Your Roll Number: ....

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examinations, August, 2023
Part I Semester II
MMATH18-201: MODULE THEORY
Unique Paper Code: 223501201

Time: 3 hours Maximum Marks: 70

Instructions: • Write your roll number on the space provided at the top of this page immediately on receipt of this question paper • Attempt **five** questions in all • **Question No. 1 is compulsory**. Attempt any four from the remaining six questions • All the questions carry equal marks • All the symbols have their usual meanings.

(1) (a) Prove that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_n, \mathbb{Z}) \cong \{0\}$. [3]

(b) Define a projective module. Give an example with justification. [3]

(c) Define isotypic module. Give example of an isotypic module and a non-isotypic module. [3]

(d) Determine $\mathbb{Z}_n \otimes \mathbb{Z}_m$ if $gcd(n, m) = 1$, $n, m \in \mathbb{N}$. [3]

(e) Define a composition series in a module M. Give examples of two different composition series in some module M. [2]

(2) (a) State and prove Short Five Lemma. [2+5]

(b) Let R be a principal ideal domain and M be a torsion module over R. Prove that $M = \oplus M_p$ where each M_p is p-primary submodule, p being prime in R. [7]

(3) (a) Prove that an abelian group M is a left R-module if and only if there exists an anti-ring homomorphism $f : R \to EndM$. [6]

(b) Let C be any category and Ens be the category of all sets. Determine (with justification) a contravariant functor from C into Ens. [3]

(c) Let R be a P.I.D. and M be a left R-module such that $M = \oplus_{i=1}^{n} M_i$. If each M_i is bounded and cyclic with pairwise co-prime bounds, prove that M is bounded and cyclic. [5]

(4) (a) Prove that a left R-module I is injective if and only if for any $f \in Hom_R(A, I)$ there exists $f' \in Hom_R(R, I)$ such that $f' = f$ on A, A being a left ideal of R. [5]

(b) For left R-modules M, N and P, prove that [6]

$$(M \oplus N) \otimes P \cong (M \otimes P) \oplus (N \otimes P).$$

(c) Prove or disprove: Subring of a Noetherian ring is Noetherian. Justify your answer. [3]

(5) (a) Prove that every submodule of a semisimple module M is complemented in M. [7]

(b) Prove that a short exact sequence $0 \to M' \xrightarrow{\lambda} M \xrightarrow{\mu} M'' \to 0$ splits if and only if λ has a right inverse. [7]

(6) (a) State and prove the Universal Property of free modules. [2+6]

(b) Let M be a semisimple left R-module and H_α denote the α-socle of M. Prove that $End_R(M)$ is (group) isomorphic to $\prod_\alpha End_R(H_\alpha)$. [6]

(7) (a) Let $\{M_i; i \in I\}$ be a family of left R-modules. State and prove the Universal Property of direct product of M_i's. [2+6]

(b) Explain by an example that an Artinian module need not be Noetherian. [6]

Your Roll Number: .....

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations, August 2023
Part I Semester II
MMATH18-202: Introduction to Topology (UPC-223501202)

Time: 3 hours | Maximum Marks: 70

Instructions: • Write your roll number on the space provided at the top of this page immediately on receipt of this question paper • **Q 1** is compulsoty • Answer **any four** questions from Q2 to Q7. • Each question carries 14 marks

---

(1) (a) Let $\mathbb{R}$ be the set of real numbers with cofinite topology. Show that the relative topology on the subset $\mathbb{Q} \subset \mathbb{R}$ is cofinite topology. [4 Marks]

(b) Let X and Y be two spaces. Give an example to show that the projection map $p : X \times Y \to Y$ need not be a closed map. [3 Marks]

(c) Prove or Disprove: (1) The unit interval $[0, 1]$ is a compact subset of $\mathbb{R}$ with topology induced from discrete metric on $\mathbb{R}$. (2) The *Sorgenfrey line* $\mathbb{R}_l$ is connected. [4 Marks]

(d) Show that every metric space is first countable. [3 Marks]

