta(t) = (\cos\theta \sin t, \sin\theta \sin t, \cos t), \ 0 \le t \le \pi.$$

If $\mathbf{v} = (p, 0, 1, 0) \in S_p$, determine $P_{\alpha_\theta}(\mathbf{v})$. [8]

(6) (a) Let $U = \{(x_1, x_2) \in \mathbb{R}^2 : {x_1}^2 + {x_2}^2 < 1521\}$. Find the length of the connected oriented plane curve $f^{-1}(0)$ oriented by $-\frac{\nabla f}{\|\nabla f\|}$, where $f : U \to \mathbb{R}$ is defined by $f(x_1, x_2) = 12x_1 + 5x_2$. [6]

(b) Let C be a curve in $\mathbb{R}^2$ which lies above the x_1- axis. Define the surface of revolution obtained by rotating the curve C about the x_1- axis and show that it is indeed a 2-surface in $\mathbb{R}^3$. [5]

(c) If a parameterized curve α in the unit n-sphere $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$ is a geodesic, then show that it is of the form $\alpha(t) = (\cos at)v + (\sin at)w$ for some orthonormal pair of vectors $\{v, w\}$ in $\mathbb{R}^{n+1}$ and some $a \in \mathbb{R}$. [3]

(7) (a) Let S be a compact, connected oriented n-surface in $\mathbb{R}^{n+1}$ such that $S = f^{-1}(c)$, where $f : \mathbb{R}^{n+1} \to \mathbb{R}$ is smooth and $\nabla f(q) \neq 0$ for all $q \in S$. Prove that the Gauss map maps S onto the unit sphere $\mathbb{S}^n$. [10]

(b) Let $\alpha(t) = (x(t), y(t))$, $t \in I$ be a local parameterizations of the oriented plane curve C. Show that the curvature κ satisfies $\kappa \circ \alpha = \frac{x'y'' - y'x''}{[(x')^2 + (y')^2]^{3/2}}$. [4]

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Your Roll Number: ....

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations, May 2023
Part- II Semester- IV
MMATH18-403(iii): Topological Dynamics
(Unique Paper Code 223502411)

Time: **3 hours** Marks: **70**

Instructions: • All notations used are standard • **Question no.** 1 is **compulsory** • Attempt any **four** questions from the remaining six questions.

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(1) Do as directed.

(a) Is product of two minimal homeomorphisms defined on a compact metric space a minimal homeomorphism? Justify. (3)

(b) Give an example to justify that the expansiveness of a homeomorphism depends upon the choice of metric in general. (3)

(c) If two $k \times k$ matrices A and B with entries in $\{0, 1\}$ are irreducible, then is it necessary that their corresponding subshifts are topologically conjugate? Justify. (3)

(d) Is Sarkovskii's theorem true for any continuous map $f : \mathbb{S}^1 \to \mathbb{S}^1$ having a periodic point of prime period three? Justify. (3)

(e) Justify that $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x + \sqrt{5}$ does not have POTP. (2)

(2) (a) Let A be a $k \times k$ matrix with entries in $\{0, 1\}$ such that no row/column of A is full of zeros. Prove that the shift map σ on X_A is transitive if and only if the digraph associated to A is strongly connected. (11)

(b) With proper justification, find stable sets of fixed points of $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$. (3)

(3) (a) For the logistic map $F_\mu : \mathbb{R} \to \mathbb{R}$ defined by $F_\mu(x) = \mu x(1 - x)$, prove that

(i) for $1 < \mu \leq 2$ and $x \in (0, 1)$, $\lim_{n\to\infty} F_\mu^n(x) = p_\mu$,

(ii) For $\mu > 1$ and $x < 0$ or $x > 1$, $\lim_{n\to\infty} F_\mu^n(x) = -\infty$. (7)

(b) Let X be a compact metric space and $f : X \to X$ be continuous. Prove that for any $x \in X$, $\omega(x) = \bigcap_{m\geq 0} \left(\overline{\bigcup_{n\geq m} \{f^n(x)\}}\right)$ and deduce that $\omega(x)$ is non-empty. Justify that if X is non-compact then $\omega(x)$ may be empty also. (7)

(4) (a) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and J, K closed and bounded intervals of $\mathbb{R}$ such that $K \subseteq f(J)$ then prove that there exists a closed and bounded interval J_0 of $\mathbb{R}$, $J_0 \subseteq J$ such that $f(J_0) = K$. (5)

