Your Roll Number: ....

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations, May 2022
Part II Semester IV

MMath18 401 (i): Advanced Group Theory (UPC 223502401)

Time: $3\frac{1}{2}$ hours — Maximum Marks: 70

Instructions: • Write your roll number on the space provided at the top of this page immediately on receipt of this question paper • **Question 1** is compulsoty • Answer **any four** questions from **Question 2** to **Question 8** • Each question carries 14 marks.

(1) (a) Let S and T be solvable subgroups of a group G with $S \triangleleft G$. Is ST solvable ? Justify [$3\frac{1}{2}$ Marks]

(b) Let G be a finite nilpotent group of order n and $m|n$. Show that G has a subgroup of order m. [$3\frac{1}{2}$ Marks]

(c) Let N be normal subgroup of a finite group G and P is a Sylow subgroup in N. Show that $G = N_G(P)N$. [$3\frac{1}{2}$ Marks]

(d) Let K be a normal subgroup of a group G, where $|G| = 105$ and $|K| = 21$. Show that G is semidirect product of K by a group Q of order 5. [$3\frac{1}{2}$ Marks]

(2) (a) Prove that a finite characteristically simple group is either simple or a direct product of isomorphic simple groups. [9 Marks]

(b) Prove that the direct product of two odd order groups is solvable. [5 Marks]

(3) (a) Let G be a group such that the higher center $\zeta^2(G) = G$. For $a \in G$, show that the map $\phi : G \to G$, defined by $\phi(x) = [a, x]$, is a homomorphism. [5 Marks]

(b) Let G be a finite solvable group of order mn, where $(m, n) = 1$. Assume that G contains a proper normal subgroup whose order is not divisible by n. [4+5 Marks]

(i) Prove that G contains a subgroup of order m.

(ii) Prove that any subgroup of G of order k, where $k|m$, is contained in a subgroup of order m.

(4) (a) Let H, K, L be subgroups of a group G and N be normal subgroup G. If $[H, K, L][K, L, H] \leq N$ then show that $[L, H, K] \leq N$. [8 Marks]

(b) Show that nilpotent group is solvable. Give an example that the converse may not be true. [6 Marks]

(5) (a) Determine *Free group* F generated by singleton set $X = \{x\}$. Is [5+3 Marks]

F indecomposable ? Justify. [6 Marks]

(b) Show that *fitting subgroup* of a finite group G is a *characteristic* subgroup of G. [8 Marks]

(6) (a) Let $\Phi(G)$ be Frattini subgroup of a finite group G. Show that g is nongenerator of G if and only if $g \in \Phi(G)$. [6 Marks]

(b) Let $H \triangleleft G$ and both H and G/H have descending chain conditions (DCC) then show G has DCC. [9 Marks]

(7) (a) Define reduced word and binary operation juxtaposition on the collection of reduced words for a given set X. Show that the collection of reduced words under the operation juxtaposition is a free group with basis X. [5 Marks]

(b) Let G be an indecomposable group having both chain conditions. Show that every normal endomorphism of G is either nilpotent or an automorphism. [9 Marks]

(8) (a) Let G be a group. Show that there is an integer n with $\zeta^n(G) = G$ iff $\gamma_{n+1}(G) = 1$, where $\zeta^i(G)$ and $\gamma_i(G)$ are higher center and lower center of group G. [5 Marks]

(b) Let π be a set of primes. Define π-number and π-group. For $|\pi| \geq 2$, give an example of a finite group G such that a π-subgroup of G need not exist.

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Department of Mathematics
University of Delhi, Delhi

M.A./M.Sc. Mathematics, Part-II, Semester-IV
Examination, May 2022

Paper: **MMATH18 401(ii) Algebraic Number Theory**
Unique Paper Code: 223502402

Time: $3\frac{1}{2}$ *Hours* *Maximum Marks: 70*

Note: • Attempt five questions in all. • Question 1 is compulsory. • All questions carry equal marks.

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1. (a) Find the discriminant of the number field $\mathbb{Q}(\theta)$, where $\theta^3 + 4\theta + 7 = 0$. (3)
   (b) Let $\alpha \in \mathcal{O}_K$, where $\mathcal{O}_K$ is the ring of integers of a number field K. Show that if $N(\alpha)$ is a rational prime, then α is irreducible. Is the converse true? Justify. (3)
   (c) Find all the associates of $\sqrt{-3}$ in the ring of integers of $\mathbb{Q}(\sqrt{-3})$. (2)
   (d) Prove that every finitely generated $\mathcal{O}_K$-submodule of the $\mathcal{O}_K$-module K is a fractional ideals of $\mathcal{O}_K$, where $\mathcal{O}_K$ is the ring of integers of the number field K. (3)
   (e) Find the prime ideal factorization of the ideal $\langle 2, 1+\sqrt{-5}\rangle$ of $\mathcal{O}_K$, where $K = \mathbb{Q}(\sqrt{-5})$. (3)

2. (a) Give an example of a number field whose conjugate fields are distinct and an example of a number field whose conjugate fields are not distinct. (4)
   (b) Let α be an algebraic number. Prove that α is an algebraic integer if and only if the minimal polynomial of α over $\mathbb{Q}$ has integer coefficients. (5)
   (c) Let K be a number field of degree n. Define an integral basis for K and show that an integral basis for K is also a $\mathbb{Q}$-basis for K. (2+3)

3. (a) Find the integral basis and discriminant of the number field $K = \mathbb{Q}(\zeta)$, where ζ is primitive p-th root of unity for some odd prime p. (10)
   (b) Let P and Q be distinct prime ideals of the ring of integers $\mathcal{O}_K$ of a number field K. Prove that $P + Q = \mathcal{O}_K$ and $P \bigcap Q = PQ$. (4)

4. (a) Prove that $\mathbb{Z}[i]$ is a unique factorization domain and use it to find all integer solutions of the equation $y^2 + 1 = 2z^3$. (5+5)
   (b) Let I be a non-zero fractional ideal of an integral integral domain R with quotient field K. Prove that $I' = \{x \in K \mid xI \subseteq R\}$ is a fractional ideal of R. (4)

5. (a) Prove that a unique factorization domain is integrally closed and hence deduce that every principal ideal domain is a Dedekind domain. (7)

(b) Let K be a number field and $\mathcal{O}_K$ its ring of integers. Prove that every non-zero fractional ideal of $\mathcal{O}_K$ is a finitely generated $\mathcal{O}_K$-module. (7)

6. (a) Prove that every discrete subgroup of $(\mathbb{R}^n, +)$ is a lattice. (6)

   (b) Let L be an n-dimensional lattice in $\mathbb{R}^n$ and $\mathbb{F}$ its fundamental domain. Prove that the translates of $\mathbb{F}$ by lattice points does not overlap. (4)

   (c) Let $K = \mathbb{Q}(\sqrt{d})$, where d is a square free integer. Calculate $\sigma : K \longrightarrow \mathbb{L}^{s,t}$, distinguishing the cases $d > 0$, $d < 0$. (4)

7. (a) Prove that an odd prime p can be expressed as a sum of two squares if and only if $p \equiv 1 \pmod 4$. (7)

   (b) Let K be a number field of degree $n = s + 2t$ with ring of integers $\mathcal{O}_K$. Prove that every non-zero ideal I of $\mathcal{O}_K$ is equivalent to an ideal B such that $N(B) \leq (2/\pi)^t \sqrt{|d_K|}$. (7)

8. (a) State and prove Dedekind's Theorem. (10)

   (b) Find the class number of $\mathbb{Q}(\zeta)$, where ζ is a primitive 7-th root of unity. (4)

Your Roll Number ...

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examination, May 2022
Part - II; Semester - IV
MMATH18-401(iii) : **Simplicial Homology Theory**
Unique Paper Code: 223502403

Time: 3 hours and 30 minutes | Max Mark: 70

Instructions: • All the symbols have their usual meaning. • Question No.1 in **Part - A** is compulsory. • Answer any four questions from remaining seven questions in **Part-B**.

Part A, Compulsory.

1. Choose the appropriate options:
   (a) Let K be the 2-skeleton of $Cl(\sigma^3)$. Then which of the following statements are true:
      A. 2nd Betti number of K is 1.
      B. There are 4 linearly independent 0-cycles with respect to integral homology group of K.
      C. K is not a triangulation of the sphere $\mathbb{S}^2$.
      D. Euler's character of K is 2.

   (b) Which of the following statements are true:
      A. Closure of $\sigma^n, n \geq 1$ is not a n-pseudomanifold.
      B. Disk $\mathbb{D}^n$, $n \geq 1$ is contractible.
      C. $\mathbb{S}^n$ is retract of $\mathbb{D}^{n+1}, n \geq 1$.
      D. $\mathbb{D}^n, n \geq 1$ is a retract of $\mathbb{R}^n$.

   (c) Which of the following statements are true:
      A. A self-homeomorphism f on $\mathbb{S}^n$, $n \geq 1$, has degree ± 1.
      B. Degree of the map $z \to -z$ on $\mathbb{S}^1$ is 1.
      C. 0th homology group of a one-point set is trivial.
      D. 0th homology group of $\mathbb{S}^n$ is $\mathbb{Z}$ for $n \geq 0$.