(2) (a) Prove that G is open in a space X iff $\overline{G \cap \overline{A}} = \overline{G \cap A}$ for every subset A of X. [7 Marks]

(b) Let $\mathcal{S}$ be a nonempty family of subsets of a set X. Determine the smallest topology on X containing $\mathcal{S}$. [7 Marks]

(3) (a) Give an example of a continuous bijection between two spaces which is not a homeomophism. [5 Marks]

(b) Let $X_\alpha, \alpha \in A$, be an infinite family of spaces. Define *box topology* and *product topology* on $\prod X_\alpha$. Give an example of a property that holds under the product topology but does not hold under the box topology on $\prod X_\alpha$. [6 Marks]

(c) Let Y be a subspace of a space X and $A \subset Y$. Show [3 Marks] that $A'_Y = A'_X \cap Y$, where A'_X and A'_Y are the derived sets of A in X and Y, respectively.

(4) (a) Let X and Y be connected spaces. If A and B are proper subsets of X and Y, respectively, then show that $X \times Y - A \times B$ is connected. [6 Marks]

(b) Let X be a sapce and for all $x, y \in X$, there exists a connected subset C_{xy} of X containing x and y. Show that X is connected. [4 Marks]

(c) Give an example to show that continuous image of a locally connected space need not be locally connected. Justify. [4 Marks]

(5) (a) Show that an infinite Hausdorff space has infinitely many disjoint open sets. [7 Marks]

(b) Show that if X is a Hausdorff space then every sequence in X converges to atmost one point in X. Is the converse true ? Justify. [3+4 Marks]

(6) (a) Show that the set $\mathbb{R}$ of real numbers with cofinite topology is not first countable. [4 Marks]

(b) Prove or Disprove: An open subspace of a separable space is seperable. [3 Marks]

(c) Show that countable product of seperable spaces is seperable. [7 Marks]

(7) (a) Let X and Y be two compact spaces. Show that $X \times Y$ is compact. [7 Marks]

(b) Let X be any space and the projection map $p : X \times Y \to Y$ be a closed map for all spaces Y. Show that X is a compact space. [7 Marks]

✠

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examinations, August 2023
Part I Semester II
MMATH18-203: FUNCTIONAL ANALYSIS
(Unique Paper Code 223501203)

Time: 3 hours — Maximum Marks: 70

Instructions: • Attempt five questions in all. **Question 1 is compulsory**. All questions carry equal marks.• The symbols used have their usual meanings.

(1) (a) If X and Y are normed spaces and $T : X \to Y$ is a bounded linear operator, show that T maps bounded sets in X to bounded sets in Y. [2]

(b) If X is a complex inner product space and $T : X \to X$ is a bounded linear operator such that $\langle Tx, x\rangle = 0$ for all $x \in X$, then show that $T = 0$. [3]

(c) Give an example, with justification, to show that in a normed space weak convergence of a sequence need not imply strong convergence. [3]

(d) If X is a normed space and $x, y \in X$ such that $f(x) = f(y)$ for all $f \in X'$, then show that $x = y$. [2]

(e) Give an example , with justification, to show that a bounded linear operator need not be closed. [2]

(f) Define the linear operator $T : l^2 \to l^2$ by $T(\xi_1, \xi_2, \dots) = (0, \xi_1, \xi_2, \dots)$. Show that T has no eigenvalues. [2]

(2) (a) If a normed space X has the property that the closed unit ball $M = \{x \in X : \|x\| \leq 1\}$ is compact, then show that X is finite dimensional. [6]

(b) Let X be a real or complex vector space and p a real valued functional on X satisfying $p(x + y) \leq p(x) + p(y)$, $p(\alpha x) = |\alpha|p(x)$ for all $x, y \in X$ and $\alpha \in \mathbb{K}$. Let f be a linear functional defined on a subspace Z of X such that $|f(z)| \leq p(z)$ for all $z \in Z$. Then show that f has a linear extension $\tilde{f}$ from Z to X such that $|\tilde{f}(x)| \leq p(x)$, $\forall x \in X$. [8]

