Your Roll Number: ....

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examinations, March 2021
Part I Semester I

MMATH18-101: FIELD THEORY

(Unique Paper Code: 223501101)

Time: 3 + 1 Hours | Maximum Marks: 70

Instructions: • Attempt five questions in all. Question 1 is compulsory. All questions carry equal marks. •

---

(1) (a) Without using the fact that $\frac{\mathbb{Q}[x]}{<x^5-2>}$ is a field, show that the element $x - 5 + < x^5 - 2 >$ is invertible in $\frac{\mathbb{Q}[x]}{<x^5-2>}$. [2]

(b) Let K be the set of all real numbers that are algebraic over $\mathbb{Q}$. Show that $K|\mathbb{Q}$ is an infinite extension. [2]

(c) Show that the algebraic closure of $\mathbb{F}_3$ is an infinite field. [2]

(d) Let $k \subset F \subset E$ be a tower of algebraic extensions such that $E|k$ is separable. Show that $E|F$ is separable. [2]

(e) Show that every finite extension of $\mathbb{F}_5$ is a cyclic extension. [3]

(f) Let F be a field of characteristic p. Show that if F contains a primitive n^{th} root of unity, then p cannot divides n. [3]

(2) (a) Show that a finite extension $E|k$ is normal if and only if E is the minimal splitting field of some polynomial over k. [2+3]

(b) Show that the algebraic closure of a countable field is countable. [4]

(c) Let $F|k$ be an extension and $\alpha \in F$. Show that $k[\alpha] = k(\alpha)$ if and only if α is algebraic over k. [1+4]

(3) (a) Let $F|k$ be a normal extension of degree n. Show that the group of all k-automorphisms on F is of order n if and only if F is generated by finitely many separable elements over k. [9]

(b) Let E be the minimal splitting field of the family of all polynomials over k. Show that E is an algebraically closed field. [5]

(4) (a) Show that the Galois group of n^{th}-cyclotomic polynomial over $\mathbb{Q}$ is same as the Galois group of $x^n = 1$ over $\mathbb{Q}$ and that it is a solvable group for all $n \geq 1$. [1+4]

(b) Show that the extension $k(x_1, x_2, \ldots, x_n)|k(e_1, e_2, \ldots, e_n)$ is a Galois extension where $e_1, \ldots, e_n$ are elementary symmetric functions in indeterminates $x_1, \ldots, x_n$. Determine the order of the Galois group also. [5]

(c) Determine the Galois group of $x^3 - 3x - 1$ over $\mathbb{Q}$. [4]

(5) (a) Let k be a finite field with n elements. Then show that

(i) $n = p^m$ for some $m \in \mathbb{N}$, where $p = char(k)$. [4]

(ii) k has primitive $(n-1)^{th}$ root of unity. [5]

(b) Show that the polynomial $x^p - x - a$ is irreducible over $\mathbb{F}_p$ for any non zero $a \in \mathbb{F}_p$. [5]

(6) (a) Let f be a separable polynomial over a field k, $K|k$ be an extension, E be the minimal splitting of f over K, and L be the minimal splitting field of f over k contained in E. Show that $Gal(E|K)$ is isomorphic to the subgroup of $Gal(L|k)$ corresponding to the subfield $K \cap L$. [5]

(b) Show that $x^4 - 2$ is irreducible over $\mathbb{Z}_3$. [4]

(c) Consider the extension $\mathbb{Q}(\omega)|\mathbb{Q}$ where ω is a primitive 7^{th} root of unity. Show that there are only finitely many subfields of $\mathbb{Q}(\omega)$ containing $\mathbb{Q}$. [5]

(7) (a) Let f be a monic polynomial over $\mathbb{Z}$ and $\bar{f}$ be the corresponding polynomial over $\mathbb{Z}_5$. Show that if the group of f over $\mathbb{Q}$ is G, then the group H of $\bar{f}$ over $\mathbb{Z}_5$ is a subgroup of G as a permutation group on the roots. [9]

(b) Show that there exists an irreducible polynomial of degree 6 over $\mathbb{Q}$ which is not solvable by radicals. [5]

✠

M.A./M.Sc. Mathematics, Part-I, Semester I
OBE Blended Mode Examination, March 2021
Paper No.: MMATH18-102
Complex Analysis
(Unique Paper Code 223501102)

Time: 3 Hours | Maximum Marks: 70

Instructions: Attempt five (5) questions in all. **Question no. 1 is compulsory**. All questions carry equal marks. The set G is an open subset of the complex plane $\mathcal{C}$ unless otherwise stated.

