Department of Mathematics, University of Delhi

M.A./M.Sc. Mathematics Open Book Examination, July 2020

Part II, Semester IV

Paper: MATH14 - 401(A) Algebraic Number Theory

Maximum Marks: **70** Time allowed: **2 Hours**

Attempt any 7 questions. Each question carry 10 marks.
Throughout K denote a number field and $\mathcal{O}_K$ its ring of algebraic integers.

1. Let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a $\mathbb{Q}$-basis of a number field K. Prove that the $n \times n$ matrix $A = \left[\alpha_i^{(j)}\right]$ is invertible, where $\alpha_i^{(j)}$ denotes the K-conjugate of α_i for $1 \leq i, j \leq n$. (10)

2. Find an integral basis and the discriminant of the number field $K = \mathbb{Q}(\theta)$ such that θ is a root of the polynomial $f(t) = t^3 - t + 2$. (5+5)

3. Find an integral basis and the discriminant of the number field $K = \mathbb{Q}(\theta)$, where $\theta \in \mathbb{C}$ and $\theta^3 + \theta^2 - \theta + 2 = 0$. (5+5)

4. Find the group of units of $\mathcal{O}_K$, where $K = \mathbb{Q}\left(\sqrt{-44}\right)$. (10)

5. Does the factorization $10 = (3+i)(3-i) = 2 \cdot 5$ implies that $\mathbb{Z}[i]$ is not a unique factorization domain? Justify. (10)

6. Find a $\mathbb{Z}$-basis for the ideal $I = \langle 2, 1+\sqrt{-5}\rangle$ of $\mathcal{O}_K$, for the number field $K = \mathbb{Q}\left(\sqrt{-5}\right)$ and use it to claculate the norm of I. (5+5)

7. Let I be an invertible fractional ideal of $\mathcal{O}_K$ and let $I' = \{\alpha \in K \mid \alpha I \subseteq \mathcal{O}_K\}$. Prove that I' is a fractional ideal of $\mathcal{O}_K$ and is the inverse of I. (6+4)

8. Prove that every non-zero ideal of $\mathcal{O}_K$ is generated by atmost two elements. (10)

9. Let L be an n-dimensional lattice in $\mathbb{R}^n$ such that $L \subseteq \mathbb{Z}^n$. Prove that the index of L in $\mathbb{Z}^n$ is equal to the volume of the fundamental domain of L. (10)

10. Give an example of a number field with class-number 1 and an example of a number field with class-number different from 1. (5+5)

Department of Mathematics, University of Delhi
M.A./M.Sc. End Semester Examination, 2020
MATH14-402A: Abstract Harmonic Analysis

SEMESTER IV

Time: 2 hours | Max. Marks: 70

Instructions:

1. Attempt any seven questions.
2. All questions carry equal marks.

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1. Let $H = \begin{pmatrix} \frac{\lambda}{1+\lambda} & 0 \\ 0 & \frac{1}{1+\lambda} \end{pmatrix}$, $0 \leq \lambda \leq 1$. Show that $p(a) = \mathrm{tr}(Ha)$ is a positive linear functional on $M_2(\mathbb{C})$. Find the norm of p.

2. Let $A = C_0(\mathbb{R})$. Let Π be a map from A into $B(L^2(\mathbb{R}))$ given by $(\Pi(f)g)(x) = f(x)g(x)$ for $f \in A$ and $g \in L^2(\mathbb{R})$. Show that Π is a representation of A and $\langle \Pi(f^*)g, h\rangle = \langle (\Pi(f))^* g, h\rangle$, where $\langle \cdot, \cdot \rangle$ is an inner product in $L^2(\mathbb{R})$. If $f(x) = e^{-x^2}$, then show that $\|\Pi(f)\| \leq 1$.

3. Let Π be a unitary representation of $\mathbb{R}$ on a Hilbert space $\mathcal{H}$. Show that ρ given by
$$\langle \rho(f)u, v\rangle = \int_{-\infty}^{\infty} f(x)\langle \Pi(x)u, v\rangle dx \qquad \forall u, v \in \mathcal{H}$$
is a $*$-representation of $L^1(\mathbb{R})$. Under what conditions ρ is cyclic? Justify.

