Department of Mathematics, University of Delhi
(M.A./M.Sc. Mathematics Examinations–December 2020)
Part II Semester III
MMATH18-301(i): Algebraic Topology
Unique Paper Code 223502301

Time: 3 hours Maximum Marks: 70

Instructions: • Question number 1 is compulsory. • Answer any other **four** questions • Each question carries 14 marks.

(1) (a) Let X be the Hawaiin earring. Show that each circle C_n with radius $1/n$ and centered at the point $(1/n, 0)$ is a retract of X. (3)

(b) Let x_0 be an interior point of the unit n-disc $\mathbb{D}^n$. Show that there exists a homeomorphism of $\mathbb{D}^n$ onto itself which sends x_0 into 0. (3)

(c) Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be a continuous map with $f(1) = 1$. Show that the induced homomorphism $f_\#$ on $\pi(\mathbb{S}^1, 1)$ is the multiplication by $\deg(f)$. (3)

(d) If a discrete group G acts freely and properly on a Hausdorff space X, show that each point $x \in X$ has an open nbd U in X such that $gU \cap U = \emptyset$ for every $g \in G$ distinct from the identity element.

(e) Let $q : \mathbb{S}^n \to \mathbb{R}P^n$ be the quotient map which identifies the antipodal points. Compute the group $\Delta(p)$. (2)

(2) (a) Let $X = \mathbb{D}^2 \cup B$, where B is the open ball of radius 1 and centered at the point (2,0) in the Euclidean space $\mathbb{R}^2$. Show that the subspace X of $\mathbb{R}^2$ is contractible. (3)

(b) Show that the unit n-disc $\mathbb{D}^n$ is a strong deformation retract of the Euclidean space $\mathbb{R}^n$. (4)

(c) Give an example of a deformation retract that is not a strong deformation retract. Justify your claims. (7)

(3) (a) Let f be a continuous map of a space X into the n-sphere $\mathbb{S}^n$. If f is not surjective, show that it is nullhomotopic. (3)

(b) Show that the exponential map $p : \mathbb{R} \to \mathbb{S}^1$ is open. (4)

(c) Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be the continuous map $z \mapsto z^n$, $n \geq 1$ an integer. Determine the induced homomorphism $f_\#$ on $\pi(\mathbb{S}^1)$. (3)

(d) Let $\phi, \psi : X \to Y$ be continuous maps. Show that the homomorphisms $\phi_\#$ and $\psi_\#$ on $\pi(X, x_0)$ differ by an isomorphism. (4)

(4) (a) Define the degree of a continuous map $\mathbb{S}^1 \to \mathbb{S}^1$. Show that two continuous maps f and g of $\mathbb{S}^1$ into itself have the same the degree if and only if $f \simeq g$. (8)

(b) Show that the quotient map $\mathbb{S}^n \to \mathbb{R}P^n$, which identifies pair of antipodal points, is a covering map. (6)

(5) (a) Show, by an example, that the composition of two covering maps need not be a covering map. Justfy your assertions. (7)

(b) Let $p : \widetilde{X} \to X$ be a covering map, where X is locally path connected. Show that each point of X has a path-connected admissible nbd. (3)

(c) State the Homotopy Lifting property of a covering map $p : \widetilde{X} \to X$. If f is a loop in X based at x_0, show that f lifts to a loop in $\widetilde{X}$ based at $\tilde{x} \in p^{-1}(x_0)$ if and only if $[f] \in p_{\#}\pi\big(\widetilde{X}, , \tilde{x}\big)$. (4)

(6) (a) Let $p : \widetilde{X} \to X$ be a covering map, where $\widetilde{X}$ is path-connected. For any $x_0 \in X$ and $\tilde{x} \in p^{-1}(x_0)$, show that the isotropy subgroup of $\pi(X, x_0)$ at $\tilde{x}$ is $p_{\#}\pi\big(\widetilde{X}, \tilde{x}\big)$. (4)

(b) Let $p : \widetilde{X} \to X$ be a covering map, where $\widetilde{X}$ is connected and X is locally path-connected. Show that the subgroups $p_{\#}\pi\big(\widetilde{X}, \tilde{x}\big)$, $\tilde{x} \in p^{-1}(x_0)$, constitute a complete conjugacy class in $\pi(X, x_0)$. (6)

(c) If $p : \widetilde{X} \to X$ is a regular covering map, and g a closed path in X, show that either every lifting of g is closed or none is closed. (4)

(7) (a) If a discrete group G acts freely and properly on a connected Hausdorff space X, prove that the orbit map $q : X \to X/G$ is a regular covering map. (6)

(b) Let X be a connected, locally path-connected and semilocally simply connected Hausdorff space. Show that, for each subgroup $H \subseteq \pi(X, x_0)$, there exists a covering space $\widetilde{X}$ of X such that $\pi\big(\widetilde{X}\big) \cong H$. (8)

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Your Roll Number: ....

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examinations, December 2020
Part II Semester III
MMATH18-301(ii): COMMUTATIVE ALGEBRA
(Unique Paper Code: 223502302)

Time: 3 + 1 Hours | Maximum Marks: 70

Instructions: • Attempt five questions in all. Question 1 is compulsory. All questions carry equal marks. • Throughout the paper all the rings are assumed to be Commutative ring with nonzero identity.

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(1) (a) Determine the radical of the ideal $< 5 + 2x + 4x^2 >$ in $\mathbb{Z}_8[x]$. [2]

(b) Let I be an ideal of a ring R such that $I_\mathfrak{m} = R_\mathfrak{m}$ for all maximal ideals $\mathfrak{m}$ of R. Show that $I^2 = I$. [3]

(c) Let R be a ring and $\mathfrak{p} \in Spec(R), \mathfrak{q} \in Max(R)$ be such that $R_\mathfrak{p} \subseteq R_\mathfrak{q}$. Show that if $R_\mathfrak{q}$ is Noetherian, then $R_\mathfrak{p}$ is Noetherian. [3]

(d) Let R be a DVR and I be a nonzero ideal. Show that there exists $m \in \mathbb{N}$ such that the nonzero elements of I are precisely the set of all elements of R whose valuation is greater than or equal to m. [3]

(e) Show that $\mathbb{Z}[\sqrt{2}]$ is a Dedekind domain. [3]

(2) Let R be a ring.

(a) If $a \in R$ is any nonzero element, then show that there exists a prime ideal of R not containing a. [5]

(b) Let $f, g \in R[x]$ be primitive polynimals. Show that $f.g$ is primitive. [5]

(c) Let $\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3 \in Max(R)$ be such that $\mathfrak{m}_1\mathfrak{m}_2\mathfrak{m}_3 = 0$. Show that the nilradical of R is zero and R is isomorphic to a finite direct product of fields. [4]

(3) (a) Give an appropriate example to justify the following statement: Let R and T be domains such that $S^{-1}R \subseteq S^{-1}T$ for some multiplicative closed subset S of R and T. Then R and T may not be comparable. [4]

(b) Let $R_\mathfrak{p} \subseteq R_\mathfrak{q}$ for some $\mathfrak{p}, \mathfrak{q} \in Spec(R)$. Determine whether $\mathfrak{q} \subseteq \mathfrak{p}$ or $\mathfrak{p} \subseteq \mathfrak{q}$. Justify your answer. [3]

(c) Let $\mathfrak{p}$ be a prime ideal of R. Show that the prime ideals of R contained in $\mathfrak{p}$ are in one to one correspondence with the prime ideals of $R_{\mathfrak{p}}$. [7]

(4) (a) Let R be a Noetherian ring. Show that R has only finitely many minimal prime ideals. [4]
(b) Show that the isolated primary components in a decomposable ideal I of a ring R are uniquely determined by I. [5]
(c) Show that the number of primary ideals in any minimal priamry decomposition of a decomposable ideal I is unique. [5]

(5) (a) Determine the integral closue of $\mathbb{Z}_{2\mathbb{Z}}$ in $\mathbb{Q}$. [3]
(b) Let $R \subseteq T$ be an integral ring extension. Show that

$$dim(R) = dim(T)$$

[3]
(c) State and prove the going down theorem. [9]

(6) (a) Let I be a non zero proper ideal in a Noetherian ring R. Show that the family of all prime ideals conaining I has finitely many minimal elements. [5]
(b) Show that an Artin ring is uniquely determined as a finite direct product of Artin local rings. [9]

(7) Let R be a Noetherian local domain of dimension one, $\mathfrak{m}$ its maximal ideal, and $k = R/\mathfrak{m}$ its residue field. Show that the following are equivalent:
(a) R is a DVR.
(b) $dim_k(\mathfrak{m}/\mathfrak{m}^2) = 1$.
(c) There exists $x \in R$ such that every nonzero ideal is of the form $< x^m >$ for some $m \geq 0$.
[14]

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Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations, OBE Blended Mode,Dec 2020
Part II, Semester III, Unique Paper Code- 223502303
MMATH18-301(iii):Representation of Finite Groups

Time: 3 hours Maximum Marks: 70

Instructions: • Attempt five questions in all.• **Question1** is compulsory. • All questions carry equal marks • Throughout this question paper G will denote a finite group and F a scalar field.