---

1. (a.) Evaluate $\displaystyle\int_\gamma \frac{1}{z^2+1}dz$, where $\gamma : [0,\pi] \to \mathcal{C}$ is defined by $\gamma(t) = e^{2it}$, for $0 < r < 1$ and $r > 1$. [3]

   (b.) If $f : D = \{z : |z| < 1\} \longrightarrow \overline{D}$ is an analytic function, show that $f'(0) \in \overline{D}$. [3]

   (c.) Determine residue(s) of the function $f(z) = \dfrac{z}{(z^2+2)^3}$ at its pole(s). [3]

   (d.) Suppose the function $f : G \longrightarrow \mathcal{C}$ is analytic and 1-1; show that $f'(z) \neq 0$ for any z in G. Does the converse hold? Justify. [3]

   (e.) Suppose $f : G \longrightarrow \mathcal{C}$ is a non constant analytic function with $f(a) = 0$ for some $a \in G$. Can a be a limit of a sequence of zeros of f in G ? Why or why not? [2]

2. (a.) Find an analytic function which maps the region $G = \{z = x + iy; -\frac{\pi}{2} < y < \frac{\pi}{2}\}$ onto the disk $B(1;1) = \{z : |z-1| < 1\}$. [8]

   (b.) Evaluate $\displaystyle\int_0^\infty \frac{cosx - 1}{x^2}dx$. [6]

3. (a.) Let $f : D \longrightarrow \overline{D}$ be an analytic function and $a \in D$. What is the maximum possible value of $|f'(a)|$? Suppose the function $f : D \longrightarrow D$ is analytic, bijective and $f(a) = 0$, show that there is a number c on the unit circle such that $f(w) = c\varphi_a(w)$ for all $w \in D$. The function φ_a is defined by $\varphi_a(z) = \dfrac{z-a}{1-\overline{a}z}$ [8]

   (b.) Prove that index of a closed curve, whether smooth or not, is always an integer. [6]

4. (a.) State Laurent Series Development (Laurent's Theorem) and classify, with justification, the three types of Isolated singularities based on Laurent's series. [8]
   (b.) Define a branch of logarithm on the region $G = \mathcal{C} \setminus [0, \infty)$. Is it possible to define branch of logarithm on the punctured plane $\mathcal{C} \setminus \{0\}$? Justify. [6]

5. (a.) Let f and g be analytic functions in an open set containing $\overline{B}(a; R)$ with no zeros on the circle γ given by $|z-a| = R$. If $|g(z)| < |f(z)|$ on γ, show that f and $f+g$ have the same number of zeros inside the circle γ. Hence or otherwise prove the Fundamental Theorem of Algebra. [8]
   (b.) Let f be an analytic function on a disk $B(a; r)$. Show that there is an $\epsilon > 0$ and a $\delta > 0$ such that ; if $\zeta \in B(f(a); \epsilon)$ the equation $f(z) = \zeta$ has a solution in $B(a; \delta)$. [6]

6. (a.) Give three advantages of having a simply connected region over an ordinary open set. Justify one of your statements. [8]
   (b.) Let f be a non-constant analytic function on a bounded open set G and is continuous on $\bar{G}$. Prove that either f has a zero in G or f assumes minimum value on ∂G. [6]

