Your Roll Number: ....

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations August 2022
Part I Semester II
MMATH18-201: Module Theory, UPC- 223501201

Time: 3 hours Maximum Marks: 70

Instructions: • Question 1 is compulsory.• Attempt any four Questions from the rest.• All questions carry equal marks.

(1) (a) Define isomorphic categories with example. [2 Marks]

(b) Does every torsion-free divisible abelian group a direct sum of copies of rationals $\mathbb{Q}$? Justify. [3 Marks]

(c) Define length of module and determine the length of the $\mathbb{Z}$-module $\mathbb{Z}_{10}$. [3 Marks]

(d) Are $\mathbb{Z}_2$ and $\mathbb{Z}_3$ projective and free $\mathbb{Z}_6$- modules respectively? Justify. [3 Marks]

(e) Is $2\mathbb{Z}\otimes_{\mathbb{Z}}\mathbb{Z}_2$ a submodule of $\mathbb{Z}\otimes_{\mathbb{Z}}\mathbb{Z}_2$? Justify your answer. [3 Marks]

(2) (a) Prove that a short exact sequence $0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$ splits if and only if α has a right inverse. [6 Marks]

(b) State and prove universal property of direct product for left R-modules. [2+6 Marks]

(3) (a) Define a category and give two examples. [2 Marks]

(b) Describe what is Five-Lemma. Provide a detailed proof of the Lemma. [2+6 Marks]

(c) Show that a zero morphism in the category R^{Mod} corresponds to a zero morphism in categpry S^{Mod} and zero module in R^{Mod} is mapped to the zero module of the category S^{Mod} under an additive functor F from R^{Mod} to S^{Mod}. [2+2 Marks]

(4) (a) Show that a left R-module is projective if and only if it is isomorphic to a direct summand of a free left R-module. [6 Marks]

(b) Show that every injective left module over an integral domain is divisible. Does the converse hold? Justify. Also give an example of a projective module which is not injective [3+3+2 Marks]

(5) (a) If M, N and P are three R-modules, then prove the following [7 Marks]
$$(M \otimes N) \otimes P \cong M \otimes (N \otimes P).$$

(b) Prove that every module is a quotient of free module. Give an example of a module which is not free. [4+3 Marks]

(6) (a) Prove that every submodule of a semisimple module M is complemented in M. [8 Marks]

(b) Define a fully invariant submodule of an R-module M. Show that the fully invariant submodules are nothing but the two sided ideals when R is considered as a right R-module. [6 Marks]

(7) (a) Prove that a finitely generated module over a principal ideal domain R is free if and only if it is torsion-free. Hence show that R^n is not bounded. [5+2 Marks]

(b) Let M be a torsion module over a principal ideal domain R. Prove that $M = \bigoplus_p M_p$, where M_p is p-primary submodule of M and p is a prime in R. [7 Marks]

(8) (a) Let N be a proper submodule of a module M of finite length. Prove that length of N is strictly less than length of M. [7 Marks]

(b) Let $0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$ be a short exact sequence of R-modules and R-homomorphisms. Prove that M is Noetherian iff both M' and M'' are Noetherian modules. [7 Marks]

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M. A./M. Sc. Mathematics Examinations, August-2022
MMATH18-202 : INTRODUCTION TO TOPOLOGY
(UPC 223501202)

Time: 3 hours | Maximum Marks: 70

Instructions: • Attempt five questions in all. • Question 1 is compulsory and answer any four questions from Q 2 to Q 8. • All questions carry equal marks.

