Your Roll Number: ....

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examination, March-April, 2022
Part I Semester I
MMATH18-101: Field Theory
Unique Paper Code: 223501101

Time: 3 hours | Maximum Marks: 70

Instructions: • Attempt five questions in all. • Question 1 is compulsory. • All questions carry equal marks.

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(1) (a) Prove that a field k is algebraically closed if and only if every non-zero polynomial over k has a root in k [3]

(b) Give an example of a tower $k \subset F \subset E$ of field extensions such that E/F and F/k are both Galois but E/k is not. [3]

(c) Is an extension of $\mathbb{Q}$ of degree p^p, p a prime, simple? Justify. [2]

(d) Let F be a field of characteristic zero containing a primitive n-th root of unity, where n is odd. Show that F also contains a primitive $2n$-th root of unity. [3]

(e) Show that the group of the equation $x^4 + x^2 + 1 = 0$ over $\mathbb{Q}$ contains a transposition. [3]

(2) (a) Let $k \subseteq F \subseteq E$ be a tower of field extensions. Prove that E/k is algebraic if and only if both E/F and F/k are algebraic. [2+4]

(b) Find the minimal splitting field of $x^6 - 5$ over $\mathbb{Q}$ and $\mathbb{R}$. [3]

(c) Let p, q be distinct prime numbers and a a non-zero rational. Show that $x^{pq} - a$ is irreducible over $\mathbb{Q}$ if and only if both $x^p - a$ and $x^q - a$ are irreducible over $\mathbb{Q}$. [5]

(3) (a) Let α be a real root of the polynomial $X^4 - p$ over $\mathbb{Q}$, where p is a prime number. Is $\mathbb{Q}(i\alpha^2)/\mathbb{Q}$ a normal extension? Justify. [3]

(b) Prove that every countable (infinite) field has an algebraic closure which is also countable. [7]

(c) Let k be a field of non-zero characteristic p and let F be an extension of k of degree n such that n is co-prime with p. Prove that F/k is a separable extension. [4]

(4) (a) Let $f(x)$ be a separable polynomial over a field k and let E be the minimal splitting field of $f(x)$ over k. Prove that there are only finitely many subfields of E containing k. Does the result holds if $f(x)$ is not separable over k? Justify. [7+3]

(b) Determine the cyclotomic polynomial $\Phi_{p^3}(x)$, where p is a prime [4] number.

(5) (a) Prove that a finite extension of a finite field is simple. [4]

(b) Let G be the group of the equation $f(x) = 0$ over a field k. Prove that if $f(x)$ is irreducible over k, then degree of $f(x)$ divides the order of G. Does the result hold if $f(x)$ is not irreducible? Justify. [3+3]

(c) Prove that $\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}$ is a Galois extension and find its Galois group. Is every extension of $\mathbb{Q}$ contained in $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ Galois? Justify. [4]

(6) (a) Find the minimal splitting field of $x^{p^n} - 1$ over $\mathbb{F}_p$. [3]

(b) Prove that the group of an equation $f(x) = 0$ over a field k acts by permutation on the set of roots of $f(x)$. Also show that this action is transitive if and only if $f(x)$ is irreducible over k. [2+5]

(c) Find a subfield F of $\mathbb{Q}(\zeta)$ such that $[F : \mathbb{Q}] = 2$, where ζ is a primitive 5th root of unity. [4]

(7) (a) Prove that if the group of an equation $f(x) = 0$, over a field k of characteristic zero, is a solvable group then the equation is solvable by radicals. [10]

(b) Prove that the group of the equation $x^5 - 5x + 1 = 0$ over $\mathbb{Q}$ contains an element of order 4. [4]

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DEPARTMENT OF MATHEMATICS

M.A./M.Sc. Part-I, Semester-I, OBE March-April 2022

MMATH18-102 : Complex Analysis (UPC 223501102)

Time: 3 hours. Maximum Marks: 70

Instruction: Attempt any ***Five*** questions. Question **No.1 is compulsory**. Each question carries 14 marks.