7. (a.) Let f be an analytic function on an open set containing $\bar{B}(a; R)$ which is one-one in $B(a; R)$ and $\Omega = f(B(a; R))$, and γ is the circle $|z-a| = R$ show that $f^{-1}(w)$ is defined for each $w \in \Omega$ by $f^{-1}(w) = \dfrac{1}{2\pi i} \displaystyle\int_\gamma \frac{zf'(z)}{f(z) - w} dz$. [8]
   (b.) State Morera's Theorem. Is the function g defined by $g(z) = \dfrac{sinz}{z}$, for $z \neq 0$ and $g(0) = 1$ an entire function? Justify. [6]

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examinations, March 2021
Part I, Semester I

MMATH18-103: MEASURE AND INTEGRATION
(Unique Paper Code 223501103)

Time: 3 hours | Maximum Marks: 70

Instructions: • Write your Name, University Roll No., College, Course and Title of the Paper with Unique Paper Code on the first page and your Roll No. and page No. on each subsequent answer sheet. •Attempt **five** questions in all. **Question 1 is compulsory**. All questions carry equal marks.• The symbols used have their usual meanings.

---

(1) (a) IF $E \subset \mathbb{R}$ and $m^*(\mathbb{R} \setminus E) = 0$, Show that E is dense in $\mathbb{R}$. [4]

(b) Does there exist a Lebesgue measurable function which is not Borel measurable? Justify your answer. [3]

(c) Let $\{\varphi_n\}$ be a sequence of measurable simple functions on $\mathbb{R}$ such that $\{\varphi_n(x)\}$ is decreasing for every $x \in \mathbb{R}$. Define $f(x) = \lim_{n\to\infty} \varphi_n(x)$, $x \in \mathbb{R}$. Can you conclude that $\int f dx = \lim_{n\to\infty} \int \varphi_n dx$? Justify your answer. [3]

(d) Define f on $\mathbb{R}$ by [2]

$$f(x) = \begin{cases} -1, & \text{if } x \text{ is a rational number,} \\ 1, & \text{if } x \text{ is an irrational number.} \end{cases}$$

Find the four derivatives of f at any $x \in \mathbb{R}$.

(e) Let $g : [a,b] \to \mathbb{R}$ be a Lipschitz function, i.e., there exists $M > 0$ such that $|g(x) - g(y)| \le M|x - y|$, $\forall x, y \in [a,b]$. Show that g is absolutely continuous on $[a,b]$. [2]

(2) (a) If $E \subset \mathbb{R}$ and $\varepsilon > 0$, show that there is an open set O containing E such that $m^*(O) \le m^*(E) + \varepsilon$. In particular, if $E = \mathbb{N}$, give explicitly such an open set O containing E. [7]

(b) Let $[\![X, \mathbb{S}, \mu]\!]$ be a complete measure space and $\{f_n\}$ be a Cauchy sequence in $L^p(\mu), 1 \le p < \infty$. Show that there exists a functon f and a subsequence $\{n_k\}$ such that $f_{n_k} \to f\, a.e., f \in L^p(\mu)$ and $\|f_n - f\|_p \to 0$ as $n \to \infty$. [7]

(3) (a) Let f be a bounded real valued function on a finite interval $[a,b]$. If f is Riemann integrable on $[a,b]$, show that f is integrable and that $R\int_a^b f dx = \int_a^b f dx$. Give an example to show that the converse is not in general true. [7]

(b) Let $V \subset [0,1]$ be a Lebesgue non-measurable set and define $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} \text{x}, & \text{if } x \in V, \\ \text{x+2}, & \text{if } x \in [0,1] \setminus V. \end{cases}$$

Prove that $f^{-1}(\{\alpha\})$ is Lebesgue measurable for every $\alpha \in \mathbb{R}$, but f is not measurable. Prove also that if φ is a simple functon on $\mathbb{R}$ such that $\varphi^{-1}(\{\alpha\})$ is a Lebesgue measurable set for every $\alpha \in \mathbb{R}$, then φ is measurable. [5 +2]

