Your Roll Number: .....

Question paper for OBE (December 2021-University of Delhi)

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations, December 16, 2021
Part II Semester III
MMATH18-301(i): Algebraic Topology (UPC- 223502301)

Time: 3 hours Maximum Marks: 70

Instructions: • Attempt five questions in all. • Question 1 is compulsory and answer any four questions from Q.2. to Q.7. • Each question carries 14 marks.

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(1) (a) Let A be retract of a Hausdorff space X. Show that A is closed in X. [3.5 Marks]

(b) Define *Universal Covering Space* of a space X. Jusity the term Universal. [3.5 Marks]

(c) Give an example that quotient space of a Hausdorff space need not be a Hausdorff. Justify. [3.5 Marks]

(d) Show that a covering map $p : \widetilde{X} \to X$ induces injective homomorphism between fundamental groups of $\widetilde{X}$ and X. [3.5 Marks]

(2) (a) Let $f : \mathbb{D}^2 \to \mathbb{S}^1$ be a continuous map such that $f(0) = 1$. For each integer m, show that there exists $\tilde{f} : \mathbb{D}^2 \to \mathbb{R}$ such that $p\tilde{f} = f$ and $\tilde{f}(0) = m$, where p is the exponential map. [9 Marks]

(b) Define *degree* of a continuous map $f : \mathbb{S}^1 \to \mathbb{S}^1$. Determine the degree of covering map $p : \mathbb{S}^1 \to \mathbb{S}^1$ defined by $p(z) = z^n$. [5 Marks]

(3) (a) Show that a continuous map $f : \mathbb{S}^n \to \mathbb{D}^{n+1}$ can be continuously extended to the disc $\mathbb{D}^{n+1}$. [5 Marks]

(b) Prove that an open interval can never be a retract of real line $\mathbb{R}$. [6 Marks]

(c) Are two constant maps homotopic ? Justify. [3 Marks]

(4) (a) Let f_1, f_2, f_3 and f_4 be paths in X such that $f_i(1) = f_{i+1}(0)$, $1 \leq i \leq 3$. Determine the paths $(f_1 * f_2) * (f_3 * f_4)$ and $(f_1 * (f_2 * f_3)) * f_4$. [6 Marks]

(b) Let X be any space. Show that $\mathbb{D}^2 \times X$ is homotopically equivalent to X. [4 Marks]

(c) Determine the fundamental group of the real projective n-space $\mathbb{R}P^n$, $n > 1$. [4 Marks]

(5) (a) Prove that there is no antipodal preserving continuous map $f : \mathbb{S}^2 \to \mathbb{S}^1$. Hence, or otherwise prove the Borsuk Ulam theorem. [9 Marks]

(b) Prove or Disprove. Any continuous map from the product of [5 Marks] projectives space $\mathbb{R}P^n \times \mathbb{R}P^n$, $n > 1$, into the circle $\mathbb{S}^1$ is null homotopic.

(6) (a) Prove or Disprove. An infinite product of covering maps is a covering map. [6 Marks]

(b) Define a *regular* covering map. For a regular covering map $p : \widetilde{X} \to X$, show that, for any closed path g in X, either every lifting of g is closed or none is closed. [8 Marks]

(7) (a) Let $p : \widetilde{X} \to X$ be an n-sheeted covering map. If $\widetilde{X}$ is contractible, X is path connected and n is a prime then prove that the fundamental group of X is isomorphic to the cyclic group of order n. [4 Marks]

(b) Define *isomorphic covering spaces* of a space X. Prove that two universal covering spaces of a space X are isomorphic. [5 Marks]

(c) Prove that $\pi_1(\mathbb{S}^1 \vee \mathbb{S}^1)$ is a free group with two generators. [5 Marks]

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Your Roll Number: ....

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examinations, December 2021
Part II Semester III
MMATH18-301(ii): COMMUTATIVE ALGEBRA
(Unique Paper Code: 223502302)

Time: 3 + 1 Hours | Maximum Marks: 70

Instructions: • Attempt five questions in all. Question 1 is compulsory. All questions carry equal marks. • Throughout the paper all the rings are assumed to be Commutative ring with nonzero identity.

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(1) (a) For any ideal I of a ring R, show that $rad(I^5) = rad(I)$. [2]

(b) Let $\mathfrak{p}$ be a nonzero prime ideal of a ring R. Show that $R_\mathfrak{p}$ cannot be an Artin ring. [3]

(c) Let K be a field and $A \subset B$ be valuation subrings of K. Show that the quotient field of A is same as quotient field of B. [3]

(d) Let R be a ring such that $rad(0) = \mathfrak{p}$ is a prime ideal. Show that for any proper ideals $I, J \in R_\mathfrak{p}$, $rad(I) = rad(J)$. [3]

(e) Let $A \subset B$ be valuation subrings of a field K. If $\mathfrak{m}, \mathfrak{n}$ are unique maximal ideals of A, B, respectively, then show that $\mathfrak{n} \subseteq \mathfrak{m}$. [3]

(2) Let R be a ring.