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1. (i) What is the image of the region $G = \{z = x + iy : log2 < x < log3\}$ under the exponential function? [2]

   (ii) Let $Tz = \dfrac{Az + B}{Cz + D}$. Find z_2, z_3, z_4 in terms of A, B, C, D such that $Tz = (z, z_2, z_3, z_4)$. [2]

   (iii) Let f be analytic in the disk $B(a; R)$ and γ is a closed rectifiable in $B(a; R)$. Justify, without proof, $\int_\gamma f = 0$. [2]

   (iv) Suppose f is analytic in a neighbourhood of a and g has a simple pole at $z = a$. Show that $\text{Res}(fg; a) = f(a)\text{Res}(g; a)$. [2]

   (v) Evaluate $\int_\gamma \frac{sinz}{z^2}dz$, where γ is given by $\gamma(t) = 1 + \frac{1}{2}e^{it}, 0 \leq t \leq 4\pi$ [2]

   (vi) Show that if γ is a closed rectifiable curve in C and $a \notin \{\gamma\}$, then $n(-\gamma; a) = -n(\gamma; a)$. [2]

   (vii) State two advantages of defining analytic function on a simply connected domain. [2]

2. (a) Define conformal mapping. Is every analytic function $f : G \longrightarrow C$ conformal? Justify. [7]

   (b) State and prove Liouville's Theorem. Suppose the real part of an entire function is bounded, will the function be constant? Justify. [7]

3. (a) Let f be an analytic function on a region G. Suppose $a_1, a_2, ...a_m$ are the points in G that satisfy the equation $f(z) = \alpha$, and $b_1, b_2, ..., b_p$ are the points in G that satisfy the equation $f(z) = \beta$. Let γ be a closed rectifiable curve in $\mathbb{C}$ which does not pass through any a_k or b_j and $\gamma \approx 0$. State and deduce the condition for the equality

$$\sum_{k=1}^{m} n(\gamma; a_k) = \sum_{j=1}^{p} n(\gamma; b_j),$$

   What happen if γ is a circle? [7]

   (b) Evaluate $\int_0^\infty \frac{x^2}{1 + x^4}dx$. [7]

4. (a) Let f be analytic in the unit disk $D = \{z : |z| < 1\}$ such that $|f(z)| \leq 1$ and let $a \in D$. What happen if $|f(a)| = 1$? Suppose $f(a) \in D$, what is the maximum possible value of $|f'(a)|$? What happen if $f(a) = 0$? [7]

   (b) Suppose a function f has essential singularity at a and let z be a complex number. Show that there is a sequence $\{a_n\}$ such that $a_n \longrightarrow a$ and $f(a_n) \longrightarrow z$. [7]

5. (a) Let G be a connected open set and f be an analytic function on G. Show that if there is a point $a \in G$ such that $f^{(n)}(a) = 0$, for all $n \geq 0$, then f is a constant. What happen if G is not connected? [7]

   (b) Show that if γ_0 and γ_1 are two rectifiable curves in G from a to b and γ_0 and γ_1 are FEP homotopic, then
$$\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz$$
for every analytic function f on G. [7]

6. (a) Let G be an open set and γ be a closed rectifiable curve in G. Suppose that $\phi : G \times \{\gamma\} \longrightarrow C$ is continuous and define $g : G \longrightarrow C$ by
$$g(z) = \int_{\gamma} \phi(w, z)dw.$$
Show that g is continuous, and if $\frac{\partial \phi}{\partial z}$ exists for each (w, z) in $\{\gamma\} \times G$ and is continuous then g is analytic and
$$g'(z) = \int_{\gamma} \frac{\partial \phi}{\partial z}(w, z)dw.$$
[7]

   (b) If G is a bounded open set, $f : \overline{G} \longrightarrow C$ is continuous and f is analytic in G, show that $max\{|f(z)| : z \in \overline{G}\} = max\{|f(z)| : z \in \partial G\}$. By taking G as appropriate vertical strip show that the boundedness of G cannot be relaxed. [7]