(4) (a) Let $[\![X, \mathbb{S}, \mu]\!]$ be a complete measure space and $\{f_n\}$ be a sequence of integrable functions on X such that $\sum_{n=1}^{\infty} \int |f_n| d\mu < \infty$. Prove that $\sum_{n=1}^{\infty} f_n$ converges a.e.,

its sum f is integrable and $\int f\,d\mu = \sum_{n=1}^{\infty} \int f_n d\mu$. Compute the integral [3 +4]

$$\lim_{n\to\infty} \int_0^\infty \frac{n \sin(x/n)}{x(x^2+1)} dx.$$

(b) Let f be a Lebesgue integrable function on $\mathbb{R}$. If $\int f\,dx = 0$, can you conclude that $f = 0$ *a.e.*? Will the conclusion be the same if f also satisfies the condition that $f \leq 0$? Justify your answers. [7]

(5) (a) Let $\{E_n\}$ be a sequence of subsets of $\mathbb{R}$ such that $E_n \supset E_{n+1} \forall n$. Find $\limsup E_n$ and $\liminf E_n$. If E_n's are also Lebesgue measurable and $m(E_1) < \infty$, show that $m(\lim E_n) = \lim m(E_n)$. [7]

(b) Defiine g on $\mathbb{R}$ by [3 +4]

$$g(x) = \begin{cases} \sin(1/x), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Is g of bounded variation on $[\frac{1}{10}, 10]$? On $[0, 10]$? Justify your answers.

(6) (a) Defiine f on $[-1, 1]$ by [7]

$$f(x) = \begin{cases} x\sin(1/x), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Find the measure of the set $E = \{x : f(x) \leq 0\}$.

(b) Let $[\![X, \mathbb{S}, \mu]\!]$ be a complete measure space. Discuss the relationship between $L^p(\mu)$ and $L^q(\mu)$, where $0 < p < q \leq \infty$. [7]

(7) (a) Show that the outer measure m^* of a closed bounded interval I is its length. [6]

(b) Let $[\![X, \mathbb{S}, \mu]\!]$ be a measure space and f, f_n $n \geq 1$ be measurable functions on X such that $f_n \to f$ *a.e*,. If $\mu(X) < \infty$, show that $f_n \to f$ in measure. Give an example with justification to show that the conclusion fails if $\mu(X)$ is not finite. [4+4]

Your Roll Number: ....

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics End Semester Examinations, March, 2021
Part I Semester I
MMATH18-104: DIFFERENTIAL EQUATIONS

Time: 3 hours **UPC: 223501104** Maximum Marks: 70

Instructions: • Section A is compulsory. • Answer any four questions from section B • All notations have usual meaning.

Section A

(1) (a) Show that the solution of the problem $y' = -x/y,\ y(0) = 1$ cannot be extended beyond the interval $-1 < x < 1$. [3]

(b) Prove that if ϕ_1, ϕ_2 are solutions of $\frac{d}{dx}(p(x)\frac{dy}{dx}) + q(x)y = 0$ then $p(\phi_1\phi_2' - \phi_1'\phi_2)$ =constant. [2]

(c) State true or false and justify. The critical point $(0, 0)$ of the non linear autonomous system [3]

$$\frac{dx}{dt} = x + y + x^2y, \quad \frac{dy}{dt} = 3x - y + 2xy^3,$$

is asymptotically stable.

(d) Show that $\Psi = \frac{e^{ik|\vec{r}-\vec{r'}|}}{|\vec{r}-\vec{r'}|}$ where $\vec{r} = (x, y, z)$ is any point and $\vec{r'}$ is a fixed point is a solution of the space form of wave equation. [3]

(e) State and prove the necessary condition for the existence of solution of the Neumann problem for Laplace equation. What can you say about uniqueness of the solution? Justify. [3]

Section B

(2) (a) State and prove the existence and uniqueness theorem for IVP for system of n first order differential equations. [7]

(b) Find the function $\Psi(x, y, x)$ such that $(\nabla^2 + K^2)\Psi = 0$ for $y \leq 0$ and $\Psi = f(x, z)$ for $y = 0$ where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$. [7]