(a) If $a \in R$ is any nonzero element, then show that there exists a prime ideal of R not containing a. [5]

(b) Let I be a proper ideal in a ring R. Show that there is one to one correspondence between $Spec(R/rad(I))$ and $\{\mathfrak{p} \in Spec(R) \mid I \subseteq \mathfrak{p}\}$. [5]

(c) Let I be an ideal in a ring R which is not contained in any maximal ideal of R. Show that $I = R$. [4]

(3) (a) Let R be a ring and M, N, P be R-modules. Show that there is one to one correspondence between R-bilinear maps from $M \times N \to P$ and R-linear maps from $M \otimes N \to P$. [7]

(b) Let M, N, P be R-modules. Show that

$$(M \oplus N) \otimes_R P \cong (M \otimes_R P) \oplus (N \otimes_R P)$$

[7]

(4) (a) Consider the ring $\mathbb{Q}_{12}$ of all rational numbers whose denominator is some power of 12. Determine all prime ideals of $\mathbb{Q}_{12}$. [5]

(b) Show that there is one to one corresondence bertween prime ideals of R contained in $\mathfrak{p}$ and prime ideals of $R_{\mathfrak{p}}$. [5]

(c) Let M be an R-module and N be its submodule. Show that $M = N$ if and only if $M_{\mathfrak{m}} = N_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$ of R. [4]

(5) (a) Show that in a Noetherian ring R the set of all zero divisors is equal to a finite union of prime ideals. [4]

(b) Show that primary ideals may not be a power of prime ideal. [5]

(c) Show that a power of prime ideal may not be primary. [5]

(6) (a) Let $A \subseteq B$ be integral domains such that B be integral over A. Show that A is a field if and only if B is a field. [6]

(b) State and prove Going down theorem. [8]

(7) (a) Let R be a ring such that $\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_n$ be maximal ideals such that $\prod_{i=1}^{n} \mathfrak{m}_i = 0$. Show that R is Noetherian if and only if R is Artinian. [7]

(b) Let R be a local domain. Show that R is a DVR if and only if every non zero fractional ideal of R is invertible. [7]

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Department of Mathematics
University of Delhi, Delhi

M.A./M.Sc. Mathematics, Part-II, Semester-III
Examination, December 2021

Paper: **MMATH18 301(iii) Representation of Finite Groups**
(Unique Paper Code: 223502303)

Time: 3 Hour *Maximum Marks: 70*

Note: • Attempt five questions in all. • Question 1 is compulsory. • All questions carry equal marks.

1. (a) Does a representation equivalent to a faithful representation faithful? Justify. (3)
   (b) Can a product of two non-zero elements of a group algebra $\mathbb{C}G$ be zero? Justify. (3)
   (c) Show that for any character χ of a group G of degree n, $\langle \chi_{\text{reg}}, \chi \rangle = n$. (2)
   (d) Does two equivalent representations of a group have the same character? Justify.(3)
   (e) Let G be a group of order $2p$, where p is a prime. Can G have an irreducible character of degree p? Justify. (3)

2. (a) Let $G = \langle a, b, c \mid a^3 = b^3 = c^2 = 1, ab = ba, c^{-1}ac = a^{-1}, c^{-1}bc = b^{-1} \rangle$ be a group having 18 elements. Prove that the map $\rho : G \longrightarrow GL(2, \mathbb{C})$ given by $a\rho = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix}, b\rho = \begin{pmatrix} \eta & 0 \\ 0 & \eta^{-1} \end{pmatrix}, c\rho = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is a representation of G, where ϵ and η are complex cube roots of unity. For which values of ϵ, η is ρ irreducible and for which values of ϵ, η is ρ faithful? (4+2+2)
   (b) Does the permutation FG-module always isomorphic to the regular FG-module for every finite group G? Justify. (6)

3. (a) Prove that a finite group G is abelian if and only if every irreducible representation of G over $\mathbb{C}$ is of degree one. (8)
   (b) Prove that $\overline{C} \in Z(\mathbb{C}G)$, where $\overline{C}$ is the class sum of a conjugacy class C of G and $Z(\mathbb{C}G)$ is the centre of the group algebra $\mathbb{C}G$. Also show that $v\overline{C} = \lambda v$ for all v in V, where V is an irreducible $\mathbb{C}G$-module and $\lambda \in \mathbb{C}$. (3+3)

4. (a) Express $\mathbb{C}G$ as a direct sum of irreducible $\mathbb{C}G$-modules for the group $G = D_6$. (7)
   (b) Find a basis of $\text{Hom}_{\mathbb{C}G}(\mathbb{C}G, \mathbb{C}G)$, where G is a finite group. (7)

5. (a) Prove that the number of linear characters of a group divides the order of the group. Deduce that that a group of order 12 cannot be simple. (5+3)
   (b) Let χ be a non-zero, non-trivial character of G such that $\chi(g) = \overline{\chi(g)}$ for all $g \in G$. Prove that χ is reducible. (6)