   (d) Let K be a simplicial complex in $\mathbb{R}^n$. Then which of the following statements are true:
      A. If K is connected then $|K|$ is path connected.
      B. Under the subspace topology from $\mathbb{R}^n$, $|K|$ is compact, Hausdroff subset of $\mathbb{R}^n$.
      C. If $v \in K$ then st(v) is open in $|K|$.
      D. For any simplex $\sigma \in K$, σ is convex hull of vertices of σ.

   (e) Choose the incorrect option:
      A. $\sigma^n, n \geq 0$ has fixed point property .
      B. If $f : \mathbb{S}^n \to \mathbb{S}^n, n \geq 1$ is the antipodal map then deg$f \neq 0$.
      C. Fixed point property is a topological property.
      D. σ^2 is a simple polyhedron.

(f) Let $X = \{v_0, v_1, \dots, v_k\}$, $k \leq n$ be a geometrically independent subset of $\mathbb{R}^n$. Then which of the following statements are correct:

A. No $p+1$ elements of X lies in a hyperplane of dimension $p-1$.

B. Barycentric coordinates of an element of the hyperplane generated by X is unique.

C. If $A \subset X$ is non empty, then A is geometrically independent.

D. If $k = 2$, then X is a subset of a plane.

(g) Let K, L be two triangulations of X, $f : X \to X$ is a continuous map and $\phi : K \to L$ is a simplicial approximation to f. Then which of the following statements are true:

A. $H_p(K) \cong H_p(L)$, $\forall p \geq 0$.

B. The induced map $|\phi| : |K| \to |L|$ is continuous.

C. The induced map $f_p^* : H_p(K) \to H_p(L)$ is a homomorphism $\forall\ p \geq 0$.

D. f and $|\phi|$ need not be homotopic.

$(2 \times 7 = 14)$

Part B, Attempt any FOUR questions from the following:

2. (a) Define interior of a k-simplex and show that it is an open set in the hyperplane spanned by the vertices of that simplex. (7)

   (b) Let $U \subseteq \mathbb{R}^n$, $n \geq 1$ be an open bounded convex set. Then prove that $\overline{U}$ and the unit disk $\mathbb{D}^n$ are homeomorphic, mapping BdU homeomorphically onto $\mathbb{S}^{n-1}$. (7)

3. (a) Define triangulation of a space X and show that triangulation need not be unique. Is unit disk $\mathbb{D}^2$ triangulable? Justify your claim. (2+4)

   (b) Define Barycentric subdivision $K^{(n)}$, $n \geq 0$ of a simplicial complex K and show that $|K^{(n)}| = |K|$ for all n. (8)

4. (a) Define incidence number $[\sigma^{p+1}, \sigma^p]$, where σ^{p+1}, σ^p are simplexes in an oriented simplicial complex K. Show that $\Sigma[\sigma^p, \sigma^{p-1}][\sigma^{p-1}, \sigma^{p-2}] = 0$ where σ^{p-2} is a $(p-2)$ face of σ^p and the summation is over all $(p-1)$-simplexes in K. (2+4)

   (b) For an oriented simplicial complex K, carefully define $C_p(K)$, $B_p(K)$, $Z_p(K)$ and $H_p(K)$. (4)

   (c) Compute $H_p(K)$, $p \geq 0$, where $K = Cl(\sigma^2)$. (4)

5. (a) Define combinatorial components of an oriented simplicial complex K. Show that the path components of $|K|$ coincides with geometric carriers of the combinatorial components of K. (6)

   (b) If a topological space X has a triangulation K and the simplicial complex K has r many combinatorial components, then show that $H_0(X) \cong \oplus\mathbb{Z}$ (r-copies). (8)

6. (a) State and prove Euler-Poincaré Theorem. (10)

   (b) Find $\chi(K)$, where (a) K is a triangulation of sphere $\mathbb{S}^2$ and (b) K is a triangulation of torus $\mathbb{T}^2$. (2+2)

7. (a) State and prove Brouwer's Fixed Point Theorem. (7)

   (b) Establish that $\mathbb{R}^m$ is not homeomorphic to $\mathbb{R}^n$, $m \neq n$. (7)

8. (a) Show that homotopic maps $f, g : \mathbb{S}^n \to \mathbb{S}^n$ have same degrees and deduce that $\mathbb{S}^n$ is not contractible, $n \geq 1$. (10)

   (b) Show that $\mathbb{S}^n$ admits a non zero tangent vector field if and only if n is odd, $n \geq 1$. (4)

YOUR EXAMINATION ROLL NO. ...

DEPARTMENT OF MATHEMATICS
M. A./M. SC. MATHEMATICS PART (II) -SEMESTER IV
FINAL EXAMINATION MAY 2022
MMATH18- 402(I) ABSTRACT HARMONIC ANALYSIS, UPC: 223502405

Time: $3\frac{1}{2}$ HOURS | Maximum Marks: 70

• Write down your examination roll no. on the top. • Attempt **five** questions in all. • **Question No. 1 is compulsory**. Attempt any four from the remaining seven questions. • All the questions carry equal marks. • All the symbols have their usual meanings.

(1) (a) Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = x^2; 0 \leq x \leq 1$ and $f(x) = 0$, otherwise. Is f left uniformly continuous? Justify your answer. (2)

(b) Prove that the left regular representation of a locally compact group G on $L^2(G)$ is unitary. (2)

(c) Define Schur's orthogonality relations. (2)

(d) For a compact group G, $f, g \in L^1(G)$ and $[\pi] \in \hat{G}$, prove that $(f * g)\hat{}(\pi) = \hat{g}(\pi)\hat{f}(\pi)$, where $\hat{f}(\pi)$ is the Fourier transform of f at π. (2)

(e) Give an example of a regular Borel measure which is not a Haar measure. (3)

(f) Give example (with justification) of a positive definite function. (3)

(2) (a) Let G be a topological group, A be a compact subset and B be a closed subset of G. Prove that AB is closed. Illustrate by an example that the result may not be true if A is assumed to be a closed set. (4+3)

(b) Let μ be a regular Borel measure on a locally compact group G. When is μ a left Haar measure? Prove that μ is a left Haar measure if and only if $\int f\, d\mu = \int L_y f\, d\mu$, for all $f \in C_c^+(G)$ and $y \in G$. (1+6)

(3) (a) Determine $C(\pi, \gamma)$ for irreducible unitary representations π and γ of a locally compact group G. (7)

(b) For a compact group G and $f \in L^2(G)$, prove that $f(x) = \sum_{[\pi]\in\hat{G}} d_\pi tr[\hat{f}(\pi)\pi(x)]$ for all $x \in G$. (7)

(4) (a) Let π be a unitary representation of a locally compact group G and ρ be the integrated $*$-representation of $L^1(G)$. Prove that if ρ is irreducible and cyclic, then so is π. (7)

(b) Let ϕ be a bounded continuous complex valued function on a locally compact group G. Prove that if ϕ is of positive type then it is positive definite. (7)

(5) (a) Let $\mathcal{E}$ be the span of $\{\pi_{ij} : \pi$ is a finite dimensional unitary representation of $G\}$, G being a compact group. Prove that $\mathcal{E}$ is a subalgebra of $C(G)$ and is closed under conjugation. (7)

(b) For $\mu, \nu \in M(G)$, discuss the existence of the convolution $\mu * \nu$. Further prove that this convolution is associative. (4+3)

(6) (a) State and prove Gelfand Raikov Theorem. (7)

(b) Let G be a locally compact abelian group. For a positive measure $\mu \in M(\hat{G})$, consider $\phi_\mu : G \to \mathbb{C}$ defined as $\phi_\mu(x) = \int_{\chi \in \hat{G}} \chi(x)\ d\mu(\chi)$ for all $x \in G$. Prove that ϕ_μ is of positive type. (7)

(7) (a) For a locally compact group G, prove that $L^1(G)$ possesses an approximate identity. (7)

(b) For a finite group G prove the following:

(i) Number of conjugacy classes in G is the cardinality of $\hat{G}$. (4)

(ii) Cardinality of G is $\sum_{[\pi] \in \hat{G}} d_\pi^2$, d_π being the dimension of π. (3)

(8) (a) Let G be a locally compact group and $M(G)$ be the Banach algebra of complex regular Borel measures on G. Prove that $M(G)$ is unital. Is $M(G)$ a commutative Banach algebra ? Justify your answer. (7)

(b) Prove that every irreducible unitary representation of $SU(2)$ is unitarily equivalent to some π_n, where $\pi_n : SU(2) \to B(H_n)$ is defined as $\pi_n(g)(f(z)) = f(zg)$, for all $g \in SU(2), f(z) \in H_n$, H_n being the space of homogenous polynomials of degree n in two variables; $n \in \mathbb{N} \cup \{0\}$. (7)

M.A./M.Sc. Mathematics Examinations (May 2022)
Part II Semester IV

MMATH18-402(ii): Frames and Wavelets
Unique Paper Code: 223502406

Time: $3\frac{1}{2}$ hr | Maximum Marks: **70**

Instructions: • Question No. 1 is **COMPULSORY** • Answer any **FOUR** questions from Question No. 2 to Question No. 8 • All questions carry equal marks • Symbols have their usual meaning.