(1) (a) Find a complete set of non-isomorphic irreducible $\mathbb{C}G$-modules for the cyclic group of order four and their dimensions. (3)

(b) Write down the characteristic table for the group $(\frac{S_3}{A_3}) \times S_3$ (3)

(c) Consider the subgroup $H = \langle(1,2,3)\rangle$ of S_4. Find all the irreducible characters of H. (3)

(d) In symmetric groups of any order $n \geq 2$, the linear characters of the quotient group by its derived subgroup are lifted to linear characters only. Prove or disprove. (3)

(e) Let ω be the fifth root of unity. Is $\omega(\omega^2 + \omega^3)$ is an algebraic integer? Justify. (2)

(2) Define permutation and regular FG-module of a group G over a field F. Are they isomorphic for the group S_4? Prove that the $\mathbb{C}G$-submodules $U_1 = \{u \in U| \quad ug = u, \forall g \in G\}$ and $V_1 = \{v \in V| \quad vg = v, \forall g \in G\}$ of isomorphic $\mathbb{C}G$-modules U and V respectively are isomorphic. Construct one example each of subgroups in S_4 for which the permutation module over F is isomorphic to the regular FS_4-module and for which they are not. (2+2+4+3+3)

(3) (a) For any arbitrary group G, does there always exists a faithful $\mathbb{C}G$-module? Justify. Show that there are groups for which there always exists a faithful irreducible $\mathbb{C}G$-module? Justify whether this existence holds in case of finite abelian groups. (2+6+3)

(b) Give an example of a faithful irreducible $\mathbb{C}G$-module of dimension two for the group D_6. (3)

(4) (a) Determine the normal subgroups of any group G using its irreducible characters and in particular, prove the necessary and sufficient condition for G to be a simple group. (4+5)

(b) Find three irreducible characters of S_4 which are obtained by lifting irreducible characters of some suitable quotient of the group S_4. (5)

(5) (a) Prove that $\langle \chi^n, \psi \rangle \neq 0$ for any arbitrary irreducible character ψ of group G, χ faithful and takes r distinct values over G, some $n \in \{0, 1, \cdots, r-1\}$. Does the result still hold when χ is not faithful? Justify. (6+5)

(b) Show that for every irreducible character ψ of a non-trivial finite group, the inner product $\langle \chi, \psi \rangle$ is not zero when χ is either trivial or regular character of G. (3)

(6) (a) Show that the number of irreducible characters of degree one is b in a non-abelian group of order ab with a and b prime integers and $a > b$. Also determine the degrees of all other irreducible characters of this group. (4+4)

(b) Give an example of a group of order ab with a and b prime integers which has no irreducible character of degree more than one. Prove that such a group is abelian. (3+3)

(7) Show that there exists a non trivial normal subgroup of a group of order pq, where p and q are prime integers. Find the exact order of a non-abelian group of order less than 75 having no nontrivial normal subgroup by showing that any even ordered group G with $|G| = 2n$, n odd integer always has a normal subgroup H with $|G : H| = 2$ using regular representation of G. (6+8)

No. of Printed Pages **2**

M.A./M.Sc. Mathematics Examinations, December-2020
Part II, Semester III
MMATH18-302(i): Fourier Analysis
Unique Paper Code-223502304

Time: 3 hr (09:30am to 12:30pm) Maximum Marks: 70

Instructions: • Attempt **Five** questions in all • Question No. 1 is **COMPULSORY** • All questions carry equal marks • Symbols have their usual meaning

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(Question No. 1 is Compulsory: 14 marks)

(1) (a) Show that if $f, g \in L^1(\mathbb{T})$, then $\|f * g\|_{L^1} \leq \|f\|_{L^1}\|g\|_{L^1}$. Here, $f * g$ is the convolution of f and g. [3 Marks]

(b) Show that if $f \in L^1(\mathbb{T})$, then $\lim_{|n|\to\infty} \hat{f}(n) = 0$. [4 Marks]

(c) Show that every topological group is homogeneous. [3 Marks]

(d) How you interpret the Fourier transform of functions in $L^1(G)$ as a convolution? Justify your answer. Here, G is a locally compact abelian group. [4 Marks]

(Answer any FOUR questions: 4×14 **marks** $= 56$ **marks)**

(2) (a) Let $f, \phi \in L^1(\mathbb{T})$, where $\phi(t) = \sum_{n=-N}^{N} a_n e^{int}$. How $\phi * f$ and n^{th} Fourier coefficient $\widehat{f}(n)$ of f are related? Justify your answer. Here, $\phi * f$ is the convolution of ϕ and f. [4 Marks]

(b) State and prove a result which gives non-zero complex homomorphism of $L^1(G)$. Here, G is an locally compact abelian group. [10 Marks]

(3) (a) Under what condition(s), the Fourier transform $\widehat{f}$, of a function $f \in L^1(\mathbb{R})$ is differentiable? Justify your answer. [4 Marks]

(b) How you express a function $f \in L^1(\mathbb{T})$ in terms of a summability kernel in $L^1(\mathbb{T})$? Explain. [10 Marks]

(4) (a) Show that if f is continuous and with compact support on $\mathbb{R}$, then $\|\widehat{f}\| = \|f\|$. [7 Marks]

(b) Show that $\{f \in L^1(\mathbb{R}) : \text{support of } \widehat{f} \text{ is compact}\}$ is dense in $L^1(\mathbb{R})$. [7 Marks]

(5) (a) Find the value of the modular function on a compact subgroup H of a locally compact group G. [4 Marks]

(b) Show that there exists a continuous function on $\mathbb{T}$ whose Fourier series diverges. [10 Marks]

(6) (a) Show that the modular function of a locally compact group G is a homomorphism from G to $\mathbb{R}_\times$. Here, $\mathbb{R}_\times$ is the multiplicative group of positive real numbers. [3 Marks]

(b) Show that if $f \in L^1(\mathbb{T})$, $t_o \in \mathbb{T}$ and $\int_0^{2\pi} \frac{|f(t)-f(t_o)|}{|t-t_o|} dt < \infty$, then $S_n(f, t_o)$, the sequence of partial sum of the Fourier series of f at t_o, converges to $f(t_o)$. [5 Marks]

(c) Show that if U is a neighborhood of the identity e of a topological group G, and F is a compact subset of G, then there is a neighborhood V of e such that $xVx^{-1} \subseteq U$ for all $x \in F$. [6 Marks]

(7) (a) Show that a left Haar measure λ on a locally compact group G is not identically zero. [4 Marks]

(b) Show that if λ is a left Haar measure on a locally compact group G, then G is compact if and only if $\lambda(G)$ is finite. [10 Marks]

DEPARTMENT OF MATHEMATICS
M. A./M. SC. MATHEMATICS PART (II) -SEMESTER III
FINAL EXAMINATION DECEMBER 2020
MMATH18- 302(II) MATRIX ANALYSIS, UPC: 223502305

Time: 3 HOURS Maximum Marks: 70

• Write your **Name, University Roll No., College and Course on the first page and your Roll No. on each subsequent answer sheet**. • Attempt **five** questions in all. • **Question No. 1 is compulsory**. Attempt any four from the remaining six questions. • All the questions carry equal marks. • All the symbols have their usual meanings.