Q 1. Prove or disprove the following:

(a) The set $\{\frac{1}{n} \mid n \in \mathbb{N}\}$ of the real line $\mathbb{R}$ is discrete. [2 Marks]

(b) Local connectedness is preserved by a continuous map. [3 Marks]

(c) Let X be a topological space and $A \subseteq X$. If $x \in \overline{A}$, then there is a net in A that converges to x. [3 Marks]

(d) A closed subset of a separable space is separable. [3 Marks]

(e) The Bolzano–Weierstrass property is preserved by a continuous map. [3 Marks]

Q 2. (a) Let $A = \{\frac{1}{n} \mid n \in \mathbb{N}\}$. Show that the collection of open intervals (a, b) and the sets $(a, b) - A$ is a basis for a topology on $\mathbb{R}$. Compare this topology with the usual topology on $\mathbb{R}$. [8 Marks]

(b) Let X and Y be topological spaces. Prove that a function $f : X \to Y$ is continuous if and only if it is continuous at each point of X. [6 Marks]

Q 3. (a) Let X_n, $n = 1, 2, \cdots$ be a family of metrisable spaces. Show that the product space $\prod X_n$ is metrisable. [8 Marks]

(b) Let X_α, $\alpha \in A$, be a family of topological spaces. Prove that $\prod X_\alpha$ is connected if and only if each X_α is connected. [6 Marks]

Q 4. (a) Define the closed topologist's sine curve. Prove that it is connected but not path-connected. [8 Marks]

(b) Let X, Y and Z be topological spaces. Show that a function $f : Z \to X \times Y$ is continuous if and only if the coordinate maps $p_X \circ f : Z \to X$ and $p_Y \circ f : Z \to Y$ are continuous. [6 Marks]

Q 5. (a) Show that a topological space X is Hausdorff if and only if the set $\Delta = \{(x, x) \mid x \in X\}$ is closed in $X \times X$. [8 Marks]

(b) Let $\varphi : A \to X$ be a net in a topological space X. Prove that [6 Marks]

$x \in X$ is a cluster point of φ if and only if there is a subnet of ϕ that converges to x.

Q 6. (a) Let $\mathscr{T} = \{\emptyset\} \cup \{\mathbb{R} - F \mid F$ is closed and bounded in $\mathbb{R}\}$. Show that $(\mathbb{R}, \mathscr{T})$ is T_1 but not T_2. [8 Marks]

(b) Show that the Sorgenfrey line $\mathbb{R}_\ell$ is first countable but not second countable. [6 Marks]

Q 7. (a) Let X be a compact Hausdorff space, and $f : X \to Y$ be a continuous surjection. Show that Y is Hausdorff if and only if f is closed. [8 Marks]

(b) Let A be a compact subset of a Hausdorff space X and $x \in X - A$. Show that there are disjoint open sets U and V such that $x \in U$ and $A \subset V$. [6 Marks]

Q 8. (a) Prove that a T_1 space is countably compact if and only if it has the Bolzano–Weierstrass property. [8 Marks]

(b) State and prove the Tychonoff theorem. [6 Marks]

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics End Semester Examinations, August, 2022
Part I Semester II

MMATH18-203: FUNCTIONAL ANALYSIS

Time: 3 hours **(UPC: 223501203)** Maximum Marks: 70

Instructions: • Section A is compulsory. • Answer any four questions from section B • All notations have usual meaning.

Section A

(1) (a) Fix $a = \{\alpha_j\} \in l^2$, define $f(x) = \sum_{n=1}^{\infty} \alpha_n \xi_n$, $x = \{\xi_m\} \in l^2$. Find the norm of the functional f. [3]

(b) Show that if $\{e_k\}_{k=1}^{\infty}$ is an orthonormal sequence in an inner product space X, then [3]

$$\sum_{k=1}^{\infty} |< x, e_k >< y, e_k >| \leq ||x||||y||, \forall x, y \in X.$$

(c) Give an example to show that in a normed space X, the weak convergence of a sequence need not imply strong convergence. [3]

(d) Show that $\lambda = 0$ is a spectral value of the right shift operator $T : l^2 \to l^2$ defined by $T(\xi_1, \xi_2, \ldots) = (0, \xi_1, \xi_2, \ldots)$. [3]

(e) Is every isometric operator on a Hilbert space unitary?Justify. [2]

Section B

(2) (a) Show that in a normed space X absolute convergence of a series implies its convergence if and only if the space X is complete. [6]

(b) Give an example of a set in l^2 which is not total. Prove that an orthonormal set in a Hilbert space H is total in H if an only if the Parseval relation holds. [6]

(c) Show that in an inner product space, $x \perp y$ if we have $||x + \alpha y|| = ||x - \alpha y||$ for all scalars α. [2]