Question No. 1 is COMPULSORY.

(1) (a) Give an example, with justification, of a Parseval frame of $\ell^2(\mathbb{N})$ which is not ω-independent. [3 Marks]

(b) Show that if $\{f_k\}_{k=1}^{\infty}$ is a Riesz basis of an infinite dimensional Hilbert space $\mathcal{H}$ with bounds A, B then $A \leq \|f_k\|^2 \leq B$ for all $k \in \mathbb{N}$. [3 Marks]

(c) Find two dual frames of the frame $\{e_1, e_1, e_2, e_2, \ldots, e_k, e_k, \ldots\}$ for an infinite dimensional Hilbert space $\mathcal{H}$. Here, $\{e_k\}_{k=1}^{\infty}$ is the canonical orthonormal basis of $\mathcal{H}$. [4 Marks]

(d) By using the Haar wavelet, give an example with justification, of a non-tight and non-exact frame for $L^2(\mathbb{R})$. [4 Marks]

(Answer any FOUR questions from Question No. 2 to Question No. 8.)

(2) (a) Give an example with justification of a sequence in an infinite dimensional Hilbert space $\mathcal{H}$ which is complete Bessel sequence but not a frame for $\mathcal{H}$. [4 Marks]

(b) Show that the pre-frame operator and analysis operator associated with a frame may not be invertible. [4 Marks]

(c) Define Riesz basis. How do you obtain a Riesz basis of an infinite dimensional Hilbert space $\mathcal{H}$ from a given Riesz basis of $\mathcal{H}$? Justify your answer [6 Marks]

(3) (a) Show that a finite collection of vectors $\{f_k\}_{k=1}^{m}$ in $\mathbb{C}^n$ is a frame for $\text{span}\{f_k\}_{k=1}^{m}$. [6 Marks]

(b) Show that if $\{f_k\}_{k=1}^{m}$ is a frame for $\mathbb{C}^n$ $(n > 1)$, then there exist $n-1$ vectors $\{h_j\}_{j=1}^{n-1} \subset \mathbb{C}^n$ such that $\{f_k\}_{k=1}^{m} \bigcup \{h_j\}_{j=1}^{n-1}$ forms a tight frame for $\mathbb{C}^n$. [8 Marks]

(4) (a) Show that $\lim_{N\to\infty} \|f - (f_0 + \sum_{j=0}^{N} w_j)\|_{L^2} = 0$ for every $f \in L^2(\mathbb{R})$, where $f_0 \in \mathcal{V}_0$ and $w_j \in \mathcal{W}_j, j = 0, 1, \ldots, N$. Here, $\mathcal{V}_0$ and $\mathcal{W}_j$ are the spaces spanned by $\{\phi(x-k) : k \in \mathbb{Z}\}$ and $\{\psi(2^j(x-k)) : k \in \mathbb{Z}\}$, respectively; where ϕ is the Haar scaling function and ψ is the Haar wavelet. [6 Marks]

(b) Define similar frames. Give an example, with justification, of two frames which are not similar. What is the relation between similar Parseval frames and unitarily equivalent frames for a finite dimensional Hilbert space $\mathcal{V}$? Explain. [1+3+4= 8 Marks]

(5) (a) State and prove the scaling relations related to the Haar multiresolution analysis. [4 Marks]

(b) Show that dual frames of a frame $\{f_k\}_{k=1}^{\infty}$ for an infinite dimensional Hilbert space $\mathcal{H}$ are precisely the families [10 Marks]

$$\{g_k\}_{k=1}^{\infty} = \left\{S^{-1}f_k + h_k - \sum_{j=1}^{\infty}\langle S^{-1}f_k, f_j\rangle h_j\right\}_{k=1}^{\infty},$$

where S is the frame operator of $\{f_k\}_{k=1}^{\infty}$ and $\{h_k\}_{k=1}^{\infty}$ is a Bessel sequence in $\mathcal{H}$.

(6) (a) Explain the Haar decomposition of a function $f_j = \sum_{k\in\mathbb{Z}} a_k^j \phi(2^j x - k)$ in $\mathcal{V}_j$, here ϕ is the Haar scaling function and $\mathcal{V}_j \subset L^2(\mathbb{R})$ is the approximation space (associated with ϕ) at level j, $j \in \mathbb{N}\cup\{0\}$. [6 Marks]

(b) State and prove the frame minus one theorem. [8 Marks]

(7) (a) What is the relation between the optimal frame bounds of a frame for an infinite dimensional Hilbert space $\mathcal{H}$ and its associated pre-frame operator and frame operator? Explain. [4 Marks]

(b) Show that a sequence $\{f_k\}_{k=1}^{\infty}$ in an infinite dimensional Hilbert space $\mathcal{H}$ is a frame for $\mathcal{H}$ with frame bounds A, B if and only if $\{f_k\}_{k=1}^{\infty}$ is complete in $\mathcal{H}$, the pre-frame operator T associated with $\{f_k\}_{k=1}^{\infty}$ is well-defined and [10 Marks]

$$A\sum_{k=1}^{\infty}|c_k|^2 \le \|T\{c_k\}_{k=1}^{\infty}\|^2 \le B\sum_{k=1}^{\infty}|c_k|^2, \ \forall \{c_k\}_{k=1}^{\infty} \in \mathcal{N}_T^{\perp},$$

where $\mathcal{N}_T$ is the null space of T.

(8) (a) Show that if $\{f_k\}_{k=1}^{\infty}$ is a frame of an infinite dimensional Hilbert space $\mathcal{H}$ and P is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{W}$, then $\{Pf_k\}_{k=1}^{\infty}$ is a frame for $\mathcal{W}$. [4 Marks]

(b) Show that every frame of an infinite dimensional Hilbert space $\mathcal{H}$ is a multiple of a sum of three orthonormal bases of $\mathcal{H}$. [10 Marks]

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Your Roll Number: ....

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations, May 2022
Part II Semester IV

MMATH18-402 (iii): Operators on Hardy Hilbert Space
Unique Paper Code : 223502407

Time: Three and a half hours Maximum Marks: 70

Instructions: • Write your roll number on the space provided at the top of this page immediately on receipt of this question paper • Question 1 is compulsory • Attempt 4 questions from the remaining seven questions • All symbols have their usual meaning.

Section A

Attempt all parts.

(1) (a) Give an example of a function which is analytic on $\mathbb{D}$ but is not in $H^2(\mathbb{D})$. (2)

(b) Let $T : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ be given by $(Tf)(z) = z^3 f(z), z \in \mathbb{D}$. Check whether or not T is an isometry. (3)

(c) Give an example of (i) an inner function which has exactly three zeros in $\mathbb{D}$, (ii) an inner function which has a countably infinite number of zeros in $\mathbb{D}$, justifying your answers. (3)

(d) Let $T = 2I + U$, where U is the unilateral shift on $\tilde{H}^2$. Show that T is a Toeplitz operator and find its symbol. (3)

(e) Find the matrix of the flip operator J w.r.t the standard orthonormal basis of $L^2(S^1)$ and determine whether or not it is a Hankel matrix. (3)

Section B

Attempt four questions in all from this section.

(2) (a) Find operators on $\tilde{H}^2(S^1)$ and l^2 that are unitarily equivalent to the bounded linear operator $T : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ given by $(Tf)(z) = z^2 f(z)$. (6)

(b) Give an orthonormal basis for H^2 and justify your claim. Hence or otherwise write down an orthonormal basis for $\phi\tilde{H}^2$, where $\phi \in L^\infty(S^1)$ satisfies $|\phi(e^{i\theta})| = 1$ a.e. (4)

(c) Show that convergence in $H^2(\mathbb{D})$ implies uniform convergence on compact subsets of $\mathbb{D}$. Analogously, does convergence in $\tilde{H}^2(S^1)$ imply convergence on compact subsets of S^1. ? (4)

(3) (a) Define the essential norm $\|\phi\|_\infty$ of a function $\phi \in L^\infty(S^1)$. If $\phi(e^{i\theta})$ takes the value 2 for $\theta \in [0, 2\pi] \setminus \{1/2, 1/4, 1/8, \ldots 1/2^{16}\}$ and 1 otherwise, find $\|\phi\|_\infty$. Show further that $|\phi(e^{i\theta})| \leq \|\phi\|_\infty$ for almost all θ. Hence or otherwise show that a function in H^2 whose boundary function is in L^∞ must be in H^∞. (6)

(b) State Fatous' Theorem. Use it to show that if $f \in H^2$ then $\lim_{r\to 1^-} f\left(re^{i\theta}\right) = \tilde{f}(e^{i\theta})$, for almost all θ, where $\tilde{f}$ is the boundary function of f. (5)

(c) Let $f(z) = \frac{1}{2}(1 - z^2), z \in \mathbb{D}$. Verify that $f \in H^2$ and find $\tilde{f}$ the boundary function of f. Is f an inner function? (3)

(4) (a) Prove that the adjoint of the bilateral shift W defined on $l^2(\mathbb{Z})$ is unitarily equivalent to the operator M on $L^2(S^1)$ given by $(Mf)(e^{i\theta}) = e^{-i\theta}f(e^{i\theta})$. (5)

(b) Is the subspace $(\tilde{H}^2)^{\perp}$ of $L^2(S^1)$ invariant under the adjoint of the bilateral shift? Justify. (2)

(c) Show that any non-zero function in H^2 may be written as a product of an inner and an outer function. Is this factorisation unique? Justify. (7)

(5) For $\phi \in L^\infty(S^1)$ let M_ϕ be defined by $M_\phi f = \phi f$ for $f \in L^2(S^1)$.