(1) (a) Prove or disprove (with justification): Every self adjoint norm on M_n is unitarily invariant. (3)
(b) Determine a matrix norm with respect to which a unitary matrix has condition number one. (3)
(c) Determine the polar decomposition of a non-zero vector a in $\mathbb{C}^n$. (3)
(d) Prove or disprove: For $x, y \in \mathbb{R}^n$, $x \prec^w y$ if and only if $x \prec_w y$. (3)
(e) Prove or disprove: If every leading principal minor of a Hermitian matrix A is nonnegative, then A is positive semidefinite. (2)

(2) (a) Let $A \in M_n(\mathbb{R})$ with determinant 1. Prove that there exists a matrix $B \in M_n(\mathbb{R})$ such that $B^{-1}A$ is a special orthogonal matrix. (7)
(b) Let $A = (a_{ij}) \in M_n$ be a positive definite matrix. Prove that for any $m \in \mathbb{N}$, the matrix $A_m = (a_{ij}^m)$ is positive definite. (7)

(3) (a) Let $\| \cdot \|_1$ and $\| \cdot \|_2$ be norms on $\mathbb{C}^n$ such that $(\max_{x \neq 0} \frac{\|x\|_1}{\|x\|_2}) = (\max_{x \neq 0} \frac{\|x\|_2}{\|x\|_1})^{-1}$. Then prove that $\| \cdot \|_1^D = c\| \cdot \|_2^D$ for some positive scalar c. (7)
(b) Let $A \in M_n$ be Hermitian and non-singular. Prove that there exists a unique positive semidefinite matrix $P \in M_n$ and a unique unitary matrix $U \in M_n$ such that $A = UP$. Further prove that A is normal if and only if $PU = UP$. (7)

(4) (a) Find T-transforms T_1, T_2 and a permutation matrix P such that $(1, 2, 3) = (6, 0, 0)T_1T_2P$. (7)
(b) Prove that $Sl_n(\mathbb{C})$ is connected. Determine the number of path components of $O(n)$. (4+3)

(5) (a) Prove that the singular values of a matrix are the continuous functions of the entries of the matrix. (7)

(b) Let $H = \begin{pmatrix} X & Y \\ Y^* & X \end{pmatrix}$ be a Hermitian matrix with $X, Y \in M_n$. Give example to show that H may not be positive semidefinite if X and Y both are positive semidefinite. Further, if X and Y are unitary matrices such that X^* and Y commute with each other, then prove that H is positive definite if and only if X is positive definite with $\rho((X^*)^2) < 1$. (7)

(6) (a) Prove that $x \prec y$ if and only if $x \prec_w y$ and $-x \prec_w -y$. Further prove that if (x_i) is such that $x_i \geq 0, \sum_{i=1}^n x_i = 1$, then $(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}) \prec (x_1, x_2, \cdots, x_n)$. (4+3)

(b) State and prove Gersgorin theorem for columns. (7)

(7) (a) If $A, B \in M_n$ commute with each other, then prove that e^A and e^B also commute with each other. Give example to show that the converse need not be true. (4+4)

(b) For a doubly stochastic matrix A, prove that $|||A|||_2 \leq 1$, $|||\cdot|||_2$ being the spectral norm. (6)

Your Roll Number: ....

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations, December 2020
Part II Semester III
MMATH18-302(iii): Theory of Bounded Operators
Unique Paper Code 223502306

Time: **4 hours** (This includes **one hour** time for downloading the paper, scanning your answer sheets and uploading back in the email for the final submission.)
Maximum Marks: **70**

Instructions: • All notations used are standard • **Question no.** 1 is **compulsory** • Attempt any **four** questions from the remaining six questions.

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(1) Prove or disprove the following statements:
  (a) Let $T : l^2 \to l^2$ be defined by $T(x_1, x_2, \cdots) = (0, x_1, x_2, \cdots)$. Then $\sigma_p(T)$ is an open subset of $\mathbb{C}$.
  (b) Let $\{\lambda_n\}_{n\in\mathbb{N}}$ be a bounded sequence of real numbers. Let $T : l^2 \to l^2$ be defined by $T(x_1, x_2, \cdots) = (\lambda_1 x_1, \lambda_2 x_2, \cdots)$. If $\frac{1}{n}\sum_{k=1}^{n} \lambda_k \to 0$ as $n \to \infty$, then T is a compact operator.
  (c) Let $T : l^2 \to l^2$ be defined by $T(x_1, x_2, \cdots) = (x_2, x_3, \cdots)$. Then T is a compact bounded linear operator.
  (d) Let T be a bounded linear self-adjoint operator on an infinite dimensional Hilbert space X. Then the null space $\mathcal{N}(T_\lambda)$ of $T_\lambda = T - \lambda I$ is finite dimensional, for every $\lambda \neq 0$.
  (e) Every self-adjoint linear operator T on a Hilbert space H is idempotent.
  (f) The operator $T : l^2 \to l^2$ defined by $T(x_1, x_2, \cdots) = (0, 0, x_1, x_2, \cdots)$ is positive.
  (g) Every compact self-adjoint operator $T : l^2 \to l^2$ is of the form $T(x_1, x_2, \cdots) = (\lambda_1 x_1, \lambda_2 x_2, \cdots)$, where $\{\lambda_n\}_{n\in\mathbb{N}}$ is a sequence of real numbers such that $\lambda_n \to 0$ as $n \to \infty$. (2×7)

(2) (a) Let $T : l^2 \to l^2$ be defined by $Tx = y$, $x = (\xi_n)$, $y = (\eta_n)$, $\eta_n = \alpha_n \xi_n$, where $\{\alpha_n\}_{n\in\mathbb{N}}$ is dense in $[0, 1]$. Find $\sigma_p(T)$ and $\sigma(T)$.
  (b) Show that a bounded linear operator T defined on a complex Banach space X is invertible if and only if T is bounded below and its range is dense in X. Use this fact to define $\sigma_{app}(T)$ and $\sigma_{comp}(T)$ and show that $\sigma(T) = \sigma_{app}(T) \cup \sigma_{comp}(T)$
  (c) State the Spectral Mapping Theorem for Polynomials for operators on Banach spaces. Use it to determine the spectrum of an orthogonal projection P defined on a complex Hilbert space H.
$(6 + 6 + 2)$

(3) (a) Let T be a bounded linear operator on a complex Banach space X. Then show that $\sigma_{app}(T)$ is a closed set. Further, prove that boundary of $\sigma(T)$ is contained in $\sigma_{app}(T)$.
  (b) Show that the identity operator on l^2 is not a compact operator. Further, is it true that for any two vectors y and z in l^2, the operator $T : l^2 \to l^2$ defined by $Tx = \langle x, y\rangle z$ is compact? Justify your claim.
  (c) Show that range of a compact linear operator $T : X \to Y$ is separable, where X and Y are normed linear spaces. $(6 + 3 + 5)$

(4) (a) Let $X = C[a, b]$, where $[a, b]$ is any compact interval. Then show that $T : X \to X$ defined by $(Tx)(s) = \int_a^b (s-t)^2 x(t)dt$ is a compact linear operator.

(b) For a compact linear operator $T : X \to Y$, where X and Y are normed linear spaces, establish that the operator $T^* : Y' \to X'$ is also a compact operator, where T^* is defined by $(T^*g)(x) = g(Tx)$, $g \in Y'$ and, X' and Y' are dual spaces of X and Y respectively.

(c) Let T be a compact linear operator on a Banach space X. Then show that every spectral value $\lambda \neq 0$ of T, if it exists, is an eigenvalue of T. What can you say about the case $\lambda = 0$? $(2 + 6 + 6)$

(5) (a) Let T be a bounded self-adjoint linear operator on a complex Hilbert space H. Then show that a number $\lambda \in \rho(T)$ if and only if there exists a $c > 0$ such that $\|T_\lambda(x)\| \geq c\,\|x\|$, for all $x \in H$, where $T_\lambda = T - \lambda I$.

(b) For any bounded self-adjoint operator T on a complex Hilbert space H, establish that $\|T\| = \max\{|m|, |M|\}$, where $m = \inf\limits_{x \in H} \{\langle Tx, x\rangle \mid \|x\| = 1\}$ and $M = \sup\limits_{x \in H}\{\langle Tx, x\rangle \mid \|x\| = 1\}$. Further, is it true that both m and M are in $\sigma(T)$? Justify your claim.

(c) Suppose bounded linear self-adjoint operators T_1 and T_2 on a complex Hilbert space H satisfies $T_1T_2 = T_2T_1$. If $T_2 \geq 0$, then show that $T_1^2T_2$ is self-adjoint as well as positive. $(6 + 6 + 2)$

(6) (a) Suppose P_1, P_2 are two projections on a Hilbert space H satisfying $P_1(H) \subset P_2(H)$. Show that then $P_2 - P_1$ is also a projection on H.

(b) Show that every positive bounded self-adjoint linear operator $T : H \to H$ on a complex Hilbert space H has a positive square root.

(c) Suppose T, S be positive operators on a complex Hilbert space H satisfying $TS = ST$. Then show that $ST = TS \geq 0$. $(2 + 6 + 6)$

(7) (a) Find the spectral family of the zero operator $T = 0 : H \to H$ on a complex Hilbert space $H \neq \{0\}$.