(3) (a) If X and Y are Banach spaces and $T \in B(X, Y)$ is surjective, then show that the image $T(B_0)$ of an open unit ball $B_0 = B(0, 1) \subset X$ contains an open ball about 0 of Y. [7]

(b) Let H_1 and H_2 be Hilbert spaces and $T \in B(H_1, H_2)$. Define the Hilbert adjoint T^* of T, show that it exists, and is a bounded linear operator with norm $||T|| = ||T^*||$. [7]

(4) (a) Let X and Y be Banach spaces and $T : D(T) \to Y$ be a closed linear operator, where $D(T) \subset X$. Prove that if $D(T)$ is closed in X, the operator T is bounded. [6]

(b) Show that the dual space of l^1 is isomorphic to l^∞. [5]

(c) Define $T : l^\infty \to l^\infty$ by $T(\{\xi_j\}_{j=1}^{\infty}) = \{\frac{\xi_j}{j}\}_{j=1}^{\infty}$. Prove that the null space of T is closed. [3]

(5) (a) Show that any two norms on a finite dimensional vector space X are equivalent. [7]

What happens if X is infinite dimensional? Justify your answer. [7]

(b) Let X be a normed space, Y a Banach space and $T : \mathfrak{D}(T) \subset X \to Y$ be a bounded linear operator. Then show that T has an extension $\tilde{T} : \overline{\mathfrak{D}(T)} \to Y$ which is bounded, linear and has norm $||T|| = ||\tilde{T}||$. [6]

(6) (a) Let f be a bounded linear functional on a subspace Z of a normed space X. Then prove that there exists a bounded linear functional $\tilde{f}$ on X which is an extension of f to X and has the same norm $||\tilde{f}||_X = ||f||_Z$ where $||\tilde{f}||_X = \sup_{\substack{x\in X\\ ||x||=1}} |\tilde{f}(x)|$, $||\tilde{f}||_Z = \sup_{\substack{x\in Z\\ ||x||=1}} |f(x)|$. [5]

(b) Let $\{T_n\}$ be a sequence of operators such that $T_n \in B(X, Y)$ where X and Y are Banach spaces. Then show that the sequence (T_n) is strongly operator convergent if and only if

a. the sequence $\{||T_n||\}$ is bounded,

b. the sequence $\{T_n x\}$ is Cauchy in Y for every x in a total subset M of X. [3]

(c) State true or false and justify: For a bounded linear operator $T : X \to Y$ where X and Y are normed spaces the range space is closed. [7]

(7) (a) State and prove uniform boundedness theorem. Give an example to show that the hypothesis of completeness of the underlying space cannot be dropped from the uniform boundedness theorem. [7]

(b) Prove that $\sigma(T)$, the spectrum of a bounded linear operator T on a complex Banach space X is compact and lies in the disc $\{\lambda \in \mathbb{C} : |\lambda| \leq ||T||\}$. [6]

(8) (a) Let Y be a proper closed subspace of a normed space X. Let $x_0 \in X - Y$ be arbitrary and $\delta = d(x_0, y)$. Then show that there exists a $\tilde{f} \in X'$ such that $||\tilde{f}|| = 1$, $\tilde{f}(y) = 0\ \forall y \in Y$, $\tilde{f}(x_0) = \delta$. [3]

(b) Show that a reflexive normed space is complete. [5]

(c) Prove that the spectrum $\sigma(T)$ of a bounded linear operator $T \in B(X, X)$ where $X \neq 0$ is a complex Banach space is non-empty.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examinations, August 2022
Part I Semester II

MMATH18-204: FLUID DYNAMICS, UPC 223501204

Time: 3 hour | Maximum Marks: 70

Instructions: • **Q.No.1** in section **A** is compulsory. • Attempt any **FOUR** questions from section **B**. • All the symbols have their usual meaning.