7. (a) Can a function f having removable singularity at $z = a$ in G be redefined so that it is analytic in G? Justify. [7]

   (b) Let a be the only zero, with multiplicity $m > 1$, of an analytic function f in G, and γ is a closed rectifiable curve in G not passing through a and $\gamma \approx 0$. Evaluate $\int_{\gamma} \frac{f'}{f}$. What happen if $z = a$ is a simple zero and γ is a circle? [7]

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examinations, March- April 2022
Part I, Semester I
MMATH18-103: MEASURE AND INTEGRATION
(Unique Paper Code 223501103)

Time: 3 hours | Maximum Marks: 70

Instructions: • Write your Name, University Roll No., College, Course and Title of the Paper with Unique Paper Code on the first page and your Roll No. and page No. on each subsequent answer sheet. •Attempt **five** questions in all. **Question 1 is compulsory**. All questions carry equal marks.• The symbols used have their usual meanings.

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(1) (a) Does there exist a Lebesgue non-measurable subset of $[0,1]$ consisting only of irrational numbers? Justify your answer. [2]

(b) If f is a Lebesgue measurable function on $\mathbb{R}$, show that ess inf $f \geq \inf f$. Give an example with justification where strict inequality occurs. [2 +2]

(c) Let $\mathbb{S}$ be a nonempty collection of subsets of a nonempty set X. Show that $\mathbb{S}$ is a ring if and only if whenever $E, F \in \mathbb{S}$, then $E \cup F$ and $E \Delta F \in \mathbb{S}$. [2]

(d) Let $[X, \mathbb{S}, \mu]$ be a measure space and f be a nonnegative integrable function on X. For $n \in \mathbb{N}$, define $E_n = \{x : f(x) > n\}$. Then show that $\mu(E_n) \to 0$ as $n \to \infty$. Can you conclude that $\lim_{n\to\infty} \int_{E_n} f \, d\mu = 0$? Justify your answer. [2+2]

(e) Define f on $\mathbb{R}$ by [2]

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is a rational number,} \\ 2, & \text{if } x \text{ is an irrational number.} \end{cases}$$

Find the four derivatives of f at any $x \in \mathbb{R}$.

(2) (a) If $E \subset \mathbb{R}$ is measurable and $\varepsilon > 0$, show that there is an open set O containing E such that $m^*(O \setminus E) \leq \varepsilon$. In particular, if $E = \mathbf{Q} \cap [0,1]$, give explicitly such an open set O containing E. [4 + 4]

(b) Let $\mathbb{A}$ denote the σ-algebra of subsets of $\mathbb{R}$ generated by the F_σ sets. Then prove that $\mathbb{A} = \mathbf{B}$, the σ-algebra of Borel sets of $\mathbb{R}$. [6]

(3) (a) Let $[\![X, \mathbb{S}, \mu]\!]$ be a complete measure space and $\{f_n\}$ be a Cauchy sequence in $L^\infty(\mu)$. Thn prove that there exists a functon f such that $f_n \to f$ *a.e.*, $f \in L^\infty(\mu)$ and $\|f_n - f\|_\infty \to 0$ as $n \to \infty$. [6]

(b) Let $f(x) = \frac{1}{x^2}$, $x \in \mathbb{R} \setminus \{0\}$. Use monotone convergence theorem to prove that f is integrable over $[1, \infty)$ but not integrable over $(0,1]$. [4 +4]

(4) (a) If f is abso;utely continuous on $[a, b]$ and $f' = 0$ a.e., then show that f is a constant. [6]

(b) Let f and g be defined on $\mathbb{R}$ by [5+3]

$$f(x) = \begin{cases} x \sin \frac{2\pi}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases}$$

and

$$g(x) = \begin{cases} x^2 \sin\frac{1}{x}, & \text{if } x \neq 0\ , \\ 0, & \text{if } x = 0. \end{cases}$$

Then show that $f \notin BV[-1,0]$ but $g \in BV[-1,0]$.