(a) Show that M_ϕ is a bounded linear operator on $L^2(S^1)$ and if further $|\phi(e^{i\theta})| = 1$ a.e. then M_ϕ is an isometry. (4)

(b) Prove that a bounded linear operator A on L^2 is equal to M_ϕ if and only if the matrix of A with respect to the standard basis in L^2 is a Toeplitz matrix. (5)

(c) Show that $\sigma(M_\phi) = \Pi(M_\phi) = ess\,ran\,\phi$. (5)

(6) (a) Show that if T_ϕ and T_ψ are Toeplitz operators, U is the unilateral shift, and P is the orthogonal projection of $L^2(S^1)$ onto $\tilde{H}^2$, then $U^*T_\psi T_\phi U - T_\psi T_\phi = P(e^{-i\theta}\psi) \otimes P(e^{-i\theta}\overline{\phi})$. (4)

(b) Find a $\psi \in L^\infty$ such that $T_\phi T_\psi = T_\psi T_\phi$ when (i) $\phi(e^{i\theta}) = e^{2i\theta} - 2e^{3i\theta}$ (ii) $\phi(e^{i\theta}) = e^{-i\theta} + e^{-2i\theta}$ (iii) $\phi(e^{i\theta}) = e^{-2i\theta} + 2e^{-i\theta} + 1 + e^{i\theta}$. (6)

(c) Characterise normal Toeplitz operators in terms of their symbols. Give a non-trivial example of a self adjoint Toeplitz operator. (4)

(7) (a) Let $T : \tilde{H}^2 \to \tilde{H}^2$ be given by $(Tf)(e^{i\theta}) = 2f(e^{-i\theta}) - \overline{f(e^{i\theta})}$. Find the matrix of T with respect to the standard basis of $\tilde{H}^2$ and check if T is a Toeplitz operator or a Hankel operator or neither of the two. (5)

(b) Prove a necessary and sufficient condition, (involving the unilateral shift) for an operator A to have a Hankel matrix with respect to the standard basis of $\tilde{H}^2$. Use it to show that $(k_{\bar{w}} \otimes k_w)$ is a Hankel operator. (9)

(8) (a) Show that for every $\phi \in L^\infty, H_\phi^* = H_{\phi^*}$. If H_ϕ is self adjoint, is it always true that $\phi = \phi^*$ a.e. ? Justify your assertion. (6)

(b) Show that a Hankel operator H is of rank one if and only if there exists $w \in \mathbb{D}$ and a constant c such that $H = c(k_{\bar{w}} \otimes k_w)$. (5)

(c) Give an example of a Hankel operator of rank two and compute its matrix with respect to the standard orthonormal basis of $\tilde{H}^2$. (3)

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examinations, May 2022
Part II, Semester IV
MMATH18-403 (i) : CALCULUS ON $\mathbb{R}^n$ (Unique Paper Code 223502409)

Time: 3.5 hours | Maximum Marks: 70

Instructions: •Attempt **five** questions in all. **Question 1 is compulsory**. All questions carry equal marks.• The symbols used have their usual meanings.

(1) (a) Let $U \subset \mathbb{R}^n$ be open, $a \in U$ and $f : U \to \mathbb{R}^m$. If there is an $m \times n$ matrix B such that $\frac{f(a+h)-f(a)-B.h}{|h|} \to 0$ as $h \to 0$, show that B is unique. [2]

(b) Let $f(x,y) = |x|^{1/2} + |y|^{1/2}$, $(x,y) \in \mathbb{R}^2$. Check the differentiability of f at $(0,0)$. Is f continuous on $\mathbb{R}^2$? Justify your answer. [2+3]

(c) Give an example, with justification, of a negatively oriented affine 2-simplex in $\mathbb{R}^2$. [2]

(d) Let $T(r,\theta,\varphi) = (x,y,z)$, where $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, $z = r\cos\theta$. If $\omega = dx \wedge dy \wedge dz$, compute the pull back ω_T. [3]

(e) If $U = \{(x,y) \in \mathbb{R}^2 : y \neq 0\}$ and $f : U \to \mathbb{R}^3$ is of class C^1 such that $Df(x,y) = \mathbf{0}$, $\forall (x,y) \in U$, can you conclude that f is a constant? Justify your answer. [2]

(2) (a) State and prove the existence theorem for partition of unity. [8]

(b) Suppose for $1 \le j \le n$, $g_j : \mathbb{R} \to \mathbb{R}$ are differentiable. Define $f : \mathbb{R}^n \to \mathbb{R}$ by $f(x_1, x_2, \cdots, x_n) = \sum_{j=1}^{n} g_j(x_j)$. Show that f is differentiable in $\mathbb{R}^n$ and find Df. [6]

(3) (a) On $\mathbb{R}^2$ define $g(x,y) = (3ye^{2x}, xe^{3y})$ and $f(x,y) = (2x - y^3, 3x + 2y, xy + y^2)$. [7]

(i) Show that there exist neighborhoods V and W of $(0,1)$ and $(3,0)$, respectively in $\mathbb{R}^2$ such that $g : V \to W$ is bijective and $g^{-1} : W \to V$ is of class C^∞.

(ii) Compute $D(f \circ g^{-1})$ at $(3,0)$.

(b) Suppose ω is a k-form in an open set $E \subset \mathbb{R}^n$, Φ is a k-surface in E with parameter domain $D \subset \mathbb{R}^k$ and Δ is the k-surface in $\mathbb{R}^k$ with parameter domain D, defined by $\Delta(u) = u$, $u \in D$. Then show that $\int_\Phi \omega = \int_\Delta (\omega)_\Phi$. [7]

(4) (a) Suppose $\sigma = [P_0, P_1, \cdots, P_k]$ is an oriented affine k-simplex ($k \ge 2$) in an open set $E \subset \mathbb{R}^n$, $0 < i < j \le k$ and $\tilde{\sigma}$ is the affine k-simplex obtained from σ by interchanging P_i and P_j. Show that for any k-form ω in E, $\int_{\tilde{\sigma}} \omega = -\int_\sigma \omega$. [6]

(b) Let $f : Q^k \to \mathbb{R}$ be continuous, where $Q^k = \{x = (x_1, x_2, \ldots, x_k) \in \mathbb{R}^k : x_i \ge 0, \forall 1 \le i \le k, \sum_{i=1}^{k} x_i \le 1\}$. Define $\int_{Q^k} f$ and show the existence of the integral. Show also that the definition does not depend on the order in which the k single integrals are carried out. [8]

(5) (a) Let E be an open subset of $\mathbb{R}^k$ containing the standard simplex Q^k, $k > 1$ and $\sigma = [0, e_1, e_2, \cdots, e_k]$ be the oriented affine k-simplex in $\mathbb{R}^k$ with parameter domain Q^k. For any $(k-1)$-form λ of class C^1 in E, prove that $\int_\sigma d\lambda = \int_{\partial\sigma} \lambda$. [9]

(b) Calculate the second order Taylor polynomial at $(\sqrt{\pi/4}, 1)$ for the function $f(x, y) = \cos^2(x^2 y)$. [5]

(6) (a) Let H be the parallelogram in $\mathbb{R}^2$ with vertices $(2,2), (4,3), (5,6)$ and $(3,5)$. Find the affine map T which sends $(0,0)$ to $(2,2)$, $(1,0)$ to $(4,3)$ and $(0,1)$ to $(3,5)$. Use T to convert the integral $I = \int_H e^{2u+v}\, du\, dv$ to an integral over the unit square S, and thus compute it. Show also that T maps a convex set to a convex set, and hence conclude that $T(S) = H$. [7]

(b) Let $T : Q^n \subset \mathbb{R}^n \to \mathbb{R}^n$ be of class C^2, one-to-one and $\det J_T > 0$ and let $E = T(Q^n)$. How do you define the positively oriented boundary of Q^n? Discuss how it is meaningful to talk about the positively oriented boundary of E and illustrate with an example. [7]

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(7) (a) Suppose ω and λ are k- and m-forms $(k, m \geq 1)$, respectively, of class C^1 in the open set $E \subset \mathbb{R}^n$. Show that $d(\omega \wedge \lambda) = d\omega \wedge \lambda + (-1)^k \omega \wedge d\lambda$. Verify this formula for the forms ω and λ in $\mathbb{R}^3$, where $\lambda = z\,dy + x\,dz$ and [7]
$\omega = 2xy^3z^4\,dx + (3x^2y^2z^4 - ze^y \sin(ze^y))\,dy + (4x^2y^3z^3 - e^y \sin(ze^y) + e^z)\,dz$.