6. (a) Does the product of two irreducible characters of a group always an irreducible character? Justify. (4)

   (b) Write down the complete character table of $A_4 \times C_3$. (10)

7. (a) State and prove the column orthogonality relation for the irreducible characters of a group G. (6)

   (b) Prove that the degree of every irreducible character of G divides the order of G. Using this prove that every group of order p^2 is abelian, p a prime number. (6+2)

M.A./M.Sc. Mathematics Examinations, December-2021
Part II, Semester III
MMATH18-302(i): Fourier Analysis
Unique Paper Code-223502304

Time: 3 hr Maximum Marks: 70

Instructions: • Attempt **Five** questions in all • Question No. 1 is **COMPULSORY** • All questions carry equal marks • Symbols have their usual meaning

(Question No. 1 is Compulsory: 14 Marks)

(1) (a) Show that if $f(x) = 2\chi_{[0,1]}(2x)$, $x \in \mathbb{R}$, then $\widehat{f}(\xi) = \widehat{\chi}_{[0,1]}(\frac{\xi}{2})$ $(\xi \in \mathbb{R})$. [3 Marks]

(b) Show that every neighborhood of the identity element e of a topological group G contains the closure of a neighborhood of e. [3 Marks]

(c) Show that a left Haar measure λ on a locally compact group G is not identically zero. [4 Marks]

(d) Show that if K is any compact subgroup of a locally compact group G and Δ is the modular function of G, then $\Delta(x) = 1$ for all $x \in K$. [4 Marks]

(Answer any FOUR questions: 4×14 **marks** $= 56$ **Marks)**

(2) (a) Show that if $f \in L^1(\mathbb{T})$ is absolutely continuous, then $\widehat{f}(n) = o(\frac{1}{n})$, $n \neq 0$. [4 Marks]

(b) How the Fejér's kernel is used in approximation of functions in $L^1(\mathbb{T})$? Explain. [10 Marks]

(3) (a) How do you interpret the Fourier transform on $L^1(G)$ as a convolution, where G is a locally compact abelian group G? Justify our answer. [4 Marks]

(b) Show that $(\mathbb{R}, +)$, the additive group of real numbers, is a topological group. Find characters of $(\mathbb{R}, +)$. Justify your answer. [4+6= 10 Marks]

(4) (a) What can you say about the identity of the Banach algebra $L^1(\mathbb{R})$ with respect to the convolution? Explain. [5 Marks]

(b) Show that if U is a symmetric neighborhood of the identity e of a topological group G, then the set $\bigcup_{n=1}^{\infty} U^n$ is an open and closed subgroup of G. [9 Marks]

(5) (a) Show that $\left\{f \in L^1(\mathbb{R}) : \operatorname{supp}\widehat{f} \text{ is compact}\right\}$ is dense in $L^1(\mathbb{R})$. [7 Marks]

(b) Show that a Radon measure μ on a locally compact group G is a left Haar measure if and only if $\int_G L_y f d\mu = \int_G f d\mu$ for every $y \in G$ and for every $f \in C_c^+(G)$. Here, L_y is the left translate of f by y which is defined as $L_y f(x) = f(y^{-1}x)$, $x \in G$. [7 Marks]

(6) (a) Show that if $f \in L^1(\mathbb{R})$ and $xf(x) \in L^1(\mathbb{R})$, then $\widehat{f}$ is differentiable and [6 Marks]

$$\frac{d}{d\xi}\widehat{f}(\xi) = \widehat{(-ixf)}(\xi).$$

(b) Show that if G is a locally compact abelian (LCA) group, then every non-zero complex homomorphism Ψ of $L^1(G)$ is obtained in the following way: $\Psi : L^1(G) \to \mathbb{C}$, $\Psi(f) = \widehat{f}(\gamma)$. Here, $\widehat{f}$ is the Fourier transform of f and γ denote the character of G. [8 Marks]

(7) (a) Under which condition(s) $\lim S_n(f, 0) = 0$? Justify your answer. Here, $f \in L^1(\mathbb{T})$ and $S_n(f,t) = \sum_{j=-n}^{n} \widehat{f}(j)e^{ijt}$ for all $n \in \mathbb{N}$. [6 Marks]

(b) Show that if G is a topological group with identity e, U an open subset of G and F a compact subset of G such that $F \subset U$, then there exists a neighborhood V of e such that $(FV) \cup (VF) \subset U$. [8 Marks]

Your Roll Number: ....

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations, December 2021
Part II Semester III
MMATH18-302(iii): Theory of Bounded Operators
Unique Paper Code 223502306

Time: **4 hours** (This includes **one hour** time for downloading the paper, scanning your answer sheets and uploading back in the email for the final submission.)
Maximum Marks: **70**

Instructions: • All notations used are standard • **Question no.** 1 is **compulsory** • Attempt any **four** questions from the remaining six questions.