(b) Use (a) to prove that if f has a periodic point of prime period three then f has a periodic point of all prime period $n \geq 1$. (9)

(5) (a) Prove that the closed unit disc $\mathbb{D} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ does not admit any expansive homeomorphism. (8)

(b) Let (X, d) be a subspace of a metric space (Y, d) and $h : X \to X$ be an expansive homeomorphism with expansive constant δ. Prove that a homeomorphic extension f of h to Y is expansive on Y with expansive constant δ if the following conditions are satisfied:

(i) $f_{|(Y-X)} : (Y - X) \to (Y - X)$ is expansive with expansive constant δ.

(ii) There exists a basis $\mathscr{B}$ of X with respect to h such that $d(x, (Y - X)) > \delta$, for every $x \in \mathscr{B}$. (6)

(6) (a) Let (X, d) be a compact metric space and $f : X \to X$ be a homeomorphism. If f^{-1} has POTP then prove that f^{-1} has canonical coordinates and f^k has POTP for any $k \geq 1$. (8)

(b) If (X, d) is a compact metric space and $f : X \to X$ is a homeomorphism satisfying that for every $\epsilon > 0$, there exists a $\delta > 0$ such that every finite $\delta-$pseudo orbit $(z_i)_{0 \leq i \leq k}$, $k \in \mathbb{N}$ is $\epsilon-$traced by some point of X then prove that f has POTP. (6)

(7) (a) Prove that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \frac{1}{\sqrt{2}}x$ is a topologically Anosov homeomorphism. (5)

(b) Let (X, d) be a compact metric space and $f : X \to X$ be an expansive homeomorphism having POTP. Prove that f is topologically stable in the class of self-homeomorphisms of X. (9)

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examinations, May 2023
Part II Semester IV

MMATH18-404(i): ADVANCED FLUID DYNAMICS

Time: 3 Hour | Maximum Marks: 70

Instructions: • Attempt FIVE questions in all. • Q.No.1 is compulsory. • Attempt any FOUR questions from Q.No.2 to Q.No.7. Each question carries 14 marks. • All the symbols have their usual meaning.

(1) (a) Show that under MHD approximation the displacement current density is negligible in the Ampere's circuital law. [3 Marks]

(b) Write the Prandtl's boundary layer equations and boundary conditions in term of stream function $\psi(x, y)$. [3 Marks]

(c) Define equation of state of a compressible substance. Write the van der Waal's equation of state for a non-ideal gas and simplify it. [3 Marks]

(d) Check whether the heat added Q and entropy S per unit mass of an ideal gas are function of state or not? Give justification for your answer. [2 Marks]

(e) Define shock Mach number, shock strength, weak and strong shock. [3 Marks]

(2) (a) Reduce the Navier Stokes equation of motion into non-dimensional form. Explain physically the obtained non-dimensional numbers. [7 Marks]

(b) State the principle of conservation of energy for a fluid flow. Derive the internal energy equation $\rho\frac{De}{Dt} = \frac{p}{\rho}\frac{D\rho}{Dt} - \nabla.\vec{q} + \overleftrightarrow{\tau} : (\nabla\vec{w})^t$. [7 Marks]

(3) (a) Show that $c_p - c_v = -T(\frac{\partial v}{\partial T})_p^2(\frac{\partial p}{\partial v})_T$. Also find the corresponding relation for the non-ideal gas with equation of state $p(v - b) = RT$. [4 Marks]

(b) Investigate the maximum mass flow through a nozzle and find the condition for maximum isentropic mass flow at the exit plane. [5 Marks]

(c) Define normal and oblique shock wave. Prove that the shock wave is compressive in nature. [5 Marks]

(4) (a) Write the mass, momentum and energy equation for one dimensional motion of an inviscid gas in the integral form and derive the corresponding shock conditions. [3+6 Marks]

(b) Derive the magnetic field intensity equation for a conducting fluid in motion. Explain physically each term of the equation. [5 Marks]

(5) (a) Derive the equation of motion of a non-viscous conducting fluid $\frac{\partial(\rho\vec{v})}{\partial t} + \rho(\vec{v}.\nabla)\vec{v} = \overrightarrow{f(ex)} + \frac{\mu}{4\pi}(\vec{H}.\nabla)\vec{H} - \nabla(p + \frac{\mu\vec{H}^2}{8\pi}) - \vec{v}\nabla.(\rho\vec{v})$. Write the corresponding equation for viscous and incompressible fluid. [7 Marks]