(b) State and prove the implicit function theorem and illustrate the theorem with an example. [7]

(8) (a) Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open, $f : U \to \mathbb{R}^m$, $f(U) \subset V$, $g : V \to \mathbb{R}^k$ and $F(x) = g(f(x))$, $x \in U$. If f is differentiable at a and g is differentiable at $b = f(a)$, then show that F is differentiable at a, and that $DF(a) = Dg(f(a)).Df(a)$. Show also that if f and g are of class C^r, then F is also of class C^r. [5+3]

(b) Define exact and closed differential forms. Show that every exact form is closed. Considering the 1-form $\omega = \frac{x\,dy - y\,dx}{x^2+y^2}$ in $\mathbb{R}^2 \setminus \{(0,0)\}$, show that a closed form need not be exact. [6]

M.A./M.Sc. Mathematics, Part-II, Semester IV
Examination, May 2022
Paper: MMATH18-403(ii) Differential Geometry
Unique Paper Code 223502410

Time: $3\frac{1}{2}$ Hours Maximum Marks: 70

Instructions: Attempt **five** (5) questions in all. **Question no. 1 is compulsory**. All questions carry equal marks.

1. (a) Determine spherical image of the cylinder $x_2^2+...+x_{n+1}^2 = 1$ for $n = 1$ and $n = 2$. [3]
   (b) Prove that the graph of a smooth function $f : U \to R$, where U is open in R^n, is an n-surface in R^{n+1}. [3]
   (c) Show that velocity vector field along a parametrized curve α on a surface S is *parallel* if and only if α is a *geodesic* on S. [3]
   (d) Show that if $\mathbf{X}$ and $\mathbf{Y}$ are vector fields tangent to an n-surface S along a parametrized curve α in S then $(\mathbf{X}\cdot\mathbf{Y})' = \mathbf{X}'\cdot\mathbf{Y}+\mathbf{X}\cdot\mathbf{Y}'$ [3]
   (e) Compute $\nabla_{\mathbf{v}} f$ where $f(x_1, x_2) = x_1^2 - x_2^2$, $\mathbf{v} = (1, 2, cos\theta, sin\theta)$. [2]
2. (a) Show that a parametrized curve α is a geodesic in the unit n-sphere in R^{n+1} if and only if it is of the form $\alpha(t) = (cosat)\mathbf{e}_1 + (sinat)\mathbf{e}_2$ for some orthonormal pair of unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ in R^{n+1}. [7]
   (b) Define the *Weingarten map* for an n-surface S in R^{n+1}, oriented by the unit normal vector field $\mathbf{N}$. Let $p \in S$ and $\mathbf{v} \in S_p$. Show that for every parametrized curve $\alpha : I \to S$ with $\dot{\alpha}(t_0) = \mathbf{v}$ for some $t_0 \in I$, $\ddot{\alpha}\cdot \mathbf{N}(p) = L_p(\mathbf{v})\cdot\mathbf{v}$. [7]
3. (a) Determine the integral curve at $p = (0,1)$ of the vector field whose associated function is given by $X(x_1, x_2) = (x_2, -x_1)$ [6]
   (b) Show that in a connected n-surface in R^{n+1} there are exactly two unit normal vector fields. [6]
   (c) Let $f : U \to R$ be a smooth function and let α be an integral curve of the gradient of f. Show that $\frac{d}{dt}(f \circ \alpha)(t) = \| \nabla f(\alpha(t)) \|^2$ [2]
4. (a) Let $\alpha : I \to S$ be a geodesic in a 2-surface S in R^3. Prove that a vector field $\mathbf{X}$ tangent to S along α is parallel along

α if and only if both $\| \mathbf{X} \|$ and the angle between $\mathbf{X}$ and $\dot{\alpha}$ are constant. [7]

(b) Determine the *Weingarten map* for the n-sphere $x_1^2 + ... + x_{n+1}^2 = 4$ in R^{n+1} oriented by the inward normal vector field $\mathbf{N}(q) = \left(q, -\frac{q}{\| q \|}\right)$ [4]

(c) Let $\varphi : U \longrightarrow R^{n+k}$ be a parametrized n-surface in R^{n+k}. Show that $d\varphi : U \times R^{n+k} \longrightarrow R^{n+k} \times R^{n+k}$ is a parametrized $2n$-surface in R^{2n+2k} [3]

5. (a) Find global parametrization of the plane curve $x_2 - ax_1^2 = c$ and determine its curvature. [7]

(b) Define *second fundamental form* at a point p on an n-surface in R^{n+1}. Prove that on a compact oriented n-surface S in R^{n+1} there exists a point p such that the second fundamental form at p is definite. [7]

6. (a) Let U be an open set in R^{n+1} and let $f : U \to R$ be smooth. Let p be a regular point of f, and let $f(p) = c$. Show that the set of all vectors tangent to $f^{-1}(c)$ at p is exactly $[\nabla f(p)]^{\perp}$. [7]

(b) Let $a > b > 0$, determine the principal curvatures and the Gaussian curvature K for the parametrized torus in R^3, $\varphi : R^2 \to R^3$ defined by $\varphi(\theta, \phi) = ((a + b\cos\phi)\cos\theta, (a + b\cos\phi)\sin\theta, b\sin\phi)$. Discuss the conditions $K > 0, K < 0$ and $K = 0$. [7]

7. (a) Find the Gaussian curvature $K : S \to R$, where S is elliptic hyperboloid $\frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} - x_3 = 0$. [7]

(b) Define 1-form on an open set $U \subseteq R^{n+1}$ with 2 examples. Show that for each 1-form w on U there exist unique functions $f_i : U \longrightarrow R$, $1 \leq i \leq n+1$ such that $w = \sum_{i=1}^{n+1} f_i dx_i$. [7]

8. (a) Let $S = f^{-1}(c)$ be an n-surface in R^{n+1} where $f : U \to R$ is such that $\nabla f(q) \neq 0$ for all $q \in S$. Suppose $g : U \to R$ is a smooth function and $p \in S$ is an extreme point of g on S. Show that there is a real number λ such that $\nabla g(p) = \lambda \nabla f(p)$. Will such a point always exist? [7]

(b) Let S be an n-plane $a_1x_1 + ... + a_{n+1}x_{n+1} = b$, let $p, q \in S$, and let $v = (p, v) \in S_p$. Show that if α is any parametrized curve in S from p to q then $P_\alpha(v) = (q, v)$. What can you say about parallel transport in an n-plane? [7]

Your Roll Number: ....

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations, May 2022
Part II Semester IV

MMATH18-403(iii): Topological Dynamics
(Unique Paper Code 223502411)

Time: **3 hours and 30 minutes** Maximum Marks: 70

Instructions: • All notations used are standard • **Question no.** 1 is **compulsory** • Attempt any **four** questions from the remaining seven questions

(1) (a) Prove that the tent map $T : [0,1] \to [0,1]$ defined by

$$T(x) = \begin{cases} 2x & \text{if } x \in [0, \frac{1}{2}] \\ 2(1-x) & \text{if } x \in [\frac{1}{2}, 1] \end{cases}$$

is topologically conjugate to the logistic map $f(x) = 4x(1-x)$, $x \in [0,1]$. (3)

(b) Is the identity map on the Cantor space Topologically Anosov? Justify your claim. (3)

(c) If X is a compact metric space and $f : X \to X$ is a continuous map, then prove that $\omega(x)$ is a non empty subset of X for every $x \in X$. (2)

(d) Let A be a $k \times k$ matrix with entries in $\{0,1\}$. Then prove that if A is an irreducible matrix then its associated digraph is strongly connected. (3)

(e) Prove that $cos : [0,1] \to [0,1]$ has pseudo orbit tracing property (POTP). (3)

(2) (a) Let A be a $k \times k$ matrix with entries in $\{0,1\}$ such that no row / column is full of zeros. Then prove that the shift map σ on X_A is transitive if and only if A is an irreducible matrix. (9)

(b) Do the graphical analysis of the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = (x + x^3)$ and draw the phase portrait of the orbit of $x = \frac{-1}{2}$. (5)

(3) (a) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map having a periodic point of prime period 3. Then prove that f has periodic point of all prime period $n \geq 1$. (9)

(b) Let p be a hyperbolic fixed point of a C^1 map $f : \mathbb{R} \to \mathbb{R}$. Prove that there exists an open interval U about p with $x \in U$, $x \neq p$ such that $|f'(p)| < 1$ implies $lim_{n\to\infty} f^n(x) = p$. (5)

(4) (a) Let (X, d) be a compact metric space. Prove that a homeomorphism $f : X \to X$ is expansive if and only if there exists an $e > 0$ such that for all $\gamma > 0$, there exists an $N \in \mathbb{N}$ satisfying that for all $x \in X$ and for all $n \geq N$, we have