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(1) Choose the right option(s):

(a) Let $T : l^2 \to l^2$ be defined by $T(x_1, x_2, \cdots) = (x_1, 0, x_3, 0, \cdots)$. Then
- (i) $\rho(T)$ is an open subset of $\mathbb{C}$.
- (ii) T is a compact operator.
- (iii) T is a partial isometry.
- (iv) $\sigma(T^*)$ is countable.

(b) Let T be a bounded, self-adjoint operator on a complex Hilbert space H. Then
- (i) T cannot be compact.
- (ii) T is positive.
- (iii) $\sigma(T) \subset \mathbb{R}$.
- (iv) $\sigma(T) = \sigma_{app}(T)$.

(c) Let T be a bounded linear operator on an infinite dimensional Hilbert space X. Then
- (i) The null space $\mathcal{N}(T)$ of T is finite dimensional.
- (ii) If T is compact, the $Range T$ is always compact.
- (iii) If T is self-adjoint, then the null space $N(T_\lambda)$ of $T := T - \lambda I$ is finite dimensional for every $\lambda > 0$.
- (iv) If T is compact, then the null space $N(T_\lambda)$ of $T := T - \lambda I$ is finite dimensional for every $\lambda > 0$.

(d) Let T be a bounded positive linear operator on a Hilbert space H. Then
- (i) there may exist several bounded linear operators A on H satisfying $A^2 = T$.
- (ii) there exists a unique positive bounded linear operator A on H satisfying $A^2 = T$.
- (iii) if A is a bounded linear operator on H satisfying $A^2 = T$, A commutes with every operator that commutes with T.
- (iv) there exists a unique positive bounded linear operator B on H satisfying $B^4 = T$.

(e) Let T be a bounded linear operator on a Banach space X which is bounded below. Then
- (i) T is injective.
- (ii) T is surjective.

(iii) T has closed range.
(iv) T is invertible.

(f) Let P be a projection on a Hilbert space H. Then
(i) $\sigma(P) = \{0, 1\}$.
(ii) P is a partial isometry.
(iii) Every non-constant polynomial in P has degree 1.
(iv) P is the limit, with respect to the operator norm, of a sequence of compact operators on H.

(g) Let T be a compact operator on an infinite dimensional Banach space X. Then
(i) $T - \lambda I$ has separable range for every $\lambda > 0$.
(ii) $\sigma_{comp}(T)$ is always empty.
(iii) T^* is also a compact operator.
(iv) T cannot be invertible.

(2x7)

(2) (a) Let $T \in \mathcal{B}(X)$, where X is a Banach space. If T is bounded below is it necessarily one - one ? What can you say about the converse implication? Justify your claim.
(b) State and prove a necessary and sufficient condition for a bounded linear operator T to be invertible in terms of it being bounded below. Further show that $\sigma(T) = \sigma_{app}(T) \cup \sigma_{comp}(T)$.
(c) Let T and X be as above. Then show that $\sigma_{app}(T)$ is a closed set. Further, prove that boundary of $\sigma(T)$ is contained in $\sigma_{app}(T)$.
$(2 + 6 + 6)$

(3) (a) Let $T : l^2 \to l^2$ be defined by $T(\zeta_1, \zeta_2, \ldots) = (\zeta_1, \frac{1}{2}\zeta_2, \frac{1}{2^2}\zeta_3, \ldots)$. Find $\sigma_p(T), \sigma_{app}(T)$ and $\sigma(T)$. Is T compact? Justify.
(b) If y and z are any two vectors in a Hilbert space H, then show that the operator $T : H \to H$ defined by $Tx = \langle x, y\rangle z$ is compact.
(c) Let $T : X \to Y$ be a compact linear operator from a normed linear space X into a Banach space Y. Suppose $\widehat{X}$ is completion of X. Then show that there exists a compact linear operator $\widehat{T} : \widehat{X} \to Y$ which is extension of T. $(4 + 5 + 5)$

(4) (a) Let $T : X \to X$ be a compact linear operator, where X is a normed linear space. Is the range space $T_\lambda(X)$ closed for any $\lambda \neq 0$, where $T_\lambda = T - \lambda I$? Justify.
(b) Is it true that if $T \in \mathcal{B}(H)$ is bounded below then T cannot be compact? Justify your claim.
(c) Let $T : X \to X$ be a compact linear operator, where X is a normed linear space. Prove or disprove that $dim\mathcal{N}(T_\lambda^n) < \infty$ for all $n \in \mathbb{N}$.
$(5 + 3 + 6)$

(5) (a) Let T be a bounded self-adjoint linear operator on a complex Hilbert space H. Then show that $\sigma(T) = \sigma_{app}(T)$.
(b) State spectral mapping theorem for polynomials. Use it to show that for $H \neq \{0\}$ and T as above, both m and M are in $\sigma(T)$, where $m = \inf_{x \in H}\{\langle Tx, x\rangle \mid \|x\| = 1\}$ and $M = \sup_{x \in H}\{\langle Tx, x\rangle \mid \|x\| = 1\}$.