(3) (a) Let $\{x_1, x_2, \cdots, x_n\}$ be a set of linearly independent vectors in a normed space X. Prove that there exists a $c > 0$ such that $\| \sum_{j=1}^{n} a_j x_j \| \geq c \sum_{j=1}^{n} |a_j|$ for any choice of scalars $a_1, a_2, \cdots, a_n$. [6]

(b) Let H be a Hilbert space, $M \subset H$ be a convex subset and (x_n) be a sequence in M such that $\|x_n\| \to d = \inf_{x \in M} \|x\|$. Show that (x_n) converges in H. [4]

(c) If T is a linear operator on a complex normed space X, define spectrum $\sigma(T)$ of T and classify $\sigma(T)$ into point spectrum, continuous spectrum and residue spectrum. What can you say if X is finite dimensional? Justify. [4]

(4) (a) If the dual space X' of a normed space X is separable, then show that X is separable. Deduce from this that l^1 is not reflexive. [5+3]

(b) Define $T : \mathbb{C}^2 \to \mathbb{C}^2$ by $T(\xi_1, \xi_2) = (\xi_1 + i\xi_2, \xi_1 - i\xi_2)$. Find T^* and show that $T^*T = TT^* = 2I$, where I is the identity operator on $\mathbb{C}^2$. [4]

(c) State Baire's category theorem. [2]

(5) (a) Let X be a Banach space, Y be a normed space and (T_n) be a sequence in $B(X, Y)$ which is strongly operator convergent with limit T. Then show that $T \in B(X, Y)$. Give an example, with justification, to show that the conclusion may fail if X is not complete. [4+4]

(b) If (e_k) is any orthonormal sequence in an inner product space X, show that for any $x, y \in X$, [3]

$$\sum_{k=1}^{\infty} |\langle x, e_k\rangle \langle y, e_k\rangle| \leq \|x\|\|y\|.$$

(c) If (x_n) is a sequence in a Banach space X such that $(f(x_n))$ is bounded for every $f \in X'$, show that $(\|x_n\|)$ is bounded. [3]

(6) (a) Let H_1 and H_2 be Hilbert spaces and $T \in B(H_1, H_2)$. Define the Hilbert adjoint T^* of T, show that it exists, and is a bounded linear operator with norm $\|T^*\| = \|T\|$. [7]

(b) Let X and Y be Banach spaces and $T \in B(X, Y)$ be surjective. Let $B_0 = B(0; 1)$ be the open unit ball in X. Assuming that $T(B_0)$ contains an open ball about 0 in Y, prove that T is an open mapping. [4]

(c) Let X be the space consisting of all complex sequences with only finitely many nonzero terms and norm defined by $\|x\| = \sup_j |\xi_j|$, $x = (\xi_j)$. Let $T : X \to X$ be the bounded linear operator defined by

$$Tx = \left(\frac{\xi_j}{j}\right), \quad x = (\xi_j) \in X.$$

Then show that T^{-1} is unbounded. [3]

(7) (a) Let f be a bounded linear functional on a Hilbert space H. Show that there exists a unique $z \in H$ such that $f(x) = \langle x, z\rangle$ for all $x \in H$ and that $\|z\| = \|f\|$. [7]

(b) Show that the resolvent set $\rho(T)$ of a bounded linear operator T on a complex Banach space X is open. [5]

(c) Let X and Y be normed spaces and $T \in B(X, Y)$. If (x_n) is a sequence in X such that $x_n \xrightarrow{w} x$ in X, then show that $Tx_n \xrightarrow{w} Tx$ in Y. [2]

✠

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examinations, August 2023
Part I Semester II

MMATH18-204, UPC 223501204: FLUID DYNAMICS

Time: 3 Hour　　　　Maximum Marks: 70

Instructions: • Q.No.1 in section **A** is compulsory. • Attempt any **Four** questions from section **B**. • Attempt FIVE questions in all. Each question carries 14 marks. • All the symbols have their usual meaning.