(i) $f^n(W^s_e(x,d)) \subseteq W^s_\gamma(f^n(x), d)$

(ii) $f^{-n}(W^u_e(x,d)) \subseteq W^u_\gamma(f^{-n}(x), d)$. (9)

(b) Prove that an irrational rotation on the unit circle is a minimal homeomorphism. (5)

(5) (a) For an expansive self-homeomorphism f of a compact metric space X, prove that the least upper bound of the set of expansive constants for f is not an expansive constant for f. Justify that if X is not compact then the set of expansive constants for f need not be a bounded set. (8)

(b) If A is a subset of a metric space X such that $X - A$ is finite, then prove that any homeomorphism $f : X \to X$ which is expansive on A, is expansive on X. Justify that if $X - A$ is not finite then the result need not be true. (6)

(6) (a) Prove that the unit circle $\mathbb{S}^1$ does not admit any expansive homeomorphism. (8)

(b) Let (X, d) be a compact metric space and $f : X \to X$ be a homeomorphism. Prove that f is expansive if and only if (X, f) has a generator. (6)

(7) (a) If X is a compact metric space then prove that the shift map $\sigma : X^{\mathbb{Z}} \to X^{\mathbb{Z}}$ has POTP. (8)

(b) Let(X, d) be a metric space and $f : X \to X$ be continuous. If f has POTP then prove that f^k has POTP for every $k \in \mathbb{N}$. (6)

(8) (a) Prove that a Topologically Anosov homeomorphism of a compact metric space X is topologically stable in the class of self-homeomorphisms of X. (9)

(b) Prove that a self-homeomorphism f of a compact metric space X is Topologically Anosov if and only if f^{-1} is so. (5)

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examinations, May 2022
Part II Semester IV, Unique Paper Code 223502412
MMATH18-404(i): ADVANCED FLUID DYNAMICS

Time: Three and half hour | Maximum Marks: 70

Instructions: • Q.No.1 in section **A** is compulsory. • Attempt any **Four** questions from section **B**. • Attempt FIVE questions in all. Each question carries 14 marks. • All the symbols have their usual meaning.

Section A

(Answer all parts, 14 Marks)

(1) (a) Find dimension of medium permeability γ in the Brinkman equation for fluid flow in porous medium [2 Marks]
$-\nabla p - \frac{\mu}{\gamma}\vec{v} + \mu\nabla^2\vec{v} + \rho\vec{g} = 0,$
where μ and $\vec{g}$ are the dynamic viscosity and the acceleration due to gravity respectively.

(b) Derive the equation of continuity for continuous distribution of charge. [2 Marks]

(c) Show that density ratio across a strong shock wave does not depend on shock wave velocity. [3 Marks]

(d) Write the Prandtl boundary layer equations for unsteady flow in term of stream function. [3 Marks]

(e) For a perfect gas prove that $\frac{\partial(T,S)}{\partial(p,v)} = 1$, the independent variables being p, v. [2 Marks]

(f) The internal energy u per unit mass of a non-ideal gas is given by $u = 3p(v-b)$. The equation of state of the gas is $p(v-b) = RT$. Evaluate the specific heats c_v and c_p of the gas. [2 Marks]

Section B

(Answer any FOUR questions, 56 Marks)

(2) (a) Derive the internal energy equation $\rho\frac{De}{Dt} = \overleftrightarrow{\tau} : (\nabla\vec{w})^t - \frac{p}{\rho}\frac{D\rho}{Dt} - \nabla.\vec{q}$. Compute the viscous dissipation term and discuss it for the two dimensional boundary layer flow. [8 Marks]

(b) Show that $c_p - c_v = -T(\frac{\partial v}{\partial T})_p^2(\frac{\partial p}{\partial v})_T$. Also find the corresponding relation for the non-ideal gas with equation of state $p(v-b) = RT$. [4 Marks]

(c) Write the differences between the first and second law of thermodynamics [2 Marks]

(3) (a) Define specific heats for a compressible substance. Derive the relation between specific heats at constant volume and constant pressure. Find the corresponding equations for the perfect gas. [2+4 Marks]

(b) Derive the wave equations for density and velocity distributions due to a small disturbance in a gas flow and find the speed of sound wave. [6 Marks]

(c) Define shock Mach number and shock strength. What do you mean by a weak and strong shock. [2 Marks]

(4) (a) Derive the jump conditions for mass, momentum and energy across an oblique shock wave. Also derive the corresponding Prandtl's relation for an oblique shock wave. [5+4 Marks]

(b) Show that normal shock wave is compressive in nature. [5 Marks]

(5) (a) Show that the MHD wave propagates with a speed $\sqrt{a^2 + V_A^2}$, where V_A is Alfven's velocity and a is speed of sound. [4 Marks]

(b) Derive the magnetic field equation in a moving conducting fluid under the Pre-Maxwell's equation and explain them physically. Define the Magnetic Reynolds number and explain physically. [4+3 Marks]

(c) Show that under MHD approximation the electric field energy is negligible in comparison to the magnetic field energy. [3 Marks]

(6) (a) Derive the following equation of motion of a viscous conducting fluid $\frac{\partial(\rho\vec{v})}{\partial t} + \rho(\vec{v}.\nabla)\vec{v} = \overrightarrow{f(ex)} + \frac{\mu}{4\pi}(\vec{H}.\nabla)\vec{H} - \nabla(p + \frac{\mu\vec{H}^2}{8\pi}) - \vec{v}\nabla.(\rho\vec{v})$. Write the corresponding equation for incompressible fluid. [4+2 Marks]

(b) Define Lorentz force in a conducting fluid. Show that the Lorentz's force per unit volume of a conducting fluid is equivalent to a hydrostatic pressure $\frac{\mu H^2}{8\pi}$ and a tension $\frac{\mu H^2}{4\pi}$ per unit area along the line of force. [1+7 Marks]

(7) (a) Derive the von Karman's momentum integral equation for steady, two dimensional boundary layer flow of an incompressible fluid [7 Marks]

(b) Define the mass and momentum thickness of a boundary layer and compare them in the case of boundary layer flow over a long flat plate. [4+3 Marks]

(8) (a) Describe the boundary layer problem over a long flat plate and derive the Blasius's equation along with the boundary conditions. [8 Marks]

(b) Compute the skin friction on both side of a wetted flat plate of the length l and the width b due to a boundary layer flow of viscosity μ, density ρ and uniform speed U_∞. [6 Marks]

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Your Roll Number: ....

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations, May 2022
Part II Semester IV

MMATH18-404(ii): COMPUTATIONAL METHODS FOR PDEs
(UPC: 223502413)

Time: 3.5 hours | Maximum Marks: 70

Instructions: • Section A is compulsory. • Answer any four questions from section B • All notations have usual meaning • Non-programmable scientific calculators are allowed

Section A

(1) (a) State true or false and justify the statement: Crank- Nicolson scheme has amplification factor less than or equal to 1 for one dimensional Ist order hyperbolic PDEs. [3 Marks]

(b) State true or false and justify: Douglas Rachford scheme cannot be used to solve a two dimensional heat equation with Neumann boundary conditions. [4 Marks]

(c) Use method of characteristics to find the solution of the problem [3 Marks]

$$u_x - u_y = 1, \quad u(x,0) = (x+1)^2.$$

(d) Consider the problem $-u_t + u_{xx} = f(x)$, $x \in (0,1)$, $u(x,0) = 1$, $u(0,t) = u(1,t) = 1$. Choose suitable finite element space for the above problem and derive the equivalent weak form. [4 Marks]

Section B

(2) (a) Show that the Crank-Nicolson scheme for the numerical solution of the problem $u_t = u_{xx}$ is unconditionally stable. Use this scheme to solve the above problem with initial and boundary conditions given by [4+5 Marks]

$$u(x,0) = \begin{cases} 2x, & 0 \le x \le 1/2 \\ 2(1-x), & 1/2 \le x \le 1, \end{cases}, \quad u(0,t) = 0, \ u(1,t) = 0,$$

where $\Delta x = 0.25, \Delta t = 0.1$. Find the solution at $t = 0.1$.

(b) Use matrix method to derive stability criteria for the numerical solution of the problem $u_t = u_{xx}$, $0 < x < 1$ with condition $u_x(0,t) = h_1(v - v_1)$, $u_x(1,t) = h_2(v - v_2)$, $t \ge 0$ where h_1, h_2, v_1, v_2 are constants and $h_1 \ge 0$, $h_2 \ge 0$. Here boundary conditions are approximated by central difference and explicit scheme is used. [5 Marks]

(3) (a) Show that for the problem $u_t(x,t) + a(x,t)u_x(x,t) = 0$, $x \in \mathbb{R}$, $t > 0$ the Lax-Wendroff scheme is second order consistent in both space as well as time variable and conditionally stable with condition $|a|\frac{\Delta t}{\Delta x} \le 1$. [7 Marks]

(b) Derive a numerical scheme based on finite differences for the follow- [7 Marks] ing two dimensional parabolic partial differential equation in polar co-ordinates

$$\begin{aligned} v_t = \frac{1}{r}(rv_r)_r &+ \frac{1}{r^2}v_{\theta\theta} + F(r,\theta,t), \quad 0 \le r < 1,\ 0 \le \theta < 2\pi,\ t > 0 \\ v(1,\theta,t) &= g(\theta,t) \quad 0 \le \theta < 2\pi,\ t > 0 \\ v(r,\theta,0) &= f(r,\theta) \quad 0 \le r \le 1 \quad 0 \le \theta < 2\pi. \end{aligned}$$

[7 Marks]

(4) (a) Consider the problem

$$-(u_{xx} + u_{yy}) = 1 \quad 0 < x < 1,\ 0 < y < 1,$$

with $u(x,y) = 0$ on the boundary of the square. Discretize the above problem using finite element technique with triangular elements and step sizes $\Delta x = 1/3,\ \Delta y = 1/2$. Obtain the resulting system of algebraic equations.