(b) Show that the Lorentz's force per unit volume of a conducting fluid is equivalent to the hydro-static pressure $\frac{\mu H^2}{8\pi}$ and a tension $\frac{\mu H^2}{4\pi}$ per unit area along the line of force. [7 Marks]

(6) (a) Explain the Alfven's wave. Show that the MHD wave propagate with the speed $\sqrt{a^2 + V_A^2}$, where V_A is Alfven's velocity. [5 Marks]

(b) Define the displacement, momentum and energy thickness of a boundary layer and find the expression for each of them on a flat plate. Compute and compare the three thicknesses over a flat plate for linear velocity distribution given by [6+3 Marks]

$$u(y) = \begin{cases} \frac{U_\infty}{\delta}, & 0 < y < \delta \\ U_\infty, & \delta < y < \infty, \end{cases}$$

where, U_∞ is uniform stream velocity.

(7) (a) Derive the Prandtl's boundary layer equations along with the boundary conditions for two dimensional viscous incompressible fluid flow over a slender body. Also write the corresponding equations for the steady flow. [8 Marks]

(b) Derive the energy integral equation for steady, two dimensional boundary layer flow of incompressible fluid. [6 Marks]

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Your Roll Number: ....

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examinations, May 2023
Part II Semester IV,UPC 223502413

MMATH18-404(ii): COMPUTATIONAL METHODS FOR PDES

Time: 3 hours Maximum Marks: 70

Instructions: • Attempt **five** questions in all • Question number **one** is compulsory and attempt any **four** from the remaining • Each question carries 14 marks • Scientific non-programming calculator for doing numerical calculations is allowed for this examination • Notations and acronyms have their usual meaning.

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(1) (a) State True or False and justify: Consistency of a finite difference scheme does not imply convergence. [1+3 Marks]

(b) Write the Lax-Wendroff multistep scheme for the equation $u_t + au_x = 0$. What is its order of accuracy? [3+1 Marks]

(c) State True or False and justify: A non-constant harmonic function in D cannot take it's maximum value in D but only on boundary ∂D. [1+3 Marks]

(d) Write the convection-diffusion equation and give it's physical interpretation. [1+1 Marks]

(2) (a) Find the solution of the initial value problem [7 Marks]

$$u_t + au_x = f(t, x), \text{ where } a \text{ is positive}$$

with $u(0, x) = 0$ and $f(t, x) = \begin{cases} 1 \; if \; -1 \le x \le 1 \\ 0 \; otherwise. \end{cases}$

(b) Consider system $\begin{pmatrix} u^1 \\ u^2 \end{pmatrix}_t + \begin{pmatrix} a & b \\ b & a \end{pmatrix} \begin{pmatrix} u^1 \\ u^2 \end{pmatrix}_x = 0$ [7 Marks]

on the interval $[0, 1]$, with a equal to 0 and b equal to 1 and with the boundary conditions u^1 equal to 0 at the left and u^1 equal to $1 + t$ at the right boundary. Find the solution for $0 \le x + t \le 3$ if the initial data are given by $u^1(0, x) = x$ and $u^2(0, x) = 1$.

(3) (a) Find the truncation error and stability by Von-Neumann analysis of the Leapfrog scheme for the equation $u_{tt} - a^2 u_{xx} = 0$, where a is non-negative real number. [3+4 Marks]

(b) Find the solution of the initial boundary value problem [7 Marks]

$$u_{tt} = u_{xx}, \; 0 \le x \le 1$$

subject to the initial conditions

$$u(x, 0) = sin(\pi x), \; 0 \le x \le 1$$
$$u_t(x, 0) = 0, \; 0 \le x \le 1$$

and the boundary conditions $u(0, t) = u(1, t) = 0, \; t > 0$
by using the scheme $\frac{1}{k^2}\delta_t^2 v_m^n = \frac{1}{h^2}\delta_x^2 v_m^n$
with $h = \frac{1}{4}$ and $\lambda = \frac{3}{4}$. Compute the solution for four time steps.