4. Find an orthonormal basis of $Z(L^2(SU(2)))$. Justify your answer.

5. Show that the function $g(x) = \int_{-\infty}^{\infty} e^{itx-|t|}dt$ is a positive definite function on $\mathbb{R}$. Is $f(x) = e^{x^2}$ on $\mathbb{R}$ a positive definite function? Justify your answer.

6. Show that every irreducible unitary representation of a compact group is finite dimensional. Give a 3-dimensional irreducible representation of $SU(2)$.

7. State and prove orthogonality relation for coordinate functions of representations on a compact group.

8. State and prove Bochner's theorem for positive type functions on $\mathbb{R} \times \mathbb{T}$.

9. Let $A = C[0,1]$, the space of continuous functions on $[0,1]$. If $x \in A$, $\|x\|_\infty \leq 1$, show that there exists $y \in A$ such that $y^2 = e - x$, where $e(u) = 1 \; \forall u \in [0,1]$.

10. For a compact group G, if $f \in L^1(G)$ satisfies

$$f(x)f(y) = \int_G f(txt^{-1}y)dt, \quad \forall x, y \in G,$$

then show that $f = 0$ or $f = \frac{1}{d}\chi_U$, where χ_U is the trace function of an irreducible representation U of G. Illustrate the result for $G = \mathbb{T}$, the circle group.

M.A./M.Sc. Mathematics Examinations, 2020
Part II Semester IV
MATH14-402(B): Frames and Wavelets

Time: 2 hours | Maximum Marks: 70

Instructions: • Answer any **SEVEN** questions. • Each question carries 10 marks.
• Symbols have their usual meaning.

(1) Given an example with justification of an infinite collection of vectors in $\mathbb{C}^n$ which constitutes a tight frame for $\mathbb{C}^n$. Let $\mathcal{F}$ be a finite collection of vectors in $\mathbb{C}^n$. How completeness of $\mathcal{F}$ and frame conditions of $\mathcal{F}$ are related? Justify your answer. [4+6 = 10 Marks]

(2) From a given frame $\{f_k\}_{k=1}^m$ for $\mathbb{C}^n$, how you construct a tight frame for $\mathbb{C}^n$? Explain. [10 Marks]

(3) How the Haar wavelet is useful in stable reconstruction of signals? Explain. [10 Marks]

(4) What is the relation between the frame operator and pseudo-inverse of pre-frame operator of a given frame for an infinite dimensional Hilbert space $\mathcal{H}$? Explain. [10 Marks]

(5) Given an example with justification of a sequence in an infinite dimensional Hilbert space $\mathcal{H}$ which satisfies lower frame condition but not a Bessel sequence. Show that if $\{f_k\}_{k=1}^\infty$ is a sequence in $\mathcal{H}$, and there exists $B_o > 0$ such that [10 Marks]

$$\left\| \sum c_k f_k \right\|^2 \leq B_o \|\{c_k\}\|_{\ell^2}$$

for all finite sequences $\{c_k\}$, then $\{f_k\}_{k=1}^\infty$ is a Bessel sequence.

(6) Define similar frames and unitarily equivalent frames. What is the relation between similar frames and unitarily equivalent frames? Explain. [2+8= 10 Marks]

(7) Show that if $\{f_k\}_{k=1}^\infty$ is a frame for an infinite dimensional Hilbert space $\mathcal{H}$ with frame bounds A, B; $\{g_k\}_{k=1}^\infty \subset \mathcal{H}$ and $\{f_k - g_k\}_{k=1}^\infty$ is a Bessel sequence with Bessel bound $B_g < A$, then $\{g_k\}_{k=1}^\infty$ is a frame for $\mathcal{H}$ with frame bounds $(\sqrt{A} - \sqrt{B_g})^2$ and $(\sqrt{B} + \sqrt{B_g})^2$. [10 Marks]

(8) Define dual of a frame. What can you say about existence of dual frame(s) of a given frame for an infinite dimensional Hilbert space $\mathcal{H}$? Explain. [1+9= 10 Marks]

(9) Give an example with justification of a frame for an infinite dimensional Hilbert space which is tight but not ω-independent. How exact frames and ω-independent frames in infinite dimensional Hilbert spaces are related? Justify your answer. [4+6= 10 Marks]