(b) Let T be a bounded linear operator on a complex Hilbert space H. Then show that there exists a partial isometry W such that $T = W|T| = (T^*T)^{1/2}$. Is this decomposition unique? Justify your claim.

(c) Let T be a compact linear operator on a complex Hilbert space H. Then show that there exists two orthonormal sets $\{e_n\}$ and $\{f_n\}$ in H and a sequence of positive numbers $\{s_n\}$ converging to 0 such that $T = \sum\limits_n s_n < ., e_n > f_n$. $(2 + 7 + 5)$

UPC NO.: 223502307 Your Roll Number: ....

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M. A./M. Sc. Mathematics Examinations, Dec 2020
MMATH18-303(i): ADVANCED COMPLEX ANALYSIS

Time: 3 hours Maximum Marks: 70

Instructions: • Attempt five questions in all. • Question 1 is compulsory. • All questions carry equal marks.

(1) (a) If $\mathscr{F}$ in $H(G)$ is normal then show that it is locally bounded. [4 Marks]

(b) Show that $\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right)=\frac{1}{2}$. [4 Marks]

(c) Prove that if u is harmonic then so are $u_x=\frac{\partial u}{\partial x}$ and $u_y=\frac{\partial u}{\partial y}$. [3 Marks]

(d) Let f be an analytic function on the disk $B(a;r)$ such that [3 Marks]

$$|f'(z)-f'(a)|<|f'(a)| \text{ for all } z\in B(a;r),\ z\neq a.$$

Show that f is one-one.

(2) (a) Let $\{f_n\}\subset C(G;\Omega)$ and suppose that $\{f_n\}$ is equicontinuous. If $f\in C(G;\Omega)$ and $f(z)=\lim f_n(z)$ for each z then show that $f_n\to f$. [8 Marks]

(b) Prove that a set $A\subseteq\mathbb{C}$ is convex if and only if for any points $z_1,\cdots,z_n$ in A and real numbers $t_1,\cdots,t_n\ \geq 0$ with $\sum_{k=1}^{n}t_k=1$, $\sum_{k=1}^{n}t_kz_k\in A$. [6 Marks]

(3) Let f be analytic on $G=\{z:\text{Re } z>0\}$, one-one, with Re $f(z)>\ 0$ for all z in G, and $f(a)=a$ for some real number a. Show that $|f'(a)|\leq 1$. [14 Marks]

(4) (a) Let K be a compact subset of the region G. Then show that there are straight line segments $\gamma_1,\cdots,\gamma_n$ in $G-K$ such that for every function f in $H(G)$, [8 Marks]

$$f(z)=\sum_{k=1}^{n}\frac{1}{2\pi i}\int_{\gamma_k}\frac{f(w)}{w-z}dw,\ \text{for all } z\in K.$$

(b) Prove that $\lim_{z\to 0}\frac{\log(1+z)}{z}=1$. [6 Marks]

(5) (a) State and prove the Mittag–Leffler theorem. [8 Marks]

(b) Let G be a region which is homeomorphic to the unit disk $\mathbb{D}$. Show that G is simply connected. [6 Marks]

(6) Suppose that $f:\overline{\mathbb{D}}\to\mathbb{C}$ is a continuous function such that both Re f [14 Marks]

and Im f are harmonic. Show that

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{it})P_r(\theta - t)dt, \text{ for all } re^{i\theta} \in \mathbb{D}.$$

Further, show that f is analytic on $\mathbb{D}$ if and only if

$$\int_{-\pi}^{\pi} f(e^{it})e^{int}dt = 0, \text{ for all } n \geq 1.$$

(7) (a) Let f be an entire function that omits two values. Prove that f is a constant. [8 Marks]

(b) Show that $\cos \pi z = \prod_{n=1}^{\infty}\left[1 - \frac{4z^2}{(2n-1)^2}\right]$. [6 Marks]

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations, December 2020
Part II Semester III

MMATH18-303(ii): ADVANCED MEASURE THEORY
(Unique Paper Code 223502308)

Time: 3 hours | Maximum Marks: 70

Instructions: • Attempt **five** questions in all. **Question 1 is compulsory**. All questions carry equal marks.• The symbols used have their usual meanings.

(1) (a) Let $X = [-1, 1]$ and $\mathcal{A}$ be the σ-algebra of all Lebesgue measurable subsets of $[-1, 1]$. Define

$$\nu(E) = \int_E x \sin\frac{1}{x}\, dx, \;\; E \in \mathcal{A}.$$

Determine a Hahn decomposition for the signed measure ν. [4]

(b) Consider Lebesgue measure on $\mathbb{R}$ and let $A = [-1, 1] \times [1, 2] \subset \mathbb{R}^2$. Then express $\mathbb{R}^2 \setminus A$ as a disjoint union of measurable rectangles of $\mathbb{R}^2$ and justify your answer. [4]

(c) Check if $A = \cup_{n=1}^{\infty}(\{n\} \times [0, \frac{1}{n}])$ is a Baire set of $\mathbb{R}^2$. [3]

(d) Consider the measure μ on $(\mathbb{R}, \mathcal{P}(\mathbb{R}))$ defined by $\mu(\emptyset) = 0, \mu(\{1\}) = 0$ and $\mu(A) = \infty$ for all other subsets of $\mathbb{R}$. Show that μ is inner regular but not outer regular. [3]

(2) (a) Let μ be a σ-finite measure and T be a bounded linear functional on $L^p(\mu)$ with $1 < p < \infty$. Show that there is a g in $L^q(\mu)$ with $1/p + 1/q = 1$ such that $T(f) = \int fg d\mu, \;\; \forall\, f \in L^p(\mu)$, by assuming that the result holds for any finite measure μ. [6]

(b) State Radon-Nikodym theorem (for measures) and prove only the uniqueness part. [1+7]

(3) (a) Let ν be a signed measure on $(X, \mathcal{A})$ and $E \in \mathcal{A}$ such that $-\infty < \nu E < 0$. Show that there is a negative set $A \subset E$ with $\nu A < 0$. [6]

(b) When is an outer measure μ^* on X said to be regular? Show that an outer measure induced by a measure on an algebra is a regular outer measure. [4]

(c) Show that the Lebesgue measure m_n on $\mathbb{R}^n$ is σ-finite and takes finite value on Bounded Borel sets. [4]

(4) (a) State Lebesgue decomposition theorem, prove only the uniqueness part and illustrate the theorem with an example. [6]

(b) Consider Lebesgue measure on $\mathbb{R}$, let $n \in \mathbb{N}, n \geq 2$ and $\mathcal{S}$ be the σ-algebra of subsets of $\mathbb{R}^n$ on which the complete product measure m_n is defined. If $E \in \mathcal{S}$, show that $rE \in \mathcal{S}$ for any real number r. [4]

(c) Let E be a Baire set in a locally compact Hausdorff space X. Show that either E or E^c is σ-bounded and that both E and E^c are σ-bounded if and only if X is σ-compact. [4]

(5) (a) Let X be a locally compact Hausdorff space. If X is also compact, show that every Baire measure μ on X is regular. [6]

(b) Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be complete measure spaces, $\mathcal{R}$ be the semi algebra of measurable rectangles in $X \times Y$ and $E \subset \mathcal{R}_\sigma$. Then show that $g : Y \to [0, \infty]$ defined by $g(y) = \mu(E_y)$ is measurable and $\int g d\nu = (\mu \times \nu)(E)$. [4]

(c) Let $\{A_n\}$ be a sequence of Lebesgue measurable subsets of $\mathbb{R}$ such that $m(A_n \cap A_l) = 0$ for all $n \neq l$. Show that $m(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} m(A_n)$. [4]

(6) (a) State and prove Tonelli's theorem. [8]

(b) Let μ be a finite measure on an algebra $\mathcal{A}$ (of subsets of X) and μ^* its induced outer measure. Show that a set $E \subset X$ is μ^*-measurable if and only if for each $\varepsilon > 0$ there is a set $A \in \mathcal{A}_\delta, A \subset E$, such that $\mu^*(E \setminus A) < \varepsilon$. [6]

(7) (a) Let X be a locally compact Hausdorff space, T be a positive linear functional on $C_c(X)$ and μ^* be its induced outer measure. Then show that $\mu^*(K) < \infty$ for every compact set $K \subset X$ and $\mu^*(K_1 \sqcup K_2) = \mu^*(K_1) + \mu^*(K_2)$ for disjoint compact subsets K_1, K_2 of X. [6]

(b) Define complex measure and give two examples with justifications, [5]

(c) Let μ and ν be two regular Borel measures on a locally compact Hausdorff space X such that $\int f d\mu = \int f d\nu$ for all $f \in C_c(X)$. Prove that $\mu = \nu$. [3]

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Your Roll Number: ....