(4) (a) Discuss the Von-Neumann stability analysis of the BTCS and Crank-Nicolson schemes for the equation $u_t = bu_{xx} + f(t,x)$, where $b > 0$. [3+4 Marks]

(b) Find the solution of the two dimensional heat conduction equation $u_t = u_{xx} + u_{yy}$ valid in $0 \leq x, y \leq 1$, $t \geq 0$. The initial and boundary conditions are given as [7 Marks]

$$u(x, y, 0) = sin2\pi x\ sin2\pi y, \quad 0 \leq x, y \leq 1$$
$$u(x, y, t) = 0, \ t > 0, \text{ on the boundary}$$

Solve the above equation using the Peaceman-Rachford ADI scheme with $h = \frac{1}{3}, \lambda = 1$. Integrate for one time step.

(5) (a) Draw a finite difference grid in the $r - \theta$ plane and use it to discretize the following equations: [2+5 Marks]

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = f(r, \theta)$$
$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = f(r, \theta)$$

where $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$.

(b) Find the order of accuracy and truncation error of a finite difference scheme with equal step length h for the equation $\nabla^2 u = 0$, where $u = u(x, y, z)$. [2+5 Marks]

(6) (a) Explain linear iterative methods in solving finite difference schemes for Laplace equation in a rectangle and analyse them to determine their relative rates of convergence. [3+4 Marks]

(b) Solve the boundary value problem [7 Marks]

$$\nabla^2 u = x^2 - 1, \quad |x| \leq 1, \quad |y| \leq 1$$
$$u = 0 \text{ on the boundary of the square}$$

using the five point difference scheme by taking the mesh size $h = \frac{1}{2}$.

(7) (a) Derive the Galerkin finite element method for the equation $A(x, y)u_{xx} + 2B(x, y)u_{xy} + C(x, y)u_{yy} + D(x, y)u_x + E(x, y)u_y + F(x, y)u + G(x, y) = 0, (x, y) \in R$ subject to the Dirichlet condition $u(x, y) = 0, (x, y) \in S$ and hence deduce it for the poisson equation $\nabla^2 u = g(x, y)$. [4+3 Marks]

(b) Find the Galerkin finite element solution of the boundary value problem [7 Marks]

$$\nabla^2 u = -1, \quad |x| \leq 1, \quad |y| \leq 1$$
$$u = 0, \qquad |x| = 1, \quad |y| = 1$$

with $h = \frac{1}{2}$ and a triangular mesh.

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Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations, May 2023
Part II Semester IV
MMATH18-404(iii): Dynamical Systems
UPC: 223502414

Time: 3 hours | Maximum Marks: 70

Instructions: • Question number **1** is compulsory and attempt **any Four** questions from the remaining. • Each question carries 14 marks.

(1) (a) Solve the initial value problem $\dot{x} = Ax$ with [2 Marks]

$$A = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}.$$

(b) Determine the nature of the critical point at the origin for the following system [2 Marks]

$$\dot{x} = y, \quad \dot{y} = -x^3 + 4xy.$$

(c) Solve the linear system $\dot{x} = Ax$, where [2 Marks]

$$A = \begin{pmatrix} a & 0 \\ 0 & -1 \end{pmatrix}.$$

(d) Define $\omega-$limit set and $\alpha-$limit set of a trajectory of the system. [2 Marks]

(e) Classify the equilibrium points (as sinks, sources or saddles) of the nonlinear system $\dot{x} = f(x)$ with [2 Marks]

$$f(x) = \begin{bmatrix} x_1 - x_1x_2 \\ x_2 - x_1^2 \end{bmatrix}.$$

(f) Sketch the phase portraits corresponding to [2 Marks]

$$\dot{x} = x - cos(x).$$

and determine the stability of all the fixed points.