Section A

(Answer all parts, 14 Marks)

(1) (a) Explain the continuum hypothesis. Write the two criteria for validity of the continuum hypothesis. [3 Marks]

(b) Write Euler's equation of motion in component form in (r, θ, z) coordinate. [2 Marks]

(c) Deduce the velocity component(q_r, q_θ) at a point (r, θ) due to axi-symmetric flow of incompressible fluid with the axis of symmetry OZ, where O is origin. [3 Marks]

(d) Explain the Magnus effect in a two dimensional fluid motion. [3 Marks]

(e) Compute the rate of volumetric strain in a viscous fluid motion. Find its value for an incompressible fluid. [3 Marks]

Section B

(Answer any FOUR questions, 56 Marks)

(2) (a) Define a material volume, control volume, streak lines, and vorticity vector. Derive relation between the local and particle rate of change in a fluid motion. [4+3 Marks]

(b) State the principle of conservation of mass. Derive the equation of continuity for a fluid motion in integral and differential form. Write the equation of continuity for a two dimensional motion in (r, θ) coordinate. [2+5 Marks]

(3) (a) Define a boundary surface in a fluid motion. Prove that the surface $\frac{x^2}{k^2t^4} + kt^2(y^2 + z^2) = a^2$, where a and k are constant, is a possible form of boundary surface of a liquid motion at time t. [1+5 Marks]

(b) Find the value of a, b and c for which the motion given by the velocity $(x + 2y + az)\hat{i} + (bx - 3y - z)\hat{j} + (4x + cy + 2z)\hat{k}$ is irrotational. [2 Marks]

(c) Derive the Bernoulli's equation of motion in its most general form. Also, deduce its particular forms. [5+1 Marks]

(4) (a) An inviscid incompressible fluid moves steadily with velocity $\vec{q} = (kx, -ky, 0)$, where k is constant, under no external body force. Compute the pressure $p(x, y, z)$ in the fluid motion when $p(0,0,0) = p_0$. [3 Marks]

(b) Consider a uniform stream of steady, irrotational flow of incompressible fluid past a stationary sphere of radius a at origin. Describe the problem with a neat diagram and find the velocity potential, fluid velocity, stagnation points and pressure at any point of the sphere. [6 Marks]

Also, find the condition when fluid break away the contact of the sphere along the equatorial line.

(c) State and prove the Kelvin's circulation theorem. [1+4 Marks]

(5) (a) Define a three dimensional doublet. Find the velocity at point due to a simple source of strength m and show that the motion is irrotational and hence find the velocity potential. [2+3 Marks]

(b) Find the image of a simple source of strength m in a rigid infinite plane. [2 Marks]

(c) Define axi-symmetric flow and the Stokes stream function. Find the Stokes stream function due to a uniform line source of strength m per unit length along line OA on OZ axis, where O is origin. Determine the stream surface due to the uniform line source. [2+5 Marks]

(6) (a) Define a two dimensional motion and complex velocity potential. Compare the dimensions of a stream function and kinematic coefficient of viscosity. [3 Marks]

(b) Deduce the physical interpretation of stream function for a two dimensional flow. [4 Marks]

(c) State the theorem of Blasius. Discuss the action of fluid pressure on a unit length of a long infinite circular cylinder of radius a placed in a uniform stream $-U\hat{i}$ with a circulation of $2\pi k$ about the cylinder. [2+5 Marks]

(7) (a) Compare the Weiss's sphere theorem, Milne Thomson's circle theorem and Butler sphere theorem in term of their assumptions, conclusion and applications. [7 Marks]

(b) Discuss the fluid motion due to a complex potential $w = \hat{i}k\, log z$. Find the equation of stream lines due to a source of strength m at $(-1, 0)$ and $(1, 0)$ and a sink of strength $2m$ at origin in two dimensional motion. [4+3 Marks]

(8) (a) Define stress components in a real fluid. Write relation between stress components and rate of strain in tensor form. [2+1 Marks]

(b) Derive Navier-Stokes equation of motion for a viscous compressible fluid. [4 Marks]

(c) Explain the Newton's law of viscosity. Discuss with necessary assumption about the solution of Navier Stokes equation of motion of a viscous fluid flow between the two parallel planes. Compute the velocity profile between the two planes and drag per unit area on the lower and upper plane. [2+5 Marks]

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