(10) Let $\{f_k\}_{k=1}^{\infty}$ be a Riesz basis of an infinite dimensional Hilbert space $\mathcal{H}$, and let $\mathcal{G} : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be the Gram matrix. Show that the optimal Riesz bounds are $a_o = \frac{1}{\|\mathcal{G}^{-1}\|}$ and $b_o = \|\mathcal{G}\|$. [10 Marks]

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M.A/M.Sc Mathematics Final Examinations- Sem. IV, 2020
Operators on Hardy Hilbert Spaces : 402 (C)

Maximum Marks : 70 **Time: 2 hrs**

Attempt seven questions in all. All symbols carry their usual meaning.

1. Show that $H^\infty(\mathbb{D}) \subset H^2(\mathbb{D})$. Give an example of a function in $H^2(\mathbb{D})$ which is not in $H^\infty(\mathbb{D})$. (10)

2. Compute T^* where $T : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ is given by $Tf(z) = zf(z), f \in H^2(\mathbb{D}), z \in \mathbb{D}$. Determine whether or not $\mathcal{M} := \{f \in H^2(\mathbb{D}) : f(0) = 0 = f'(0)\}$ is a reducing subspace for T. (10)

3. What is meant by an outer function in $H^2(\mathbb{D})$? Give two examples of outer functions. Further show that an outer function can not have any zeros in $\mathbb{D}$. (10)

4. Let $T : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ be given by $T(z^n) = z^{(n+3)}, n = 0, 1, \ldots$. Show that T is an isometry and find the Wold decomposition of T. (10)

5. Let S be the unilateral shift and $T = I + S^2$. Show that T on $H^2(\mathbb{T}) = \{f \in L^2(\mathbb{T}) : \langle f, e_n \rangle = 0, n = -1, -2 \ldots\}$ is a Toeplitz operator and express it in the form T_ϕ for an appropriate ϕ. (10)

6. Let $\phi \in L^\infty(\mathbb{T})$ be given by $\phi(e^{i\theta}) = 2 + 3e^{i\theta} + e^{2i\theta}$. Find $Ker\, T_{\overline{\phi}}$ where $T_{\overline{\phi}}$ represents the Toeplitz operator on $H^2(\mathbb{T})$ with symbol $\overline{\phi}$ (10)

7. Determine whether the operators T_ϕ and T_ψ acting on $H^2(\mathbb{T})$ are (i) normal (ii) self adjoint, where $\phi(e^{i\theta}) = -1 + e^{i\theta} + e^{-i\theta}$ and $\psi(e^{i\theta}) = 2i(e^{i\theta} + e^{-i\theta}) + 1$. Do T_ϕ and T_ψ commute? Justify. (10)

8. Show that $\|f\|_2 = \|\overline{f}\|_2 = \|f^*\|_2$ for every $f \in L^2$, where $f^*(e^{i\theta}) := \overline{f(e^{-i\theta})}$. Further prove that if $f \in L^\infty$ then $\overline{f}$ and $f^* \in L^\infty$ and $\|f\|_\infty = \|\overline{f}\|_\infty = \|f^*\|_\infty$. (10)

9. Show that the flip operator J on $L^2(\mathbb{T})$ is unitary and self adjoint and that
$$PJ = JW(I - P)W^*$$
where W is the bilateral shift and P is the projection of $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$. (10)

10. Let $w \in \mathbb{D}$, S be the unliateral shift on $H^2(\mathbb{T})$ and $k_w(e^{i\theta}) = \frac{1}{1-\overline{w}e^{i\theta}}$. Show that $S^*(k_{\overline{w}} \otimes k_w) = (k_{\overline{w}} \otimes k_w)S$. Hence or otherwise prove that $k_{\overline{w}} \otimes k_w$ has a Hankel matrix with respect to the standard orthonormal basis of $H^2(\mathbb{T})$. (10)

University of Delhi
M.A/M.Sc. Mathematics Examination, Part-II, Sem-IV, July 2020
MATH14 403(A), Calculus on $\mathbb{R}^n$

Time : 2 hours — Maximum Marks : 70

Attempt any **seven** questions. Each question carries 10 marks. Unless otherwise mentioned, U will denote an open subset of $\mathbb{R}^m$.