(5) (a) Prove that there exists a subset of $\mathbb{R}$ which is not Lebesgue measurable. [7]

(b) Let $[\![X, \mathbb{S}, \mu]\!]$ be a complete measure space and $\{f_n\}$ be a sequence of real valued measurable functions on X. We say $\{f_n\}$ is *almost uniformly fundamental* if for every $\varepsilon > 0$, there is a set $E \in \mathbb{S}$ with $\mu(E) < \varepsilon$ such that $\{f_n\}$ is uniformly Cauchy in E^c (i.e., for given $\delta > 0$, there is $N \in \mathbb{N}$ such that $|f_n(x) - f_m(x)| < \delta$, $\forall n \geq N$ and for all $x \in E^c$.) Show that $\{f_n\}$ is almost uniformly fundamental if and only if there is a real valued measurable function f such that $f_n \to f$ a.u.. [7]

(6) (a) Modify the proof of Lebesgue's dominated convergence theorem suitably to prove the following: Suppose $\{f_n\}, \{g_n\}$ are sequences of integrable functions on $\mathbb{R}$, such that $f_n \to f$ a.e. and $g_n \to g$ a.e., suppose f, g are integrable, $|f_n| \leq |g_n|$, $\forall n$ and $\int g_n\, dx \to \int g\, dx$. Then $\int f_n\, dx \to \int f\, dx$. [5]

(b) Suppose $\{f_n\}$ is sequence of integrable functions on $\mathbb{R}$, such that $f_n \to f$ a.e. and f is integrable. Then show that $\lim_{n\to\infty} \int |f_n - f|\, dx = 0$ if and only if [4]

$$\lim_{n\to\infty} \int |f_n|\, dx = \int |f|\, dx.$$

(c) Let $[\![X, \mathbb{S}, \mu]\!]$ be a measure space, $A, B \in \mathbb{S}$ such that $\mu(A \Delta B) = 0$. If f is an extended real valued function on X which is integrable over A, then show that f is integrable over B and that $\int_A f\, d\mu = \int_B f\, d\mu$. [5]

(7) (a) Let f be a bounded function on the finite interval $[a, b]$. If f is continuus a.e., show that f is Riemann integrable over $[a, b]$. [6]

(b) Let $[\![X, \mathbb{S}, \mu]\!]$ be a measure space and $f, f_n,\ n \geq 1$ be real valued measurable functions on X such that $f_n \to f\, a.e.,$. If $\mu(X) < \infty$, show that $f_n \to f$ in measure. Give an example with justification to show that the conclusion fails if $\mu(X)$ is not finite. [4+4]

Your Roll Number: ....

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics End Semester Examinations April 2022
Part I Semester I
MMATH18-104: Differential Equations
UPC: 223501104

Time: 3 Hours | Maximum Marks: 70

Instructions: • Attempt five questions in all. • Question 1 is compulsory. • All question carry equal marks. • All notations have usual meaning.

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(1) (a) Show that $IVP : y' = \frac{\cos y}{1-x^2}$, $y(0) = y_0$, ($|y_0| < \infty$) has a solution for $|x| < 1$. [2 Marks]

(b) Assume that $u(x, y)$ satisfies the Laplace equation in a bounded domain ω and is continuous on $\Omega = \omega \cup B$. Then show that u attains its minimum on the boundary B of ω. [3 Marks]

(c) Does there exist a unique solution ψ of the equation $y^{(3)} = x^2 + yy' + y'^2$ such that $\psi(0) = 1$, $\psi'(0) = -3$, $\psi''(0) = 0$? Explain precisely why or why not. [3 Marks]

(d) Show that $\frac{dG(x,t)}{dx}\Big|_{t=x_-}^{t=x_+} = \frac{1}{p(x)}$ is equivalent to $\frac{dG(x,t)}{dx}\Big|_{x=t_-}^{x=t_+} = -\frac{1}{p(t)}$. [2 Marks]

(e) Find the eigenvalues and eigenfunctions of the system: [4 Marks]
$y'' + \lambda y = 0;\ y(0) = y(\pi),\ y'(0) = y'(\pi)$.