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations, December 2020
Part II Semester III
MMATH18-303(iii): General Topology
(Unique Paper Code 223502309)

Time: **4 hours** (This includes **one hour** time for downloading the paper, scanning your answer sheets and uploading back in the email for the final submission.)
Maximum Marks: 70

Instructions: • All notations used are standard • **Question no.** 1 is **compulsory** • Attempt any **four** questions from the remaining six questions .

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(1) (a) If $f : X \to Y$ is a proper surjection and Y is a Lindelöf space then prove that X is also a Lindelöf space. (3)
(b) Justify that the condition of closedness on the sets A and B in the Uryshon Lemma is essential. (3)
(c) If X is a regular space and $A \subset X$ is closed then prove that X/A is Hausdorff. (2)
(d) If each open subspace of a paracompact space is paracompact then show that every subspace is paracompact. (3)
(e) Show that the cone CX of any space X is a connected space. (3)

(2) (a) Let $\backsim$ be an equivalence relation on a space X. If X is locally connected then prove that $X/\backsim$ is also locally connected. (5)
(b) Let X be a compact Hausdorff space. If A is closed a subset of X then prove that X/A is homeomorphic to the one point compactification of $(X - A)$. (5)
(c) Let $f : X \to Y$ be an open identification map and $A \subset X$ be an f-saturated set. Prove that $g : A \to f(A)$, $a \longrightarrow f(a)$, is an identification map. (4)

(3) (a) Let X be a non-compact, locally compact, Hausdorff space. Prove that its one point compactification X^* is a compact Hausdorff space in which X is dense. (5)
(b) Prove that the product $\prod_{\alpha \in \mathcal{A}} X_\alpha$ of a family of spaces X_α is locally compact if and only if each X_α is locally compact and all but finitely many X_α's are compact. (5)
(c) Prove that a locally compact dense subset of a Hausdorff space X is open in X. (4)

(4) (a) Let $f : X \to Y$ be continuous. If f is closed and $f^{-1}(y)$ is compact for every $y \in Y$ then prove that f is a proper map. (5)
(b) Prove that a first countable, Hausdorff, countably compact space is a regular space. (5)
(c) Establish that any Hausdorff compactification $\widetilde{X}$ of a complete regular space X to which every continuous map of X into a compact Hausdorff space has an extension, is homeomorphic to βX, the Stone Cěch compactification of X. (4)

(5) (a) Using Uryshon Metrization Theorem prove that the continuous image of a compact metric space in a Hausdorff space is metrizable. (6)
(b) Prove that a space is completely regular if and only if it can be embedded in a cube. (8)

(6) (a) If X is a normal space, $A \subset X$ is closed and $f : A \to (0,1)$ is continuous then prove that there exists a continuous function $g : X \to (0,1)$ such that $g_{|A} = f$. (6)

(b) Let X be a regular space with basis $\mathcal{B}$ which is countably locally finite. Prove that X is normal and every closed set in X is a G_δ set. (8)

(7) (a) Prove that every open covering of Euclidean space $\mathbb{R}^n$ has a partition of unity subordinate to it. (8)

(b) Prove that every F_σ-set in a paracompact Hausdorff space is paracompact. (6)

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examinations, December 2020
Part II Semester III

MMATH18-304(i): COMPUTATIONAL FLUID DYNAMICS

Time: 3 hours Maximum Marks: 70

Instructions: • Attempt **five** questions in all • Attempt all parts of question number 1 which is compulsory • Each question carries 14 marks as mentioned within brackets • Symbols have their usual meaning.

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(1) (a) When do you say that a two level finite difference scheme is stable w.r.t $||.||$? What are its necessary and sufficient conditions? [4 Marks]

(b) Does the power-law scheme for steady one-dimensional convection-diffusion equations is efficient then the Hybrid and the Upwind schemes? Justify. [4 Marks]

(c) Find the finite difference approximation for $\frac{\partial^2 v}{\partial t \partial x}$ at the point (m, n) and find its order of accuracy. [3 Marks]

(d) State the Scarborough condition. How important is this for boundedness property of a discretization scheme? Explain. [3 Marks]

(2) (a) Find the order of accuracy and truncation error of the Crank-Nicolson scheme for the equation $u_t + au_x = f$, where a is a positive constant and f is any function of t and x. [7 Marks]

(b) Find the solutions of [7 Marks]

$$u_t + u_x = 0$$

subject to the initial conditions

$$u(x,0) = \begin{cases} 0, & x < 0 \\ x/2, & 0 \le x \le 2 \\ 2 - \frac{x}{2}, & 2 \le x \le 4 \\ 0, & x > 4 \end{cases}$$

using the leap-frog scheme with $h=\frac{1}{2}$ and $\lambda=\frac{1}{2}$ where $\lambda=\frac{k}{h}$. Find the solutions upto two time levels.

(3) Derive the Peaceman-Rachford ADI method for two-dimensional heat conduction equation $u_t = u_{xx} + u_{yy}$. Find the solution of the two dimensional heat conduction equations [7+7 Marks]

$$u_t = u_{xx} + u_{yy}$$

subject to the initial condition

$$u(x, y, 0) = sin(\pi x) sin(\pi y), \quad 0 \le x, y \le 1$$

And the boundary condition

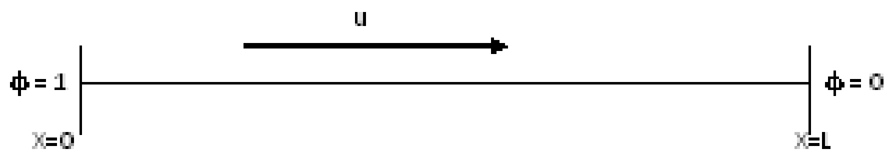
$$u = 0 \quad on \ \ the \ \ boundary, t \ge 0$$

Using the Peaceman-Rachford ADI method. With $h = \frac{1}{4}$ and $\lambda = \frac{1}{8}$ and integrate for one time step.

(4) (a) Derive fully implicit finite volume discretized equation for one-dimensional unsteady heat conduction equation: [7 Marks]

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + S$$

(b) A property Φ is transported by means of convection and diffusion through one-dimensional domain sketched as follows: Use the required [7 Marks]



governing equations; the boundary conditions are $\Phi_0 = 1$ at $x = 0$ and $\Phi_L = 0$ at $x = L$. Divide the domain into 3 three control volumes and use the hybrid scheme for convection-diffusion, calculate the distribution of Φ as a function of x for $u = 2.5m/s$. Note that $F = F_e = F_w = 2.5$, $D = D_e = D_w = 0.5$, a Piclet number $Pe_w = Pe_e = \frac{\rho u \delta x}{\Gamma} = 5$.

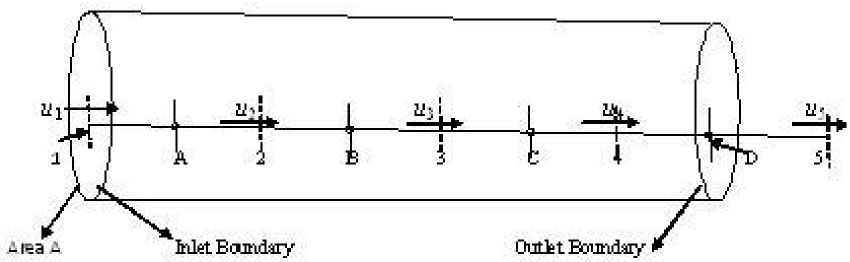
(5) Consider the problem of heat conduction that includes sources other than those arising from boundary conditions which is governed by the equation [14 Marks]

$$\frac{d}{dx}\left(k \frac{dT}{dx}\right) + q = 0,$$

Draw a diagram of a large plate of thickness $L = 4\ cm$ with constant thermal conductivity k=0.5 W/m.K and uniform heat generation q=1000 kW/m^3. The two faces A and B are at temperatures of 300^{o}C and 400^{o}C respectively. Assuming that the dimensions in the y- and z- directions are so large that temperature gradients are significant in the x-direction only. Divide the whole domain into three control volumes, calculate the steady state temperature distribution.