(1) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

Show that the partial derivatives $D_1 f$ and $D_2 f$ of f exist at every point of $\mathbb{R}^2$ but are not bounded in any neighborhood of $(0,0)$.

(2) Define $f, g : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x,y) = (x^2+y^2, \sin xy)$ and $g(u,v) = (uv, e^v)$. Use chain rule to compute $D(g \circ f)$ and verify it by explicitly computing $g \circ f$.

(3) Let $V = \{(x,y) \in \mathbb{R}^2 : 0 < x^2+y^2 < 1$ and $f : V \to \mathbb{R}^3$ be a C^1 function such that $Df(x,y) = \mathbf{0}$ for all $(x,y) \in V$. Can you conclude that f is a constant? Justify your answer. What if we replace V by $\tilde{V} = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 1, y \neq 0\}$? Justify.

(4) Let $f : \mathbb{R}^5 \to \mathbb{R}^2$ be a C^1 function such that for $a = (1,2,3,0,5)$, $f(a) = \mathbf{0}$ and

$$Df(\mathbf{a}) = \begin{bmatrix} 1 & 2 & 1 & 3 & 4 \\ 0 & 3 & 5 & 2 & 1 \end{bmatrix}.$$

Show that there is a neighborhood W of $(1,3,0)$ in $\mathbb{R}^3$ and a C^1 map $\varphi : W \to \mathbb{R}^2$ such that $\varphi(1,3,0) = (2,5)$ and for $(x_1,x_2,x_3) \in W$, $f(x_1, \varphi_1(x), x_2, x_3, \varphi_2(x)) = 0$. Compute $D\varphi(1,3,0)$.

(5) Show that the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x,y) = (x^2-y^2, 2xy)$ is one to one on $V = \{(x,y) \in \mathbb{R}^2 : x > 0, y > 0\}$, find $f(V)$, and if g is the inverse function of f, compute $Dg(0,3)$.

(6) Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(t) = \begin{cases} \frac{1+\cos t}{2}, & -\pi \leq t \leq \pi \\ 0, & \text{otherwise,} \end{cases}$$

$\varphi_{2m}(t) = f(t+m\pi)$, for $m \geq 1$ and $\varphi_{2m+1}(t) = f(t-m\pi)$, for $m \geq 0$. Show that (φ_m) is a sequence of C^1 functions on $\mathbb{R}$ with compact support and that $\sum_{m=1}^{\infty} \varphi_m(t) = 1, \ \forall t \in \mathbb{R}$.

(7) Let H be the parallelogram in $\mathbb{R}^2$ whose vertices are $(2,1), (4,4), (5,5), (3,2)$. Find the affine map T which sends $(0,0)$ to $(2,1)$, $(1,0)$ to $(4,4)$ and $(0,1)$ to $(3,2)$. Use T to convert the integral

$$\alpha = \int_H e^{2y_1 - y_2} dy_1 dy_2$$

into an integral over $I^2 = [0,1] \times [0,1]$ and thus compute it.

(8) Find the value of the 3-form $\omega = z^3 dx \wedge dy \wedge dz$ on the 3-surface Ψ in $\mathbb{R}^3$, where $D = \{(r, \theta, \varphi) : 0 \le r \le 2, 0 \le \theta \le \pi, 0 \le \varphi \le 2\pi\}$ and $\Psi(r, \theta, \varphi) = (r \sin\theta \cos\varphi, r \sin\theta \sin\varphi, r \cos\theta)$. What happens if z^3 is replaced by z^k for some odd natural number k?

(9) Prve that the 1-form

$$\omega = \frac{-x_2}{x_1^2 + x_2^2} dx_1 + \frac{x_1}{x_1^2 + x_2^2} dx_2$$

on $\mathbb{R}^2 \setminus \{(0,0)\}$ is closed. Prove also that ω is not exact by showing that $\int_C \omega \neq 0$ for some suitable closed curve C.