[7 Marks]

(b) Consider the problem $u_{tt} = u_{xx} + 2$ together with initial and boundary conditions

$$u(x,0) = \sin\pi x,\ \frac{\partial u}{\partial t}(x,0) = 0,\ 0 \le x \le 1, u(0,t) = u(1,t) = 0,\ t \ge 0.$$

Find the numerical solution of the above problem using central time central space scheme ($\Delta x = 0.25,\ \Delta t = 0.1$) for two time levels ($t = 0.1,\ 0.2$).

[9 Marks]

(5) (a) Let $u(x,y)$ be the solution to $-\nabla^2 u = f$ on the unit square with Dirichlet boundary conditions and $U_{l,m}$ be the solution to $-\nabla_h^2 v = -(\frac{1}{\Delta x^2}\delta_x^2 + \frac{1}{\Delta y^2}\delta_y^2)v = f$ with $U_{l,m} = u(x_l, y_m)$ on the boundary. Then show that

$$||u - v||_\infty \le Ch^2 ||\partial^4 u||_\infty$$

where $||\partial^4 u||_\infty$ is the maximum magnitude of all the fourth order derivatives of $u(x,y)$ over the interior of the square.

[5 Marks]

(b) Solve $u_t + u_x = 0,\ x \in [-1,3]$ with initial condition $u(0,x) = \begin{cases} \cos^2(\pi x), & -1/2 \le x \le 1/2 \\ 0 & \text{otherwise} \end{cases}$ using FTBS scheme for $\Delta x = 1, \Delta t/\Delta x = 0.8$. Find the solution at $t = 0.8$.

(6) (a) Consider the problem [7 Marks]

$$u_t - u_{xx} = f(x,t),\ 0 < x < 1, u(x,0) = u(0,t) = u(1,t) = 0.$$

Derive a finite element approximation for the above problem using Backward Euler scheme for time discretization and linear basis functions. Obtain the resulting system of algebraic equations.

(b) Perform eigen value analysis of the Gauss Seidel iteration matrix associated with the following discrete problem corresponding to Poisson's equation with Dirichlet boundary conditions [7 Marks]

$$-\frac{1}{\Delta x^2}\delta_x^2 U_{jk} - \frac{1}{\Delta y^2}\delta_y^2 U_{jk} = F_{jk}, \quad j = 1,\ldots,M_x - 1,\ k = 1,\ldots,M_y - 1$$

$$U_{0k} = f_{0k},\ k = 1,\ldots,M_y - 1, \quad U_{M_x k} = f_{M_x k},\ k = 1,\ldots,M_y - 1$$

$$U_{j0} = f_{j0},\ j = 1,\ldots,M_x - 1, \quad U_{jM_y} = f_{jM_y},\ j = 1,\ldots,M_x - 1.$$

(7) (a) Derive a second order accurate numerical scheme for the solution of two dimensional Laplace equation with Neumann boundary conditions. Use [10 Marks]

this scheme to solve the problem

$$u_{xx} + u_{yy} = 0, 0 \le x, y \le 1, \ u(x,0) = 2x, \ u(x,1) = 2x - 1, \ 0 \le x \le 1,$$
$$(u_x + u)(0,y) = 2 - y, \ u(1,y) = 2 - y, \ 0 \le y \le 1$$

with $\Delta x = \Delta y = 1/3$ and obtain the resulting system of algebraic equations.

(b) State and Prove Gerschgorin Circle theorem. [4 Marks]

(8) (a) For the linear advection equation $u_t + au_x + bu_y = 0$, where a, b are positive constants, discuss CFL condition and stability results for the scheme [6 Marks]

$$U_{jk}^{n+1} + \frac{R_x}{2}\delta_{x0}U_{jk}^{n+1} + \frac{R_y}{2}\delta_{y0}U_{jk}^{n+1} = U_{j,k}^{n}.$$

Also show that the above scheme will be unconditionally consistent with $O(\Delta t + (\Delta x)^2 + (\Delta y)^2)$.

(b) Show that the truncation error of the Crank- Nicolson scheme for the numerical solution of the partial differential equation [8 Marks]

$$u_t = \nu(u_{xx} + u_{yy} + u_{zz})$$

is of $O(\Delta t + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2)$. Derive an ADI scheme using CN scheme for the numerical solution of the above problem.

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations, May 2022
Part II Semester IV
MMATH18-404(iii): Dynamical Systems: (UPC 223502414)

Time: 3.5 hours Maximum Marks: 70

Instructions: • Attempt **any FIVE** questions. • Question 1 is compulsory. • Each question carries 14 marks.

(1) (a) Show that the system [2 Marks]

$$\dot{x} = y, \quad \dot{y} = -x^3 + 4xy.$$

has a critical point with an elliptic domain at the origin.

(b) Sketch the phase portraits corresponding to [2 Marks]

$$\dot{x} = x - \cos(x).$$

~~and determine the stability of all the fixed points.~~

(c) Use appropriate Liapunov function to determine the stability of the equilibrium points of the following system [2 Marks]

$$\dot{x_1} = -x_1 + x_2 + x_1x_2, \quad \dot{x_2} = x_1 - x_2 - x_1^2 - x_2^3.$$

(d) Define the following terms: center-focus and stable focus. [2 Marks]

(e) Determine the stability of the equilibrium points of the system $\dot{x} = f(x)$ with [2 Marks]

$$f(x) = \begin{bmatrix} x_1^2 - x_2^2 - 1 \\ 2x_2 \end{bmatrix}$$

(f) Write the system [2 Marks]

$$\dot{x} = -y - xy, \quad \dot{y} = x + x^2,$$

into polar-coordinates.

(g) State the Hartman-Grobman Theorem. [2 Marks]

(2) (a) Classify the equilibrium points (as sinks, sources or saddles) of the nonlinear system $\dot{x} = f(x)$ with [4 Marks]

$$f(x) = \begin{bmatrix} -4x_2 + 2x_1x_2 - 8 \\ 4x_2^2 - x_1^2 \end{bmatrix}.$$

(b) Define the following systems: Hamiltonian system with n degrees of freedom on E, Gradient system on E and orthogonal system to a planar system. [10 Marks]
For the Hamiltonian function $H(x, y) = x^2 + 2y^2$ sketch the phase portraits for both Hamiltonian system and gradient system orthogonal to Hamiltonian system on same phase plane.

(3) State and prove the Stable Manifold Theorem. [14 Marks]

(4) Use "The Local Center Manifold Thoerem" to find the approximation for the flow on local center manifold for the system [14 Marks]

$$\dot{x_1} = x_1y - x_1x_2^2, \quad \dot{x_2} = x_2y - x_2x_1^2, \quad \dot{y} = -y + x_1^2 + x_2^2,$$

Show that origin is unstable for this system.

(5) Solve the two-dimensional system [14 Marks]

$$\dot{y} = -y, \quad \dot{z} = z + y^2$$

and show that the successive approximations $\Phi_k \longrightarrow \Phi$ and $\Psi_k \longrightarrow \Psi$ as $k \longrightarrow \infty$ for all $(y, z) \in \mathbb{R}^2$. Define $H_0 = (\Phi, \Psi)^T$ and use this homeomorphism to find

$$H = \int_0^1 L^{-s} H_0 T^s ds.$$

Use the homeomorphism H to find the stable and unstable manifolds

$$W^s(0) = H^{-1}(E^s) \text{ and } W^u(0) = H^{-1}(E^u)$$

for this system.

(6) (a) Find the *Poincaré map* of the system [4 Marks]

$$\dot{x} = -y + x(1 - x^2 - y^2), \quad \dot{y} = x + y(1 - x^2 - y^2).$$

(b) Solve the initial value problem $\dot{x} = Ax$ with $x(0) = x_0$ where A is given by [10 Marks]

$$A = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0- & 0 & 0 & -1 \\ 2 & 0 & 1 & 0 \end{bmatrix}.$$

(7) (a) Define the flow ϕ_t of the non-linear system $\dot{x} = f(x)$. Also define the term "invariant set with respect to the flow ϕ_t". [4 Marks]

(b) Determine the flow $\phi_t : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ for the nonlinear system $\dot{x} = f(x)$ with [10 Marks]

$$f(x) = \begin{bmatrix} -x_1 \\ x_2 + x_1^2 \end{bmatrix}$$

and show that the set $S = \{x \in \mathbb{R}^2 | x_2 = -x_1^2/3\}$ is invariant with respect to the flow ϕ_t.