(6) Give an assessment of the central difference, Upwind and Hybrid difference in schemes for 1-D convection-Diffusion problem. [6+8 Marks]
What is the essence of a staggered grid? Elucidate with the help of a neat diagram. Compute the convective flux/unit mass F and the diffusive conductance D at cell faces of the u and v control volume faces.

(7) Consider the steady, one-dimensional flow of a constant-density fluid through a duct with constant cross-sectional area. Using staggered grid shown in the figure below, where the pressure p is evaluated at the main nodes $I = A, B, C$ and D, whilst velocity u is calculated at the backward staggered nodes i=1,2,3 and 4. [14 Marks]



The problem data are as follows:
. Density $\rho = 1.0kg/m^3$ is constant.
. Duct area A is constant,
.Multiplier d in $u' = d(p'_I - p'_{I+1})$ is assumed to be constant; we take d=1.0.
Boundary conditions: $u_1 = 10m/s$, $P_D = 0Pa$.
. Initial guessed velocity field: say $u_2^* = 8.0m/s$, $u_3^* = 11.0m/s$, $u_4^* = 7.0m/s$.
Use the SIMPLE algorithm and these problem data to calculate pressure corrections at nodes I=A to D and obtain the corrected velocity fields at nodes i=2 to 4.

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Your Roll Number: ....

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations, December 2020
Part II Semester III
MMATH18-304(ii) UPC: 223502311
COMPUTATIONAL METHODS FOR ODEs

Time: 3 hours | Maximum Marks: 70

Instructions: • Section A is compulsory • Answer **any four** questions from Section B • Each question carries equal marks • Non-programmable scientific calculators are allowed. • Notations have their usual meaning

Section A

(1) (a) Given $\rho(\xi) = \xi^2(\xi - 1)$ find an explicit multistep method. Write the method explicitly. [4]

(b) Is the linear multistep method $y_{n+2} - y_n = h[f(x_{n+2}, y^*_{n+2}) + f(x_n, y_n)]$, where $y^*_{n+2} - 3y_{n+1} + 2y_n = \frac{h}{2}[f(x_{n+1}, y_{n+1}) - 3f(x_n, y_n)]$ consistent when applied to the intial value problem $y' = f(x, y),\ y(t_0) = y_0$? Justify. [4]

(c) Define order and error constant of a linear multistep method. [3]

(d) Derive variational formulation of the problem $y'' = \frac{3}{2}y^2,\ 0 < x < 1,\ y(0) = 4,\ y(1) = 1$. [3]

Section B

(2) (a) Let $\theta \in [0, 1]$ be a constant and denote $t_{n+\theta} = (1 - \theta)t_n + \theta t_{n+1}$. Consider the generalized midpoint method [4+3]

$$y_{n+1} = y_n + hf(t_{n+\theta}, (1 - \theta)y_n + \theta y_{n+1})$$

for the numerical solution of the problem $y' = f(x, y)$. Show that the method is absolutely stable when $\theta \in [1/2, 1]$. Determine the region of absolute stability of the method when $0 \le \theta < 1/2$.

(b) Convert the following higher order equation to a system of first order equations [2+5]

$$y'''(t) + 4y''(t) + 5y'(t) + 2y(t) = 2t^2 + 10t + 8,\ y(0) = 3,\ y'(0) = -1, y''(0) = 3.$$

Solve the resulting system of equations using second order Taylor series method to find the solution at $y(0.4)$ using $h = 0.2$.

(3) (a) Find the order of the implicit Runge-Kutta method [8]

$$y_{j+1} = y_j + \frac{1}{4}(K_1 + 3K_2),\ K_1 = hf(t_j, y_j),\ K_2 = hf\left(t_j + \frac{2}{3}h,\ y_j + \frac{1}{3}(K_1 + K_2)\right)$$

for the solution of the initial value problem $y' = f(t, y),\ y(t_0) = y_0$. Also, determine its interval of absolute stability.

(b) The method $y_{j+1} = y_j + \frac{h}{2}(y'_{j+1} + y'_j) + \frac{h^2}{12}(y''_j - y''_{j+1})$ is written for the numerical solution of the initial value problem $y' = f(t, y),\ y(t_0) = y_0$. Use the above method to solve the following initial value problem [6]

$$y' = 3t + 2y,\ y(0) = 1,\ h = 0.1.$$

Determine an approximation to $y(0.1)$.

(4) (a) Using the Routh-Hurwitz criterion, find the interval of absolute stability of the method [5]

$$y_{j+1} = y_j + \frac{h}{12}(23y'_j - 16y'_{j-1} + 5y'_{j-2})$$

when applied on to the test equation $y' = \lambda y,\ \lambda < 0$. Is it absolutely stable? Justify.

(b) The formula $y_{j+3} = y_j + \frac{3h}{8}(y'_j + 3y'_{j+1} + 3y'_{j+2} + y'_{j+3})$ with a small steplength h is used for solving the equation $y' = -y$. Investigate the convergence properties of the method. [7]

(5) (a) Consider the boundary value problem [7]

$$y'' + 2y = x,\ 0 < x < 1,\ y(0) = 0,\ y(1) = 0.$$

Determine the coefficients of the approximate solution of the form $w(x) = x(1 - x)(a_1 + a_2x)$ using Ritz method.

(b) Solve the initial value problem $y' = -2ty^2,\ y(0) = 1$ with $h = 0.1$ on the interval $[0, 0.2]$ using P-C method [7]

$$P:\ y_{j+1} = y_j + \frac{h}{2}(3y'_j - y'_{j-1})$$

$$C:\ y_{j+1} = y_j + \frac{h}{2}(y'_{j+1} + y'_j)$$

as $P(EC)^mE,\ m = 2$.

(6) (a) Use the Numerov method to replace the boundary value problem [7]

$$y'' + (1 + x^2)y + 1 = 0, \quad y(\pm 1) = 0,$$

by a set of linear difference equations with $h = 0.25$. Write these equations in matrix form as $My = b$. Find $M,\ b$ and solve for y.

(b) Obtain the Ritz finite element solution of the boundary value problem [7]

$$y'' - y = 2x,\ 0 < x < 1,\ y(0) + y'(0) = 1,\ y(1) = 0.$$

using linear shape functions and two finite elements.

(7) (a) Solve the boundary value problem [7]

$$y'' = xy, \quad y(0) + y'(0) = 1,\ y(1) = 1$$

with $h = 1/3$ using the second order method $y_{j-1} - 2y_j + y_{j+1} = h^2 f_j$. Use second order approximations for the derivative boundary conditions.

(b) Consider the two point boundary value problem $y'' = f(x, y, y'),\ x \in (a, b)$ with boundary conditions [7]

$$a_0y(a) - a_1y'(a) = \gamma_1,\ b_0y(b) + b_1y'(b) = \gamma_2,$$

where $a_0,\ b_0,\ a_1,\ b_1,\ \gamma_1$ and γ_2 are constants. Derive a fourth order numerical scheme for the solution of the above problem.

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Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations, December 2020
Part II Semester III

MMATH18-304(iii): (MATHEMATICAL PROGRAMMING)

Unique Paper Code: 223502312

Time: 3 Hours | Maximum Marks: 70

Instructions: • Attempt five questions in all. • Question 1 is compulsory. • All questions carry equal marks.

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(1) (a) Find the set of all saddle points of the function $L(x, y_1, y_2) = x^2y_1 - y_2^2$ on $[-1, 1] \times [-2, -1] \times \mathbb{R}$. [3 Marks]

(b) If $(x, y) = (2, 2)$ is an optimal solution of the problem [4 Marks]

$$\text{Minimize } z = y + ax^2 + y^2$$
$$\text{s.t. } x + y \geq 4$$
$$x \geq 0$$

then what is the value of a?

(c) Find a separating hyperplane separating the convex sets $C = \{(x_1, x_2, x_3) : x_1 + x_2 \geq 2$ and $D = \{(x_1, x_2, x_3) : x_1^2 + 2x_2^2 + x_3^2 \leq 1$. Also find a supporting hyperpalne to the set D at $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. [4 Marks]

(d) Is the function $f(x, y, z) = 3x^2 + z^2 - 2xy$ convex on $\mathbb{R}^3$? Justify. [3 Marks]

(2) (a) Find the critical points of the function $f(x, y) = x^2y - xy^2 + 9xy$ defined on $\mathbb{R}^2$ and determine their nature. [7 Marks]

(b) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined as [7 Marks]

$$f(x, y) = \begin{cases} \dfrac{x^2y}{x^2 + y^2}, & \text{if } xy > 0, \\ 0, & \text{if } xy \leq 0. \end{cases}$$

Is this function continuous at (0,0)? Is it Gâteaux differentiable at $(0, 0)$? Is it Fréchet differentiable at $(0, 0)$? Justify.