(10) In $\mathbb{R}^3$ consider the forms

$$\omega = y^2 z dx + y^2 z^2 dy - 3xy^2 dz$$
$$\eta = yz^2 dx + x^3 z^2 dy - y^2 x^2 dz.$$

Verify by direct computation, $d(d\omega) = 0$ and $d(\omega \wedge \eta) = (d\omega) \wedge \eta - \omega \wedge (d\eta)$.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examinations, July 2020
Part -II Semester- IV

MATH 14-403(C): TOPOLOGICAL DYNAMICS

Time: 2 Hours **Maximum Marks: 70**

Instructions: • All notations are standard. • Attempt any SEVEN out of following TEN questions.

Q. 1 Consider the homeomorphism $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$. Find stable and unstable sets of fixed points $0, \pm 1$. Do the graphical analysis of f and draw the phase portrait of the orbit of $x = \frac{-1}{2}$ [10 Marks]

Q. 2 Let X be a compact metric space and $f : X \to X$ be continuous. Prove that for every $x \in X$, $\omega(x, f)$ is a non-empty closed f-invariant subset of X. Also prove that $\omega(f^n(x), f) = \omega(x, f)$ for each $n \in \mathbb{N}$ and for every $x \in X$. [10 Marks]

Q. 3 Let A be a $k \times k$ matrix with entries in $\{0, 1\}$ such that no row / column is full of zeros. Then prove that the shift map σ on X_A is transitive if and only if A is an irreducible matrix. [10 Marks]

Q. 4 Consider the logistic map $F_\mu : \mathbb{R} \to \mathbb{R}$ defined by $F_\mu(x) = \mu x(1 - x)$, $1 < \mu \leq 2$. Prove that for $x \in (0, 1)$, $lim_{n\to\infty} F_\mu^n(x) = p_\mu$. [10 Marks]

Q. 5 Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map having a periodic point of prime period 3. Then prove that f has periodic point of all prime period $n \geq 1$. [10 Marks]

Q. 6 Prove that Dyadic Solenoid is a compact connected metric space admitting an expansive homeomorphism. [10 Marks]

Q. 7 Let (X, d) be a metric space and $f : X \to X$ be a homeomorphism. Prove that expansiveness of f does not depend upon the choice of the metric d if X is compact. Give an example to justify that it depends upon the choice of metric d if X is not compact. [10 Marks]

Q. 8 Let(X, d) be a metric space and $f : X \to X$ be continuous. If f has pesudo orbit tracing property (POTP) then prove that f^k has POTP for every $k \in \mathbb{N}$. Also prove that POTP is preserved under any finite product. [10 Marks]

Q. 9 Let (X, d) be a compact metric space. Then prove that the identity map $Id : X \to X$ has POTP if and only if X is totally disconnected. [10 Marks]

Q. 10 Let (X, d) and (Y, ρ) be compact metric spaces and $f : X \to X$ and $g : Y \to Y$ be homeomorphisms such that (X, f) is topologically conjugate to (Y, g). If f is topologically Anosov then prove that g is also topologically Anosov. Is composition of two topologically Anosov homeomorphisms a topologically Anosov homeomorphism? Justify your claim. [10 Marks]

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations, July 2020
Part -II Semester- IV

MATH 14-403(C): TOPOLOGICAL DYNAMICS

Time: 2 Hours **Maximum Marks: 70**

Instructions: • All notations are standard. • Attempt any SEVEN out of following TEN questions.

Q. 1 Let X be a compact metric space and $f : X \to X$ be continuous. Prove that for any $x \in X$, $\omega(x, f) = \cap_{m \geq 0}(\overline{\cup_{n \geq m}\{f^n(x)\}})$. Also prove that $\omega(f^i(x), f^k) = f^i(\omega(x, f^k))$, for any $k \in \mathbb{N}$ and for all $i \geq 0$. [10 Marks]

Q. 2 Let A be a $k \times k$ matrix with entries in $\{0, 1\}$. Prove that A is an irreducible matrix if and only if $A \vee (A * A) \ldots\ldots \vee \underbrace{(A * \ldots * A)}_{n-times} = J$, for some $n \in \mathbb{N}$. Also prove that if the digraph associated to A is strongly connected, then A is irreducible. [10 Marks]

Q. 3 Do the graphical analysis of the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = (x + x^3)$ and draw the phase portrait of the orbit of $x = \frac{-1}{2}$. Also find attracting and repelling fixed points of the function $f(\theta) = \theta + \epsilon\, sin 2\theta$, $0 < \epsilon < \frac{1}{2}$ and $\theta \in [0, 2\pi]$. [10 Marks]