(g) For the Hamiltonian function $H(x, y) = ysin(x)$, determine both Hamiltonian system and gradient system. [2 Marks]

(2) (a) Write the following system in polar coordinates and determine the nature of origin [4 Marks]

$$\dot{x} = -y + x(1 - x^2 - y^2), \quad \dot{y} = x + y(1 - x^2 - y^2).$$

(b) State "The Local Center Manifold Theorem". Determine the center manifold near the origin of the system [10 Marks]

$$\begin{aligned} \dot{x_1} &= x_1y - x_1x_2^2, \\ \dot{x_2} &= x_2y - x_2x_1^2, \\ \dot{y} &= -y + x_1^2 + x_2^2. \end{aligned}$$

(3) (a) Define the term supercritical Hopf bifurcation. [2 Marks]

(b) What do you mean by Supercritical Pitchfork bifurcation for one-dimensional flow. Write its normal form. Also, discuss the stability analysis and whole mechanism by making graphs. [8 Marks]

(c) Make bifurcation diagram for supercritical pitchfork bifurcation. [4 Marks]

(4) (a) Find the stable, unstable and center subspaces E^s, E^u and E^c of the system $\dot{x} = Ax$ with the matrix [4 Marks]

$$A = \begin{bmatrix} -1 & -3 \\ 0 & 2 \end{bmatrix}.$$

(b) State and prove "The Stable manifold Theorem". [10 Marks]

(5) (a) Consider the system [4 Marks]

$$\dot{x} = -y + x^3 + xy^2$$
$$\dot{y} = x + y^3 + x^2y.$$

Show that origin is an unstable focus for this nonlinear system.

(b) Solve the two-dimensional system [10 Marks]

$$\dot{y} = -y, \quad \dot{z} = z + y^2$$

and show that the successive approximations $\Phi_k \longrightarrow \Phi$ and $\Psi_k \longrightarrow \Psi$ as $k \longrightarrow \infty$ for all $(y_1, y_2, z) \in \mathbb{R}^3$. Define $H_0 = (\Phi, \Psi)^T$ and use this homeomorphism to find

$$H = \int_0^1 L^{-s} H_0 T^s ds.$$

(6) (a) Find all the fixed points of the system [6 Marks]

$$\dot{x} = -x + x^3, \ \dot{y} = -2y.$$

Use linearization to classify them. Then, check your conclusions by deriving the phase portrait for the full non-linear system.

(b) Consider the following two-dimensional system [8 Marks]

$$\dot{x} = -y + ax(x^2 + y^2), \ \dot{y} = x + ay(x^2 + y^2).$$

where a is parameter. Show that small non-linear term can change a center into a spiral.

(7) (a) Find the *Poincaré map* of the system [7 Marks]

$$\dot{x} = -y + x(1 - x^2 - y^2), \quad \dot{y} = x + y(1 - x^2 - y^2).$$

(b) Solve the initial value problem $\dot{x} = Ax$, where [7 Marks]

$$A = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 2 & 0 & 1 & 0 \end{bmatrix}.$$

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Your Roll Number: ....

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examinations, May 30, 2023
Part II Semester IV

MMATH18-405(i): CRYPTOGRAPHY

(Unique Paper Code: 223503401)

Time: 2 Hours | Maximum Marks: 35

Instructions: • Attempt five questions in all. • Question 1 is compulsory.
• All questions carry equal marks.

(1) Explain the following:

(a) Stream ciphers. [2]

(b) Affine linear block ciphers. [2]

(c) Vernam one-time pad. [1]

(d) Discrete logarithms. [1]

(e) Public key encryption [1]

(2) Let $n \geq 3$ be an odd composite number. Show that the set $\{1, 2. \ldots, n-1\}$ contains at most $(n-1)/4$ numbers that are prime to n and not witness for the compositeness of n. [7]

(3) Describe RSA encryption as a block cipher. Also, show that $m^{ed} \equiv d\ (mod\ n)$ where (n, e) is the public RSA key and d is the corresponding private key. [7]

(4) Discuss the following:
(a) ElGamal Encryption scheme. [3.5]
(b) Rabin Encryption scheme. [3.5]

(5) Explain the following ciphers with examples:

ECB & CBC [3.5 + 3.5]

(6) Explain the working of S-boxes and round key generations $K_1, K_2, \ldots, K_{16}$ in DES algorithm. [7]

(7) Describe the encryption and decryption in Feistel cipher in detail. [7]

Department of Mathematics, University of Delhi

M.A./M.Sc. Mathematics Final Examination, 2023

Part II Semester IV

MMATH18 405(ii): SUPPORT VECTOR MACHINES

Unique Paper Code: 223503402

Time: 2 hours | Maximum Marks: 35

Instructions: • Attempt five questions in all. • Question 1 is compulsory.
• Answer any four questions from Q. 2 to Q. 7. • All questions carry equal marks.
• All notations have standard meaning.