(3) Find the Karush-Kuhn-Tucker (KKT) points of the following problem [14 Marks]

$$\text{Minimize } x - x^2 + y^2$$
$$\text{s.t. } x + y = 4$$
$$x \geq 0, y \geq 0.$$

Find the nature of the KKT points and also solve it geometrically.

(4) Find the dual of the following problem [14 Marks]

$$\text{Minimize } z = 2x_1 - 4x_2 + x_1^2 + x_2^2$$
$$x_1 + 3x_2 \leq 6$$
$$x_1 \geq 0.$$

Also find the saddle point of the Lagrangian.

(5) Solve the following problem using convex simplex method [14 Marks]

$$\text{Minimize } z = x_1 - x_2 + 2x_1^2 + x_2^2$$
$$\text{s.t. } 4x_1 + x_2 \leq 4$$
$$x_1 + 3x_2 \leq 6$$
$$x_1 \geq 0, x_2 \geq 0.$$

Also solve the above problem geometrically.

(6) Solve the following problem using Wolfe's method [14 Marks]

$$\text{Maximize } z = 2x_1 - 4x_2 - x_1^2 - 2x_2^2$$
$$\text{s.t. } x_1 + x_2 \leq 1$$
$$x_1 \geq 0, x_2 \geq 0.$$

Also solve the above problem geometrically.

(7) (a) Solve the following problem geometrically and using exterior penalty function method [7 Marks]

$$\text{Minimize } f(x_1, x_2) = 2x_2 + x_1^2$$
$$\text{s.t. } x_1 + x_2 \geq 1$$

starting from the point $(x_1, x_2) = (0, 0)$ with $\mu = 1, \beta = 10$ and $\epsilon = 0.1$.

(b) Solve the problem using barrier function method [7 Marks]

$$\text{Minimize } f(x) = 3x - 1$$
$$\text{s.t. } x \geq -2$$

starting from the point $x = 0$ with $\mu = 1, \beta = \frac{1}{10}$ and $\epsilon = 0.01$.

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Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations, December 2020
Part II Semester III

MMATH18-304(iv): METHODS OF APPLIED MATHEMATICS
(Unique Paper Code 223502313)

Time: 3 hours | Maximum Marks: 70

Instructions: • Attempt **five** questions in all. • Question 1 is compulsory. • All questions carry equal marks. • All symbols have their usual meaning unless otherwise mentioned.

(1) (a) Obtain the expressions for resolvent and iterated kernels for Volterra's integral equation. [4 Marks]

(b) Solve the integral equation $\phi(x) = 1 + x + \int_0^x e^{-2(x-t)}\phi(t)dt$. [2 Marks]

(c) Show that equation $\phi(x) = \frac{1}{2}\int_0^1 a(x)a(t)(1+\phi^2(t))dt$, $(a(x) > 0$ for all $x \in [0,1])$ has no real solutions if $\int_0^1 a^2(x)dx > 1$. [3 Marks]

(d) If $\alpha(x)$ is continuous in $[a,b]$, and if $\int_a^b \alpha(x)h''(x)dx = 0$ for every function $h(x) \in \mathcal{D}_2(a,b)$ such that $h(a) = h(b) = 0$ and $h'(a) = h'(b) = 0$, then prove that $\alpha(x) = c_0 + c_1 x$ for all x in $[a,b]$ with c_0 and c_1 as constants. [3 Marks]

(e) State the Buckingham π-theorem. [2 Marks]

(2) (a) Show that the differential of a differentiable functional is unique. Find the extremals of the functional $\int_{x_0}^{x_1}(y^2 + y'^2 - 2y\sin x)dx$. [4+3 Marks]

(b) Show that regular perturbation fails on the boundary value problem: $\epsilon y'' + (1+\epsilon)y' + y = 0$, $0 < t < 1$, $0 < \epsilon << 1$ with $y(0) = 0$ and $y(1) = 1$. If t is near zero, show that $\epsilon y''(t)$ is large; if $t = O(1)$, then find the solution. Find the exact solution and hence, find the inner and outer approximations. [7 Marks]

(3) (a) Solve the following integro-differential equation: [5 Marks]

$$\phi^{''}(x) - 2\phi^{'}(x) + \phi(x) + 2\int_0^x \cos(x-t)\phi^{''}(t)dt + 2\int_0^x \sin(x-t)\phi^{'}(t)dt = \cos x;$$

with $\phi(0) = \phi^{'}(0) = 0$.

(b) Derive the *Abel's* generalised equation for a particle moving under the gravity and also obtain the solution. [5 Marks]

(c) Using the Fredholm determinants, find the resolvent kernel for the kernel $K(x,t) = \sin x - \sin t$. [4 Marks]

(4) (a) Define the reciprocal kernels. Obtain the Volterra solution of nonhomogeneous Fredholm integral equation of second kind. [2+4 Marks]

(b) Solve the integral equation [4+4 Marks]

$$\phi(x) = x + \lambda \int_0^{2\pi} |\pi - t| \sin x \, \phi(t) dt.$$

Also find the iterated kernels for the kernel $K(x, t) = e^x \cos t$, $a = 0$, $b = \pi$.

(5) (a) Using the Hilbert-Schmidt method, solve [7 Marks]

$$\phi(x) = (x + 1)^2 + \int_{-1}^{1} (xt + x^2 t^2) \phi(t) dt.$$

(b) Define Fredholm alternatives. Prove that eigenfunctions of a symmetric kernel corresponding to different eigenvalues are orthogonal. [2+5 Marks]

(6) (a) Investigate the solvability of $\phi(x) - \lambda \int_0^{2\pi} |x - \pi| \phi(t) dt = x$ for different values of parameter λ. [4 Marks]

(b) Use the Green's function to reduce the BVP [6 Marks]

$$y''' + \lambda y = 2x; \; y(0) = y(1) = 0, \; y'(0) = y'(1);$$

into an integral equation.

(c) Prove that the characteristic numbers of a symmetric kernel are real. [4 Marks]

(7) (a) Find the characteristic numbers and eigenfunctions of the homogeneous integral equation if the kernel is given by [7 Marks]

$$\kappa(x, t) = \begin{cases} \sin x \ \cos t, & 0 \le x \le t, \\ \sin t \ \cos x, & t \le x \le \frac{\pi}{2}. \end{cases}$$

(b) A physical system is described by a law $f(E, P, A) = 0$, where E, P and A are energy, pressure, and area, respectively. Show that $PA^{3/2}/E = \text{const}$. [7 Marks]

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Department of Mathematics, University of Delhi

M.A./M.Sc. Mathematics Open Book Examination 2020

Part-II, Semester-III

MMATH18-305(i) Coding Theory(Unique Paper Code 223503301)

Time: 3 Hrs **Max. Marks:35**

Attempt **5 questions** in all. **Question 1** is compulsory. All questions carry equal marks. Notations and symbols have their usual meaning. Marks are indicated against each question.

1. (a) Prove that the number of inequivalent binary codes of length n and containing just two codewords is n. (1)

   (b) Find the dimension and rate of ternary $(3,9,2)$-code. (1)

   (c) Show that there is a one-to-one correspondence between cosets and syndromes. (1)

   (d) Determine the number of codewords of weight 3 in Hamming code $Ham(r,2)$. (1)

   (e) Show that a code having minimum distance 4 can be used simultaneously to correct single error and detect double errors. (1)

   (f) Prove that $A_q(n,n) = q$. (1)

   (g) Find the largest possible dimension of $C(7,k,3)$-linear code. (1)

2. (a) Prove that a q-ary $(q+1,M,3)$-code satisfies the relation $M \leq q^{q-1}$. (2)

   (b) If d even and $n < 2d$, then show that $A_2(n,d) \leq \lfloor \frac{d}{(2d-n)} \rfloor$. (3)

   (c) Prove that a nonzero codeword polynomial of minimum degree in a cyclic code C is unique. (2)

3. (a) Let C is a binary Hamming code of length n and $\hat{C}$ is its extended code of length $n+1$. For a binary symmetric channel with symbol error probability p, (4)

find $P_{corr}(C)$ and $P_{corr}(\hat{C})$ in terms of p and n. Hence, show that $P_{corr}(C) = P_{corr}(\hat{C})$.