Q. 4 Let (X, d) be a compact metric space and $f : X \to X$ be an expansive homeomorphism having POTP. Prove that f has canonical co-ordinates and for $\epsilon > 0$, a number less than an expansive constant for f, we have $W^s(x, d) = \cup_{n \geq 0}(f^{-n}(W^s_\epsilon(f^n(x), d)))$. [10 Marks]

Q. 5 Prove that the tent map $T : [0, 1] \to [0, 1]$ defined by

$$T(x) = \begin{cases} 2x & \text{if } x \in [0, \frac{1}{2}] \\ 2(1 - x) & \text{if } x \in [\frac{1}{2}, 1] \end{cases}$$

has dense set of periodic points. Use this to conclude that the logistic map $f(x) = 4x(1 - x)$, $x \in [0, 1]$ also has dense set of periodic points. [10 Marks]

Q. 6 Prove that the set of points having converging semiorbits of an expansive self-homeomorphism of a compact metric space is a countable set. Conclude that any closed arc does not admit any expansive homeomorphism. [10 Marks]

Q. 7 If A is a subset of a metric space X such that $X - A$ is finite, then prove that any homeomorphism $f : X \to X$ which is expansive on A, is expansive on X. Justify that if $X - A$ is not finite then the result need not be true. [10 Marks]

Q. 8 If X is a compact metric space then prove that the shift map $\sigma : X^{\mathbb{Z}} \to X^{\mathbb{Z}}$ has pseudo orbit tracing property (POTP). [10 Marks]

Q. 9 Prove that an irrational rotation on the unit circle is a minimal homeomorphism but does not have POTP. What can you say about minimality of a rational rotation? Justify your claim. [10 Marks]

Q. 10 Prove that a topologically Anosov homeomorphism of a compact metric space X is topologically stable in the class of self-homeomorphisms of X. [10 Marks]

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations, July 2020
Part -II Semester- IV

MATH 14-403(C): TOPOLOGICAL DYNAMICS

Time: 2 Hours **Maximum Marks: 70**

Instructions: • All notations are standard. • Attempt any SEVEN out of following TEN questions.

Q. 1 Find for which values of distinct $a, b \in \mathbb{R}$, $f, g : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = ax$ and $g(x) = bx$ are topologically conjugate. Is minimality preserved under topological conjugacy? Justify your claim. [10 Marks]

Q. 2 Let A be a $k \times k$ matrix with entries in $\{0, 1\}$. Prove that X_A is a closed subset of Σ_k. Also prove that if A is an irreducible matrix then its associated digraph is strongly connected. [10 Marks]

Q. 3 Let $f : \mathbb{R} \to \mathbb{R}$ be a C^1 map and p be a hyperbolic fixed point of f. Prove that there exists an open interval U about p with $x \in U$, $x \neq p$ such that $|f^{'}(p)| > 1$ implies there exists a $k > 0$ satisfying $f^k(x) \notin U$ and $|f^{'}(p)| < 1$ implies $lim_{n\to\infty} f^n(x) = p$. [10 Marks]

Q. 4 Let (X, d) be a compact metric space. Prove that a homeomorphism $f : X \to X$ is expansive if and only if there exists an $e > 0$ such that for all $\gamma > 0$, there exists an $N \in \mathbb{N}$ such that for all $x \in X$ and for all $n \geq N$, we have
(i) $f^n(W^s_e(x, d)) \subseteq W^s_\gamma(f^n(x), d)$
(ii) $f^{-n}(W^u_e(x, d)) \subseteq W^u_\gamma(f^{-n}(x), d)$. [10 Marks]

Q. 5 Show that the function $f : [1, 5] \to [1, 5]$ defined by

$$f(x) = \begin{cases} 2x + 1 & \text{if } x \in [1, 2] \\ 7 - x & \text{if } x \in [2, 3] \\ -2x + 10 & \text{if } x \in [3, 4] \\ 6 - x & \text{if } x \in [4, 5] \end{cases}$$

has a periodic point of prime period 5 but does not have any periodic point of prime period 3. [10 Marks]