(c) Is it true that residual spectrum of a bounded self-adjoint linear operator on a Hilbert space is empty? Justify your claim.

$(6 + 6 + 2)$

(6) (a) Show that a monotonically increasing, commuting sequence $\{T_n\}$ of self-adjoint operators over a complex Hilbert space H satisfying $T_n \leq K$, where K is also a self-adjoint operator, is strongly operator convergent provided $T_nK = KT_n$ for all n. Also, show that the limit operator is self-adjoint.

(b) Does every positive bounded self-adjoint linear operator $T : H \to H$ satisfying $0 \leq T \leq I$ have a unique positive square root? Justify your claim.

$(7 + 7)$

(7) (a) Show that a monotonically increasing sequence $\{P_n\}$ of projections on a complex Hilbert space H strongly operator converges to a projection P. Further, show that P projects H onto $\overline{\cup_{n=1}^{\infty}(P_n(H))}$.

(b) Show that for a $T \in \mathcal{B}(H)$, where H is a Hilbert space, we can find a partial isometry W satisfying $T = W \mid T \mid$. Is this decomposition unique? Justify your claim.

(c) Let T be a compact linear operator on a complex Hilbert space H. Then show that there exists two orthonormal sets $\{e_n\}$ and $\{f_n\}$ in H and a sequence of positive numbers $\{s_n\}$ converging to 0 such that $T = \sum_n s_n < ., e_n > f_n$. $(4 + 5 + 5)$

M.A./M.Sc./ Mathematics/ Semester III/OBE
December 2021
Paper: MMATH18- 303(i)
Advanced Complex Analysis (UPC 223502307)

Time: 3 Hours | Maximum Marks: 70

Instruction: Attempt five (5) questions in all. **Question No.1 is compulsory**.

In this paper $\mathcal{C}$ is the complex plane, $D = \{z : |z| < 1\}$ is the unit disk and $\mathcal{R}$ is the real line.

1. (a.) Let $f(z) = z - z^2$. Determine one value of r so that $f(D)$ contains a disk of radius r. [3]
   (b) Suppose $f \in H(G)$ does not assume the values $1 + i$ and -2. Determine an analytic function g which never assume any integer. [3]
   (c.) Are the complex plane $\mathcal{C}$ and the disk $B(1; 2)$ conformally equivalent? Why? [2]
   (d.) Prove that a harmonic function has the MVP. [3]
   (e.) Discuss the convergence of $\prod_{n=2}^{\infty} \frac{1}{n^p}$, $p > 0$. [3]

2. (a.) Let $\{f_n\} \subset H(G)$ be a sequence of one-one functions which converges to f. What can you say about f? Justify. [7]
   (b.) If $u : G \longrightarrow \mathcal{R}$ is harmonic, show that $u_x - iu_y$ is analytic. What happen when $\{z : u_x(z) = u_y(z)\}$ has a limit point in G? Justify. [7]

3. (a.) Let $\{f_n\} \subset H(G)$ and $\sum_{n=1}^{\infty} f_n(z)$ converges uniformly on compact subsets to f. Show that $f^{(k)}(z) = \sum_{n=1}^{\infty} f_n^{(k)}(z)$ for each $k \geq 0$ [7]
   (b.) Let G be a simply connected set which is not the whole plane and let $c \in G$. Show that there is $f \in H(G)$ such that $f(c) = 0, f'(c) > 0, f$ is one-one and $f(G) \subset D$. [7]

4. (a.) Show that a family $\mathcal{F}$ in $H(G)$ is *normal* if and only if $\mathcal{F}$ is *locally bounded.* [7]

   (b.) Define absolute convergence of an infinite product. Discuss the convergence of $\prod_{n=1}^{\infty}\left|1+\frac{i}{n}\right|$ and $\prod_{n=1}^{\infty}\left(1+\frac{i}{n}\right)$. [7]

5. (a.) Let G be a simply connected region, $f \in H(G)$ and $\phi_1, ..., \phi_n$ are non-vanishing analytic functions on G and let $\partial_\infty G = A \cup B_1 \cup ... \cup B_n$. Suppose there is a number M such that (a) $limsup_{z\to a}|f(z)| \leq M$ for all $a \in A$ and (b) $limsup_{z\to b}|f(z)||\phi_k(z)|^{\eta_k} \leq M$ for each $b \in B_k$ and each $\eta_k > 0$ for $k = 1, ..., n$. Show that $|f(z)| \leq M$ for all $z \in G$. [7]

   (b.) Show that the Poisson kernel $P_r(\theta)$ satisfies $lim_{r\to 1^-} P_r(\theta) = 0$ uniformly in θ for $\pi \geq |\theta| \geq \delta$ for each $\delta > 0$. [7]

6. (a.) Show that if f is analytic on an open set containing the disk $\overline{B}(a; R)$ then

   $|f(a)|^2 \leq \frac{1}{\pi R^2}\int_0^{2\pi}\int_0^R |f(a + re^{i\theta})|^2 r\ dr\ d\theta$ [6]

   (b.) Define a *Dirichlet Region* and show that the unit disk $D = \{z : |z| < 1\}$ is a Dirichlet Region. [8]

7. (a.) Show that a sequence $\{f_n\}$ converges to f in $H(G)$ iff $\{f_n\}$ converges to f uniformly on all compact subsets of G. [6]

   (b.) Let f be analytic on a region containing closure of the disk $B(0; R)$. Show that the range of f contains a disk of radius $R|f'(0)|L$, where L is the Landau's constant. [8]

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Your Roll Number: ....