(b) Consider the ring $(Z_5 = \{0,1,2,3,4\}, +_5, \times_5)$. Let $f(x) = 2+5x+3x^2+2x^3$ and $g(x) = 1+4x+5x^2$. Find (3)
(i) $f(x)+g(x)$.
(ii)$f(x).g(x)$.

4. (a) A bit word 0101 is to be transmitted. Construct the even parity seven-bit Hamming code for this data. Further, a 7 bit Hamming code is received as 1011101. Assume even parity and state whether the received code is correct or not? If wrong state the bit in error. (4)

(b) Let C be a code over $GF(3)$ generated by (3)

$$\begin{bmatrix} 1 & 2 & 0 & 2 & 1 & 0 \\ 2 & 0 & 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 2 & 1 & 2 \end{bmatrix}.$$

Find a parity-check matrix and minimum distance of the code C.

5. (a) Obtain a $(7,4)$-cyclic code generated by $g(x) = x^3+x^2+1$. Also, determine whether the received sequence $v = (1001100)$ is valid or not? If not correct it. (4)

(b) Write down a parity-check matric for the 7-ary $[8,6]$-Hamming code and use it to decode the received vectors 34524102 and 12054334. (3)

6. (a) Let C be a binary $[7,4]$-code with the generator matrix (4)

$$G = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

Determine the set of codewords.

(b) Find $C^{\perp}$ for $C = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$. (3)

7. (a) Let C_1 and C_2 be cyclic codes of length n over F and let $g_1(x)$ and $g_2(x)$ be their generator polynomials, respectively. Show that the set (4)

$$\{c(x) \in F_n[x] : c(x) \equiv g_2(x)c_1(x) \pmod{(x^n - 1)} \quad \text{for some} \quad c_1(x) \in C_1\}$$

is a cyclic code of length n over F. Also, find its generator polynomial.

(b) (i) Write out the multiplication table for $F_2[x]/(x^2+1)$. Explain why $F_2[x]/(x^2+1)$ is not a field. (3)

(ii) Find the condition for $(n,2,n)$-code to be a perfect code.

(iii) For a linear (n,k,d)-code over $GF(q)$, show that $d \leq n-k+1$.

Your Roll Number: ....

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examinations, December 2020
Part II Semester III

MMATH18-305(ii): STOCHASTIC CALCULUS FOR FINANCE

Time: 3 Hours | Maximum Marks: 35

Instructions: • Attempt five questions in all. Question 1 is compulsory. All questions carry equal marks.

• **Notations:** $(\Omega, \mathcal{F}, P)$ denote a probability space in all of the following questions. The stochastic process $(W(t) : t \geq 0)$ denotes a one-dimensional standard Brownian Motion and $\{\mathcal{F}_t^W\}_{t\geq 0}$ denotes the natural filtration of the Brownian motion. $\mathcal{B}(\mathbb{R})$ denotes the Borel σ-algebra on $\mathbb{R}$.

(1) (a) Let $\Omega = \mathbb{R}$. Define [1 Marks]

$$P(A) = \alpha\delta_1(A) + \beta\delta_2(A) + \gamma\delta_3(A),$$

where δ_a denotes the Dirac measure in $a \in \mathbb{R}$ and $\alpha, \beta, \gamma \in \mathbb{R}$. Determine the set of all $\alpha, \beta, \gamma \in \mathbb{R}$ such that P defines a probability measure on $(\Omega, \mathcal{B}(\mathbb{R}))$.

(b) Calculate $P(|W(t)| < x)$ in terms of CDF of standard normal. [1.5 Marks]

(c) Prove that the Black-Scholes formulae for the price of European call and put options on a non-dividend paying stock satisfy put-call parity. [1.5 Marks]

(d) Suppose $u(t, x)$ satisfies the diffusion equation $\frac{\partial u}{\partial t} = -\frac{1}{2}\frac{\partial^2 u}{\partial x^2}$ with terminal condition $u(T, x) = h(x)$. Using Feynman-Kac theorem, find the stochastic representation of the solution $u(t, x)$. Compute $u(0, x)$ for $h(x) = x^2$. [1.5 Marks]

(e) Compute $E[(S(t))^2]$ where $S(t)$ is given by [1.5 Marks]

$$S(t) = S(0)e^{(r-\frac{1}{2}\sigma^2)t+\sigma W(t)}.$$

(2) Let $\Omega = [0, 1)$ and let $\mathcal{D} = \{[0, \frac{1}{2}), [\frac{1}{3}, 1)\}$.

(a) Determine the σ-algebra $\sigma(\mathcal{D})$ of Ω which is generated by $\mathcal{D}$. [2.5 Marks]

(b) Define a function [2.5 Marks]

$$X : \Omega \to \mathbb{R}, \quad X(\omega) = \begin{cases} 2, & \text{if } \omega \in [0, \frac{1}{3}), \\ 3, & \text{if } \omega \in [\frac{1}{3}, \frac{1}{2}), \\ -1.1, & \text{else}, \end{cases}$$

Is X a $\sigma(\mathcal{D})$-$\mathcal{B}(\mathbb{R})$ measurable function? Justify your answer.

(c) Define a function [2 Marks]

$$Y : \Omega \to \mathbb{R}, \quad Y(\omega) = \omega^2.$$

Is Y a $\sigma(\mathcal{D})$-$\mathcal{B}(\mathbb{R})$ measurable function? Justify your answer.

(3) For a fixed $T > 0$, define the process $M = (M(t) : t \geq 0)$ by

$$M(t) := E[W(T)^3 | \mathcal{F}_t^W].$$

(a) Show that $(M(t) : t \in [0, T])$ defines a martingale with respect to $\{\mathcal{F}_t^W\}_{t\in[0,T]}$. [3 Marks]

(b) Determine $M(t)$ for $t > T$ and prove or disprove that $(M(t) : t > 0)$ is a martingale w.r.t. $\mathcal{F}_t^W$. Note that the values for $t > T$ are included. [2 Marks]

(c) Show using the properties of Brownian motion that for $t \in [0, T]$ it follows that [2 Marks]

$$M(t) = 3(T - t)W(t) + W^3(t).$$

(4) Let Y_1 be a $\mathcal{F}_1^W$-measurable function with $E[Y_1^2] = 1$ and Y_2 be a $\mathcal{F}_2^W$-measurable function with $E[Y_2^2] = 2$. Define a stochastic process $(\Phi(t) : t \in [0, 4])$ by [7 Marks]

$$\Phi(t) = \begin{cases} 3, & \text{if } t \in [0, 1], \\ Y_1, & \text{if } t \in (1, 2], \\ Y_2, & \text{if } t \in (2, 3], \\ 0, & \text{if } t \in (3, 4]. \end{cases}$$

Give a reason why $(\Phi(t) : t \in [0, 4])$ is an adapted stochastic process. Write the stochastic integral

$$\int_0^4 \Phi(t)\, \mathrm{d}W(t)$$

as the sum of three random variables. Calculate the mean and variance of the stochastic integral

$$\int_0^4 \Phi(t)\, \mathrm{d}W(t).$$

(5) Define the process $(Y(t) : t \geq 0)$ by [7 Marks]

$$Y(t) := (W(t) + t) \exp\left(-W(t) - \tfrac{1}{2}t\right)$$

Write Y as an Itô process, that is find $a(t)$ and $b(t)$ such that

$$\mathrm{d}Y(t) = a(t)\, \mathrm{d}t + b(t)\, \mathrm{d}W(t).$$

Also, find $E[Y(t)]$ and $Var[Y(t)]$.

(6) Consider option on an underlying stock which gives a continuous dividend of its dividend yield q during the life of the option. The stochastic equation for the stock becomes [7 Marks]

$$\frac{\mathrm{d}S}{S} = (\mu - q)\, \mathrm{d}t + \sigma\, \mathrm{d}W(t)$$

where μ and σ are the drift rate and the volatility respectively and are constant. From time t to $t+\, \mathrm{d}t$, the change in the portfolio value of a portfolio containing one unit of stock will have an additional effect of

$qS\,dt$ due to dividend payment . Let r denote the risk-free interest rate with continuous compounding. Derive the modified Black-Scholes equation for the European call option on the underlying asset S. Write the stochastic representation of the price as an expectation using the Feynman-Kac theorem.

(7) Let S represent the non-dividend paying stock in the Black-Scholes model given by [7 Marks]

$$S(t) = S(0)e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W(t)}$$

under the real world measure P. Let Q denote the risk-neutral measure. Write the dynamics of S under Q. Compute the time zero price of an option which pays M Rupees at maturity T if $K_1 \leq S(T) \leq K_2$ and 0 otherwise.

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