Section A

(Answer all parts, 14 Marks)

(1) (a) Derive relation between local and particle rates of change. What is a convective derivative? [2 Marks]

(b) Define stress components, stress matrix and stress tensor at a point in viscous fluid. [3 Marks]

(c) Describe the forces acting in fluid motion. [3 Marks]

(d) Compute the dimension of stream function in two dimensional motion and kinematic coefficient of viscosity. [2 Marks]

(e) Define axi-symmetric flows and Stokes stream function. [2 Marks]

(f) Derive the complex potential due to a line source of strength m at point $z = a$. [2 Marks]

Section B

(Answer any FOUR questions, 56 Marks)

(2) (a) The components of mass flux vector $\rho\vec{q}$ in a fluid motion are $\rho u = 4x^2y, \rho v = xyz, \rho w = yz^2$. Compute the net out flux of mass through a closed surface formed by the planes $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$. [4 Marks]

(b) Define boundary surface for a fluid motion. Write the equation of continuity for the following motion (i) three dimensional flow in (R, θ, z) (ii) axi-symmetric flow about Oz axis in (r, θ, ψ)(iii) spherically symmetric flow. [5 Marks]

(c) Write the difference between the streamlines and pathlines. Consider a two dimensional velocity field given by $\vec{V} = \frac{V_0}{l}(-x\hat{i} + y\hat{j})$, where V_0 and l are constants. (i) At what location the speed is equal to V_0 ? (ii) Determine the streamlines, (iii) Determine the acceleration field for this flow. [5 Marks]

(3) (a) State the principle of conservation of momentum of an inviscid fluid in a volume V. Derive the Euler's equation of motion for an inviscid fluid. [5 Marks]

(b) An infinite mass of ideal incompressible fluid is attracted towards the origin by a force $kr^{-\frac{3}{2}}$ per unit mass, where r is distance of any point from origin. Initially the fluid is at rest and there is cavity in form of sphere $r = a$ in it.If there is no pressure at infinity or in the cavity, show that the cavity will be filled after an interval of time $\left(\frac{4a^5}{25k^2}\right)^{\frac{1}{4}}$. [6 Marks]

(c) If S is the spherical surface lying wholly within a steady irrotational incompressible fluid, then prove that the mean value of velocity potential over the sphere is independent of radius of the sphere. [3 Marks]

(4) (a) State and prove the Butler sphere theorem. [5 Marks]

(b) Find the equations of the streamline due to uniform line sources of strength m through the points $(-c, 0), (c, 0)$ and a uniform line sink of strength $2m$ through the origin. [4 Marks]

(c) Prove that the sum of the normal stresses in a viscous fluid across any three perpendicular planes at a point is invariant. Define mean pressure at the point. [5 Marks]

(5) (a) Define a three dimensional doublet. Find the velocity potential and fluid velocity at point due to the doublet. [5 Marks]

(b) State and prove the Weiss's sphere theorem. [5 Marks]

(c) Find the Stokes stream function and stream surface due to a uniform distribution of simple source along a line. [4 Marks]

(6) (a) Define complex potential, complex velocity, speed of fluid and stagnation point. [3 Marks]

(b) Define two dimensional image system. Find the image of a line source of strength m in a circular cylinder of section $|z| = a$. [5 Marks]

(c) A long infinite cylinder of radius a is placed in a uniform stream $-U\hat{\imath}, \hat{\imath}$ is a unit vector along $+ve$ x-axis, and a circulation of $2\pi k$ round the cylinder is produced. Discuss the Magnus effect and calculate the forces and couple acting on the cylinder. [6 Marks]

(7) (a) Find the rate of dilation in a viscous fluid. Write the tensor form of relation between stress and rate of strain. [3 Marks]

(b) Derive the Navier-Stokes equation of motion for a viscous fluid flow. Write the corresponding equation for the incompressible fluid. [5 Marks]

(c) Find the velocity profile for steady flow of viscous incompressible fluid under no body forces through a tube of uniform circular cross-section of radius a. Find the volume of fluid discharged over any section per unit time when pressure difference at two points on the axis of tube distant l apart is p. [6 Marks]

✠