(2) (a) Show that a unique solution exists for the initial-value problem $y' = 1 + y^2$, $y(0) = 0$ on the interval $|x| \le \frac{1}{2}$. [5 Marks]

(b) Suppose that $p(x) > 0$ and $p(x)$ is continuous in $(0, \infty)$. Prove that every nontrivial solution of $y'' + p(x)y = 0$ has infinitely many zeros in $(0, \infty)$. [4 Marks]

(c) If A is an $n \times n$ constant matrix, then $\Psi(t) = e^{tA}$ is a fundamental matrix of the system $Y' = AY$ on J. If $\Phi(t) = E$, then show that $\Phi(t) = e^{(t-t_0)A}E$. [5 Marks]

(3) (a) Solve by the method of eigenvalues : $y_1' = y_1 + y_2 + 2t$, $y_2' = 4y_1 + y_2 - \sin t$. [7 Marks]

(b) Using integral transformation, determine the solution of the equation $\frac{\partial z}{\partial x} - \frac{\partial^2 z}{\partial y^2} = 0$, which approach to 0 as $y \to \infty$ and satisfies the conditions [7 Marks]
(i) $z = f(x)$ when $y = 0$, $x > 0$, (ii) $z = 0$ when $y > 0$, $x = 0$.

(4) (a) Consider the differential equation $y^{(n)} = f(x, y, y', ..., y^{(n-1)})$, where the function f is continuous and satisfies the Lipschitz condition in a domain Ω of real $(n+1)-$dimensional $(x, y, y', ..., y^{(n-1)})$-space and $(x_0, c_0, c_1, ..., c_{n-1})$ is a point in Ω. Then show that [7 Marks]

there exists a unique solution ϕ such that $\phi(x_0) = c_0$, $\phi'(x_0) = c_1, ..., \phi^{(n-1)}(x_0) = c_{n-1}$ defined on some interval $|x - x_0| \leq h$ about $x = x_0$.

(b) Determine the solution of diffusion equation $\frac{\partial\theta}{\partial t} = k\nabla^2\theta$ using the Green's function subject to the conditions $\theta(\mathbf{r}, \mathbf{t}) = \phi(\mathbf{r}, \mathbf{t})$ and $\theta(\mathbf{r}, \mathbf{0}) = \mathbf{f}(\mathbf{r})$. [7 Marks]

(5) (a) If V is a Lyapunov function such that $-\nabla V(X).F(X)$ is positive definite in Ω, then show that origin is asymptotically stable. [5 Marks]

(b) Obtain the Poisson integral formula for Dirichlet problem defined on a circle. [6 Marks]

(c) Determine the asymptotic behavior of the solution of following system near the critical point: [3 Marks]
$\dot{x} = -x - x^2 + xy, \quad \dot{y} = -y + xy - y^2.$

(6) (a) Using the Green's function, find the solution of three dimensional Laplace equation for a semi-infinite space. [6 Marks]

(b) Find the Green's function for the BVP: [4 Marks]
$L[y] = (1 - x^2)y'' - 2xy' = 0$, $y(0) = 0$, $y'(1) = 0$.

(c) Derive the Lagrange identity for $L[y] = 0$ and hence, deduce for self-adjoint operator. [4 Marks]

(7) (a) Consider the IVP: $y_1' = y_2^2 + 1$, $y_2' = y_1^2$; $y_1(0) = 0, y_2(0) = 0$. Compute a bound M, a Lipschitz constant K and first three approximations. [4 Marks]

(b) Find the fundamental matrix of the system $Y' = AY$ if [3 Marks]

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}.$$

(c) State and prove the Helmholtz's second theorem. [7 Marks]

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