(1) (a) Define hard $\bar{\varepsilon}$-band hyperplane for a linear regression problem. [2 Marks]

(b) Define graph kernel based on the path. [2 Marks]

(c) Write the dual of the primal problem (P) considered in Q. 3. [3 Marks]

(2) Consider the convex programming problem (CP) [7 Marks]

$$\text{Minimize } f_0(x)$$
$$\text{subject to } f_i(x) \leq 0, i = 1, 2, ..., m,$$
$$h_i(x) = a_i^T x_i - b_i = 0, i = 1, 2, ..., p,$$

where $f_0, f_i, i = 1, ..., m$ are continuous convex functions on $\mathbb{R}^n$ and $a_i \in \mathbb{R}^n, b_i \in \mathbb{R}, i = 1, ..., p$. Derive its dual problem. Is the dual a convex programming problem? State weak duality for (CP).

(3) Consider the linearly separable problem (P) [7 Marks]

$$\text{Minimize } \frac{1}{2}\|w\|^2$$
$$\text{subject to } y_i((w.x_i) + b) \geq 1, i = 1, 2, ..., l,$$

corresponding to the training set $T = \{(x_1, y_1), (x_2, y_2), ..., (x_l, y_l)\}$. Prove that (P) has a solution (w^*, b^*) such that $w^* \neq 0$. Also, prove that there exists a $j \in \{i|y_i = 1\}$ and a $k \in \{i|y_i = -1\}$ such that $(w^*.x_j) + b^* = 1$ and $(w^*.x_k) + b^* = -1$, respectively.

(4) (a) Give an algorithm for linearly separable support vector classification. [5 Marks]

(b) What do you mean by a support vector in linearly separable support vector classification? [2 Marks]

(5) (a) Explain how one can construct a hard $\bar{\varepsilon}$-band hyperplane using the classification method through an example. [4 Marks]

(b) Consider the problem (P_w) [3 Marks]

$$\text{Minimize } \frac{1}{2}\|w\|^2$$
$$\text{subject to } (w.x_i) + b - y_i \leq \varepsilon, i = 1, 2, ..., l,$$
$$y_i - (w.x_i) - b \leq \varepsilon, i = 1, 2, ..., l,$$

corresponding to the training set $T = \{(x_1, y_1), (x_2, y_2), ..., (x_l, y_l)\}$. Suppose that ε_{inf} is the optimal value of the following problem

$$\text{Minimize } \varepsilon$$
$$\text{subject to } -\varepsilon \leq y_i - (w.x_i) - b \leq \varepsilon, i = 1, 2, ..., l.$$

Prove that if $\varepsilon > \varepsilon_{inf}$ then the problem (P_w) has a solution and the solution with respect to w is unique.

(6) (a) If $K_1(x, x')$ and $K_2(x, x')$ are kernels on $\mathbb{R}^n \times \mathbb{R}^n$ then prove that the product $K(x, x') = K_1(x, x').K_2(x, x')$ is also a kernel. [3 Marks]

(b) Prove that Gaussian radial basis function with parameter σ given by $K(x, x') = exp(-\|x - x'\|^2/\sigma^2)$ is a kernel. [4 Marks]

(7) (a) Suppose $\alpha^* = (\alpha_1^*, ..., \alpha_l^*)^T$ is a solution to the problem [7 Marks]

$$\text{Minimize } \frac{1}{2}\sum_{i=1}^{l}\sum_{j=1}^{l} y_i y_j K(x_i, x_j)\alpha_i\alpha_j - \sum_{j=1}^{l}\alpha_j$$
$$\text{subject to } \sum_{i=1}^{l} y_i\alpha_i = 0,$$
$$0 \leq \alpha_i \leq C, i = 1, 2, ..., l,$$

corresponding to the training set $T = \{(x_1, y_1), (x_2, y_2), ..., (x_l, y_l)\}$. If $g(x) = \sum_{i=1}^{l} y_i\alpha_i^* K(x_i, x) + b^*$, then prove that

(i) support vector x_i corresponding to $\alpha_i^* \in (0, C)$ satisfies $y_i g(x_i) = 1$,

(ii) support vector x_i corresponding to $\alpha_i^* = C$ satisfies $y_i g(x_i) \leq 1$,

(iii) nonsupport vector x_i satisfies $y_i g(x_i) \geq 1$.

Also give a geometrical interpretation of conditions in (i)-(iii).