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations, December 2021
Part II Semester III
MMATH18-304(ii) UPC: 223502311
COMPUTATIONAL METHODS FOR ODEs

Time: 3 hours | Maximum Marks: 70

Instructions: • Section A is compulsory • Answer **any four** questions from Section B • Each question carries equal marks • Non-programmable scientific calculators are allowed. • Notations have their usual meaning

Section A

(1) (a) Given the first characteristic polynomial $\rho(\xi) = (\xi - 1)(\xi - 1/2)$, find the implicit multistep method. [3]

(b) State true and False and Justify the statement: The method $y_{n+1} = y_n + hf(t_{n+1/2}, \frac{1}{2}(y_n + y_{n+1}))$ where $t_{n+1/2} = (t_n + t_{n+1})/2$ is unconditionally consistent for the numerical solution of the problem $y' = f(x, y)$. [3]

(c) Find the interval of absolute stability of the classical second order Runge-Kutta method. [3]

(d) Define order and the error constant for the linear multistep method for the solution of a first order initial value problem. [2]

(e) Give a first order scheme for the numerical solution of $y'' + y' - y = 0,\ 0 < x < 1, \quad y'(0) = 2,\ y(1) = 1$. [3]

(2) (a) Find the range of α for which the linear multistep method [7]

$$y_{j+1} - \alpha(y_j - y_{j-1}) - y_{j-2} = \frac{h}{2}(3 + \alpha)(y'_j + y'_{j-1})$$

is absolutely stable when applied on the equation $y' = \lambda y,\ \lambda < 0$. Is it consistent? Justify.

(b) Compute $y(0.5),\ y'(0.5),$ and $y''(0.5)$ with Taylor series method of order two and step length $h = 0.5$ when $y(x)$ is the solution of the initial value problem [7]

$$y''' + 2y'' + y' + 3y = \sin x, \quad 0 \le x \le 1, \quad y(0) = 0,\ y'(0) = 1,\ y''(0) = 2,$$

after reducing it to the system of first order equations.

(3) (a) Consider the Modified Euler method [7]

$$y_{j+1} = y_j + p_1 hf(x_j, y_j)$$
$$y_{j+2} = y_{j+1} + p_2 hf(x_{j+1}, y_{j+1}),\ j = 0,\ 2,\ 4, \ldots$$

where $p_1,\ p_2 > 0$ and $p_1 + p_2 = 2$ for the problem

$$y' = \alpha y,\ y(0) = 1, \quad \alpha < 0.$$

Show that the method is stable for $-\frac{2}{p_1 p_2} < \alpha h < 0$ if $1 - \frac{1}{\sqrt{2}} < p_1,\ p_2 < 1 + \frac{1}{\sqrt{2}}$.

(b) Consider the following Runge-Kutta method for the differential equation $y' = f(x, y)$: [7]

$$y_{j+1} = y_j + \frac{1}{6}(K_1 + 4K_2 + K_3)$$

where $K_1 = hf(x_j, y_j)$, $K_2 = hf(x_j+h/2, y_j+K_1/2)$, $K_3 = hf(x_j+h, y_j-K_1+2K_2)$, Compute $y(0.4)$ when $y'(x) = (y + x)/(y - x)$, $y(0) = 1$ and $h = 0.2$.

(4) (a) Solve the initial value problem $y'' = 2t + 4y$, $y(0) = 1$, $y'(0) = -1$, $t \in [0,\ 0.6]$ using Numerov method with $h = 0.2$. Use the Taylor series method of order four to obtain the starting value. [7]

(b) Using a second order method, find the solution of the BVP [7]

$$(1 + x^2)y'' + 2xy' - y = 1 + x^2, \quad y(0) = 0,\ y'(1) = 1,$$

with $h = 1/2$ and extrapolate to get better estimate of the value of $y(1)$.

(5) (a) Solve the boundary value problem [7]

$$y'' + y = x, \quad 0 < x < 1 \quad y(0) = 4,\ y(1) = 1,$$

using the Ritz finite element method with linear piecewise polynomials for four elements of equal lengths.

(b) Consider the non linear second order differential equation $y'' = f(x, y)$, $a < x < b$ subject to the mixed boundary conditions [7]

$$a_0y(a) - a_1y'(a) = \gamma_1$$
$$b_0y(b) + b_1y'(b) = \gamma_2,$$

and derive a second order numerical scheme for its solution.