(8) (a) Define the term bifurcation. [2 Marks]

(b) What do we mean by Saddle-node bifurcation. Write its normal form. Discuss the stability analysis and whole mechanism by making graphs. [8 Marks]

(c) Make bifurcation diagram for saddle-node bifurcation. [4 Marks]

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Department of Mathematics
University of Delhi, Delhi
M.A./M.Sc., Part II Semester IV
MMATH18-405(i) CRYPTOGRAPHY

Time: 2 hr. 30 min. M.M. - 35

Instruction: Question one is compulsary of section A and attempt any four questions from section B.

Section A

1. (a) Determine the number of keys in an Affine cipher over $\mathbb{Z}_m$ for $m = 1225$. (01)

   (b) Write the definition of symmetric and asymmetric key cryptosystem. (01)

   (c) How S-boxes work in DES? (01)

   (d) Let $p = 17$ be the prime and $g = 3$ be the primitive root in $\mathbb{Z}_{17}$. Alice chooses private key $a(= 7) \in \{0, 1, \cdots, p-2\}$ and Bob chooses private key $b(= 4) \in \{0, 1, \cdots, p-2\}$. Then what will be the shared key of Alice and Bob using Diffie-Hellman key exchange protocol? (02)

   (e) What is $\{53\}^{-1}$ in $GF(2^8)$? (02)

Section B
Attempt any four questions

2. The following ciphertext were formed using keyword columnar transposition ciphers. Cryptanalyze with the given crib. `IDHTE NCLEX MECEH ACLHX AHPAO OAROA NTABF HDEFB SSAKT POATL IUESR OSBRL`, with the crib `PEACH BASKET`. (07)

3. Let $p = 53, g = 2, g^b \equiv 30 \mod 53$ be Bob's public ElGamal key. Alice uses it to generate the ciphertext $(24, 37.)$ Then what will be the corresponding plaintext? (07)

4. Let the 26 keys in the Shift Cipher are used with equal probability $1/26$. Then prove that for any plaintext probability distribution, the Shift Cipher has perfect secrecy. (07)

5. Let the output of subbyte operation of one round of AES is;

87	F2	4D	97
EC	6E	4C	90
4A	C3	46	E7
8C	D8	95	A6

Then fill the complete output after mix coloumns followed by shift rows in following state.

	40	A3	4C
37		70	9F
94	E4		42
ED	A5	A6	

(07)

6. Consider an RSA cipher with primes $p = 3$ and $q = 11$, encryption exponent $e = 7$, and correspondences $A = 0, B = 1, C = 2, \cdots, Z = 25$. Use this cipher to encrypt BOONE. (07)

7. Suppose that $l = m = 4$. Let π_S be defined as follows, where the input (i.e., z) and output (i.e., $\pi_S(z)$) are written in hexadecimal notation,

z	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
$\pi_S(z)$	8	4	2	1	C	6	3	D	A	5	E	7	F	B	9	0

Further, let π_P be defined as follows:

z	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\pi_P(z)$	1	5	9	13	2	6	10	14	3	7	11	15	4	8	12	16

Then using SPN network find the first round output of the plaintext x= `0010 0110 1011 0111`. The key for this SPN network is `0011 1010 1001 0100 1101 0110 0011 1111` and key schedule is K_{4r-3}. (07)

8. Suppose we are told that the plaintext BREATHTAKING yields the ciphertext RUPOTENTOIFV where the Hill cipher is used (but m is not specified). Determine the encryption matrix. (07)

M.A/M.Sc. Mathematics 2022
Part II Semester-IV
MMATH18-405(ii)- Support Vector Machines
UPC 223503402

Time: 2½ hours Maximum Marks: 35

Instructions: *Question 1 is **compulsory**. Answer any **four** questions from Question 2 to 8. All questions carry equal marks. Symbols have their usual meaning.*

1. (a) Prove that the quadratic function

$$f(x) = \frac{1}{2}x^T Hx + r^T x + \delta,$$

where $H \in \mathbf{R}^{n \times n}$ is a semi-definite matrix, $r \in \mathbf{R}^n$, and $\delta \in \mathbf{R}$, is convex. (2)

(b) What is difference between the classification and regression problem? (1)

~~(c) State the weak duality theorem. (2)~~

(d) Prove that the sum $K_1 + K_2$ is kernel given K_1 and K_2 are kernels. (1)

(e) Define adjacent matrix of a graph. (1)

2. (a) State the strong duality theorem. (1)

(b) Write the KKT conditions for the convex optimization problem:

$$\begin{aligned} \min \quad & f_0(x), \quad x \in \mathbf{R}^n \\ \text{s.t.} \quad & f_i(x) \le 0, \quad i = 1, \ldots m \\ & h_i(x) = a_i^T x - b_i = 0, \quad i = 1, \ldots m. \end{aligned}$$

Further, prove that for the above CPP satisfying Slater's condition. It is necessary and sufficient for its solution x^* to satisfy KKT conditions. (2+4)

3. (a) Write the algorithm for maximal margin method for linearly separable problems. (2)

(b) Prove that for a linearly separable problem, the solution to the CPP

$$\begin{aligned} \min_{w,b} \quad & \frac{1}{2}||w||^2 \\ \text{s.t.} \quad & y_i((w \cdot x_i) + b) \ge 1, \quad i = 1, \ldots l \end{aligned}$$

where $x_i \in \mathbf{R}^n$ and $y_i \in \mathcal{Y} = \{-1, 1\}$, exists and satisfies:

(i) $w^* \neq 0$.

(ii) there exists $j \in \{i | y_i = 1\}$ such that $(w^* \cdot x_j) + b^* = 1$. (5)

4. (a) Prove that the solution to the general classification problem in $\mathbf{R}^n$:

$$\begin{aligned} \min_{w,b,\zeta} \quad & \frac{1}{2}||w||^2 + C\sum_{i=1}^{l} \zeta_i \\ \text{s.t.} \quad & y_i((w \cdot x_i) + b) \ge 1 - \zeta_i, \quad i = 1, \ldots l \\ & \zeta_i \ge 0, \quad i = 1, \ldots l \end{aligned}$$

where $\zeta = (\zeta_1, \ldots, \zeta_l)^T$ and $C > 0$ is the penalty parameter, exist with respect to (w, b). Is the solution *w.r.t.* w unique? Justify. (2+1)

(b) For the above classification problem with training set T={(−1, −1), (2, 1)} and $C < \frac{1}{2}$, find the solution to the primal problem with respect to b. (4)

5. For the convex optimization problem:

$$\begin{aligned} \min_{w,b} \quad & \frac{1}{2}||w||^2 \\ \text{s.t.} \quad & (w \cdot x_i) + b - y_i \le \epsilon, \quad i = 1, \dots l \\ & y_i - (w \cdot x_i) - b \le \epsilon, \quad i = 1, \dots l \end{aligned}$$

where $x_i \in \mathbf{R}^n$ and $y_i \in \mathcal{Y} = \mathbf{R}$. If ϵ_{inf} is the optimal value of the problem

$$\begin{aligned} \min_{w,b,\epsilon} \quad & \epsilon \\ s.t. \quad & -\epsilon \le y_i - ((w \cdot x_i) + b) \le \epsilon, \quad i = 1, \dots l, \end{aligned} \quad \text{then}$$

(i) Find the dual of above primal problem. (4)

(ii) Prove that if $\epsilon > \epsilon_{inf}$, then the primal problem and its dual both have solutions. (3)

6. (a) Write the algorithm for linear ϵ-support vector regression. (3)

(b) Write the algorithm for classification machine based on the nonlinear separation. (4)

7. Consider the convex optimization problem:

$$\begin{aligned} \min_{w,b,\zeta} \quad & \frac{1}{2}||w||^2 + C\sum_{i=1}^{l} \zeta_i \\ \text{s.t.} \quad & y_i((w \cdot \Phi(x_i)) + b) \ge 1 - \zeta_i, \quad i = 1, \dots l \\ & \zeta_i \ge 0, \quad i = 1, \dots l \end{aligned}$$

where $x_i \in \mathbf{R}^n$, $y_i \in \mathcal{Y} = \{-1, 1\}$, and $\Phi : \mathbf{R}^n \to \mathcal{H}$ is defined as $\Phi(x) = \mathbf{x}$.

(a) Find the dual of above primal problem. (3)

(b) Suppose α^* is a solution to the dual of above primal problem and if there exists a component of α^*, $\alpha_j^* \in (0, C)$, then prove that the solution (w^*, b^*) of the primal problem can be obtained as:

$$w^* = \sum_{i=1}^{l} \alpha_i^* y_i \mathbf{x}_i \quad \text{and}$$

$$b^* = y_j - \sum_{i=1}^{l} \alpha_i^* y_i (\mathbf{x}_i . \mathbf{x}_j)$$

(4)

8. (a) Prove that the function $K(x, x') = exp\left(-\frac{||x - x'||^2}{\sigma^2}\right)$ is a kernel. (3)

(b) Discuss the properties of support vector machines based on C-support vector classification. (4)