(3) (a) Prove that the unique solution Φ of the non homogeneous linear vector differential equation $\frac{dX}{dt} = A(t)X + F(t)$ that satisfies the initial condition $\Phi(t_0) = X_0$ where $t_0 \in [a, b]$ can be expressed as [5]

$$\Phi(t) = \Psi(t)\Psi^{-1}(t_0)X_0 + \Psi(t)\int_{t_0}^{t} \Psi^{-1}(u)F(u)du$$

where $\Psi(t)$ is an arbitrary fundamental matrix of the corresponding homogeneous vector differential equation

$$\frac{dX}{dt} = A(t)X.$$

(b) Prove that if there exist a Lyapunov's function $V(X)$ in an open region Ω about the origin for a non-linear autonomous system $\dot{X} = F(X)$ such that $-\nabla V(X).F(X)$ [4+5] is positive definite in Ω then origin is asymptotically stable. Show that the critical point of the system

$$\dot{x} = 8x - y^2$$
$$\dot{y} = -y + x^2$$

are $(0,0),\ (2,4)$. Determine the stability of the origin from its associated linear system.

(4) (a) Find the system of three first order equations equivalent to the third order equation $y''' = 4x^2 + 3yy' + (y'')^2$. Explain precisely why or why not there exist a unique solution of the given equation under the conditions $y(0) = 1, y'(0) = -3, y''(0) = 0$. [5]

(b) Consider the equation $\frac{d^2x}{dt^2} + q(t)x = 0$ where q is continuous on $a \leq t \leq b$ and such that $0 < m < q(t) < M$ for some constants m and M. Let ϕ_1 be a solution of the above equation having consecutive zeros at $t_1,\ t_2\ (a \leq t_1 < t_2 < b)$. Show that $\frac{\pi}{\sqrt{M}} < t_1 - t_2 < \frac{\pi}{\sqrt{m}}$. [4]

(c) Solve the intial boundary value problem [5]

$$\frac{\partial \theta}{\partial t} = \kappa \frac{\partial^2 \theta}{\partial x^2},\ x \geq 0,\ t > 0,\ \theta(0,t) = \phi(t),\ t > 0,\ \theta(x,0) = f(x),\ x \geq 0.$$

(5) (a) Find the solution of the equation $\frac{\partial^4 z}{\partial x^4} - \frac{\partial^2 z}{\partial y^2} = 0,\ -\infty < x < \infty,\ y \geq 0$ satisfying the condition (i) z and its partial derivatives tend to zero as $x \to \pm\infty$, (ii) $z = f(x)$, and $\frac{\partial z}{\partial y} = 0$ at $y = 0$. [6]

(b) Find the solution of the Llaplace equation $\nabla^2 \Psi = 0$ for the points outside a finite region V when we are given the value of both Ψ, and $\frac{\partial \Psi}{\partial n}$ on the boundary S of V. [8]

(6) (a) Find the general solution of $y_1' = 2y_1 + y_2 - y_3,\ y_2' = y_1 + 2y_2 - y_3,\ y_3' = y_1 + y_2 - 3y_3$. [6]

(b) If the surface $y = 0$ of the semi-infinite solid $y \geq 0$ is maintained at a temperature $\phi(x, y, z, t)$ for $t > 0$ and if the initial temperature of the solid is $f(x, y, z)$ determine the distribution of temperature in the solid. [8]

(7) (a) Suppose $u(x, y, z)$ is a harmonic function in a domain $D \in \mathbb{R}^3$ and is continuous on $\bar{D}$. Then show that u attains its maximum on the boundary B of D. [5]

(b) Find the eigen values and eigen functions of the Sturm-Liouville problem [5]

$$\frac{d}{dx}\left[x\frac{dy}{dx}\right] + \frac{\lambda}{x}y = 0, \quad y'(1) = 0,\ y'(e^{2\pi}) = 0 \text{ where } \lambda > 0.$$

(c) Solve the non homogeneous BVP $y'' = -x^4,\ y(0) = 0,\ y(1) + y'(1) = 4$ using the method of Green's function. [4]

✠