(6) (a) Find the variational problem corresponding to the boundary value problem $y'' = y$, $0 < x < 1$, $y'(0) = 0$, $y(1) = 1$. Solve the given problem using finite element method with $h = 1/3$. [7]

(b) Perform convergence analysis of the central difference scheme for the numerical solution of the boundary value problem $y'' + p(x)y' + q(x)y = 0, x \in [0, 1]$ $y(0) = y_0$, $y(1) = y_1$ where $q(x) > q_0 > 0$. [7]

(7) (a) Solve the initial value problem $y' = -4ty^2$, $y(0) = 1$ with $h = 0.1$ on the interval $[0, 0.2]$ using P-C method [7]

$$P: \ y_{j+1} = y_j + \frac{h}{2}(3y'_j - y'_{j-1})$$
$$C: \ y_{j+1} = y_j + \frac{h}{2}(y'_{j+1} + y'_j)$$

as $P(EC)^mE$, $m = 2$.

(b) Consider the two step method [7]

$$y_{n+1} = \frac{1}{2}(y_n + y_{n-1}) + \frac{h}{4}[4y'_{n+1} - y'_n + 3y'_{n-1}],\ n \geq 1$$

for the numerical solution of the problem $y' = f(x, y)$. Show that the method has order 2 and also find the leading term in the truncation error.

✠

Roll Number: ....

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations, December 2021
Part II Semester III

MMATH18-304(iv): METHODS OF APPLIED MATHEMATICS
(Unique Paper Code 223502313)

Time: 3 hours | Maximum Marks: 70

Instructions: • Attempt **five** questions in all. • Question 1 is compulsory. • All questions carry equal marks. • All symbols have their usual meaning unless otherwise mentioned.

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(1) (a) Determine the resolvent kernel for Voltrra-type integral equation with kernel $K(x,t) = \frac{\cosh x}{\cosh t}$. [3 Marks]

(b) Solve the integral equation $\int_0^x \frac{\phi(t)dt}{(x-t)^{\frac{1}{2}}} = e^x$. [2 Marks]

(c) Define Fredholm determinants. [2 Marks]

(d) If $J[y] = \int_0^1 (x^2 - y^2 + y'^2)dx$. $y \in C^2[0,1]$, find ΔJ and $\delta J[y,h]$ when $y(x) = x$ and $h(x) = x^2$. [4 Marks]

(e) If $\alpha(x)$ and $\beta(x)$ are continuous in $[a,b]$, and if $\int_a^b (\alpha(x)h(x) + \beta(x)h'(x))dx = 0$ for every function $h(x) \in \mathcal{D}_1(a,b)$ such that $h(a) = h(b) = 0$ then prove that $\beta(x)$ is differentiable, and $\beta'(x) = \alpha(x)$ for all x in $[a,b]$. [3 Marks]

(2) (a) Solve the convolution-type equation $2\phi(x) - \int_0^x \phi(t)\phi(x-t)dt = \sin x$. [5 Marks]

(b) Find the iterated kernels of the kernel $K(x,t) = e^{|x|+t}$; $a = -1$, $b = 1$. [5 Marks]

(c) Solve the integral equation [4 Marks]

$$\phi(x) - \lambda \int_0^{2\pi} \sin(x-t)\phi(t)dt = \cos x.$$

(3) (a) Find the characteristic numbers and eigenfunctions of the homogeneous integral equation if the kernel is given by [7 Marks]

$$\kappa(x,t) = \begin{cases} \sin x \ \sin(t-1), & -\pi \le x \le t, \\ \sin t \ \sin(x-1), & t \le x \le \pi. \end{cases}$$

(b) Show that the integral equation [7 Marks]

$$\phi(x) = f(x) + \frac{1}{x}\int_0^{2\pi} \sin(x+t)\phi(t)dt$$

possesses no solution for $f(x) = x$, but it has infinitely many solution when $f(x) = 1$.

(4) (a) Use the Green's function to reduce the BVP [6 Marks]

$$y''' - \lambda y = e^x;\; y(0) = y(1) = 0,\; y'(0) = y'(1);$$

into an integral equation.

(b) Show that integral equation: [4 Marks]

$$\phi(x) - \lambda \int_0^1 (3x-2)t\phi(t)dt = 0$$

has no characteristic numbers and eigenfunctions.

(c) Find the general solution of the Euler equation corresponding to the functional $J[y] \doteq \int_a^b f(x)\sqrt{1+y'^2}dx$, and investigate the case when $f(x) = \sqrt{x}$. [4 Marks]

(5) (a) Using the Hilbert-Schmidt method, solve [7 Marks]

$$\phi(x) = x + \lambda \int_0^1 K(x,t)\phi(t)dt,$$

where

$$K(x,t) = \begin{cases} x(t-1), & 0 \le x \le t, \\ t(x-1), & t \le x \le 1. \end{cases}$$

(b) Define variational derivative of a functional. Prove that the all iterated kernels of a symmetric kernel are also symmetric. [2+5 Marks]

(6) (a) Find the extremal of functional $J[y] = \int_a^b (y^2 - y'^2 - 2y\cosh x)dx$. [6 Marks]

(b) Investigate the solvability of $\phi(x) - \lambda\int_0^\pi \cos^2 x\phi(t)dt = 1$ for different values of parameter λ. [4 Marks]

(c) A physical phenomenon is described by the quantities P, l, m, t and ρ, representing pressure, length, mass, time, and density, respectively. If there is a physical law $f(P, l, m, t, \rho) = 0$ relating these quantities, show that there is an equivalent physical law of the form $g(l^3\rho/m, t^6P^3/m^2\rho) = 0$. [4 Marks]

(7) (a) Define weak and strong extremum for a functional. State and prove the necessary condition for the extremals of functional $\int_a^b F(x, y_1, ..., y_n, y_1', ..., y_n')dx)$. [2+6 Marks]

(b) Use singular perturbation method to find a uniform approximate solution of the problem $\epsilon y'' + t^{1/3}y' + y = 0, 0 < t < 1, 0 < \epsilon << 1$ with $y(0) = 0$ and $y(1) = e^{-3/2}$. [6 Marks]

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examinations, December 2021
Part II, Semester III
MMATH18-305(i): CODING THEORY

Time: 3 Hours M.M. - 35

Instruction: Attempt five questions in all. Question no. one is compulsary. All the questions carry equal marks.

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1. (a) Let C be a binary perfect code of length n with minimum distance 7. Then n will be;
   1. $n = 7$ or $n = 23$
   2. $n = 8$ or $n = 23$
   3. $n = 7$ or $n = 10$
   4. $n = 17$ or $n = 32$

   (01)

   (b) The generator polynomial of Reed Solomon is of the form;
   1. $g(x) = \prod_{i=1}^{d-1}(x - \alpha)$
   2. $g(x) = \prod_{i=1}^{d-1}(x - \alpha^i)$
   3. $g(x) = \prod_{i=1}^{d-1}(\alpha)$
   4. $g(x) = \prod_{i=1}^{d-1}(x)$

   where α is primitive in $\mathbb{F}_q$. (01)

   (c) Write the parity check matrix for Ham(3,3). (01)

   (d) Find the least common multiple and greatest common divisor of the following polynomials over $\mathbb{F}_2$:

   $$f_1(x) = 1+x^2, \quad f_2(x) = 1+x+x^2+x^4, \quad f_3(x) = 1+x^2+x^4+x^6.$$

   (01)

   (e) Determine which of the following codes are linear over F_q:
   1. $q = 2$ and $C = \{1101, 1110, 1011, 1111\}$,
   2. $q = 3$ and $C = \{0000, 1001, 0110, 2002, 1111, 0220, 1221, 2112, 2222\}$,
   3. $q = 2$ and $C = \{00000, 11110, 01111, 10001\}$
   4. All of above.

   (01)

(f) What is sphere-covering bound? (01)

(g) Consider the cyclic code $C = \{000, 110, 101, 011\}$; then find out the mapping $\pi(C)$ with irreducible polynomial (x^3-1) over $\mathbb{F}_2[x]$. (01)

2. (a) Show that if a binary group block code has odd minimum weight, adding a parity-check digit in the code increases the minimum weight by 1. (03)

   (b) Show that if the columns of the generator matrix G for a binary code consist of all 2^k-1 nonzero $k-$tuples, than all nonzero code words have weight $2^k - 1$. (04)

3. Construct a code consisting of eight words of length 7 such that any two distinct codewords have distance at least 4. For a Binary symmetric channel with error probability p, determine the probability that a received word is decoded correctly. (07)

4. (a) Show that a [15,8,5] code does not exist. (03)

   (b) For each n such that $4 \leq n \leq 12$, compute the Hamming bound and the sphere-covering bound for $A_2(n, 4)$. (04)

5. Factorize $x^5 - 1$ into irreducible polynomials over $GF(2)$ and hence determine all the cyclic binary codes of length 5.

   (07)

6. Consider the binary $[15, 7]$-cyclic code generated by $g(x) = 1 + x + x^6 + x^7 + x^8$. Decode the word 110011101100010 for this cyclic code.

   (07)

7. Let C_i be an $[n, k_i, d_i]$-linear code over $\mathbb{F}_q$, for $i = 1, 2$. Define

$$C := \{(\mathbf{a}+\mathbf{x}, \mathbf{b}+\mathbf{x}, \mathbf{a}+\mathbf{b}+\mathbf{x}) : \mathbf{a}, \mathbf{b} \in C_1, \mathbf{x} \in C_2\}.$$

   (a) Show that C is a $[3n, 2k_1 + k_2]$-linear code over $\mathbb{F}_q$. (01)

   (b) If G_i is a generator matrix of C_i , for $i = 1, 2$, find a generator matrix of C in terms of G_1 and G_2. (03)

   (c) If H_i is a parity-check matrix of C_i , for $i = 1, 2$, find a parity-check matrix of C in terms of H_1 and H_2. (03)