Q. 6 Prove that the closed unit disc in the plane does not admit any expansive homeomoephism. [10 Marks]

Q. 7 For an expansive self-homeomorphism f of a compact metric space X, prove that the least upper bound of the set of expansive constants for f is not an expansive constant for f. Justify that if X is not compact then the set of expansive constants for f need not be a bounded set. [10 Marks]

Q. 8 Consider $\mathbb{R}$ with the usual metric. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is a contraction map then f has pseudo orbit tracing property (POTP). [10 Marks]

Q. 9 If (X, d) is a compact metric space and $f : X \to X$ is a homeomorphism such that f^k has POTP for some $k \in \mathbb{N}$ then prove that f has POTP. [10 Marks]

Q. 10 Prove that a self-homeomorphism f of a compact metric space X is topologically Anosov if and only if f^{-1} is so. Give example (with justification) of a Topologically Anosov homeomorphism which is topologically stable in the class of self-homemorphisms of a compact metric space. [10 Marks]

Your Roll Number: .....

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examinations, July 2020
Part II Semester IV

MATH14-404(B): ADVANCED FLUID DYNAMICS

Time: 2 hours Maximum Marks: 70

Instructions: • Answer ANY **7** questions. • Each question carries 10 marks.• All the symbols have their usual meaning.

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(1) Compute the ratio of inertial force to viscous per unit mass and identify the non-dimensional number. Compare the transport and diffusion phenomena in the motion of conducting fluid under magnetic field. [10 Marks]

(2) Calculate the isothermal and isentropic compressibilities in a non-deal gas given by $p(v-b) = RT$. Show that non-idealness of the gas decreases the compressibilities. Derive the isentropic compressibility in term of the Mach number. Using the Mach number, classify the fluid flow. [10 Marks]

(3) For a non-deal gas with equation of state $p(v-b) = RT$, the internal energy u per unit mass is $u = 3p(v-b)$, where p and v are the pressure and volume. Assuming that $\gamma(= c_p/c_v)$ is constant, show that $p^{3\gamma}(v-b)^4 = F(T)$, where F is an arbitrary function. [10 Marks]

(4) Show that $p(v-b)^\gamma$ =constant for an isentropic flow of a non-ideal gas given by $p(v-b) = RT$. Show that entropy of an isolated system performing internal irreversible processes will increase. [10 Marks]

(5) Show that a shock wave is compressive in nature. Deduce the flow variables behind a very strong shock wave in term of shock speed. [10 Marks]

(6) Under suitable assumptions, derive the wave equation in term of velocity potential due to a small disturbance. Find the speed of the small disturbance and explain the wave motion physically. [10 Marks]

(7) What are the Maxwell's electromagnetic field equations and constitutive relations for a conducting fluid in motion. A charge can not exists in a conductor at rest under pre-Maxwell's equations. Justify? [10 Marks]

(8) Describe the interaction between the bulk motion of conducting fluid and electromagnetic field. Discuss the basic equations of motion of an inviscid conducting fluid under an electric and magnetic field. [10 Marks]

(9) Estimate the non-dimensional boundary layer thickness at leading distance x. Compute the skin friction on a flat plate of length l. [10 Marks]

(10) Perform scale analysis on the basic non-dimensional equations of motion for two dimensional unsteady laminar flow of lightly viscous incompressible fluid over cylindrical body of slender section and derive the Prandlt boundary layer equations. Explain the interaction of boundary layer flow with outer flow. [10 Marks]

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Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations, July 2020
Part II Semester IV
MATH14-404 (E): Dynamical Systems

Time: 2 hours | Maximum Marks: 70

Instructions: • Attempt **any seven** questions from given 10 questions. • Each question carries 10 marks.

(1) Classify the equilibrium points of the Lorenz equation $\dot{x} = f(x)$ with

$$f(x) = \begin{bmatrix} x_2 - x_1 \\ \mu x_1 - x_2 - x_1 x_3 \\ x_1 x_2 - x_3 \end{bmatrix}$$

for $\mu > 0$. At what value of the parameter μ do two new equilibrium points "bifurcate" from the equilibrium point at the origin? Also, explain the following terms: stable focus, stable node and topological saddle equilibrium points.

(2) Solve the initial value problem $\dot{x} = Ax$, where

$$A = \begin{bmatrix} -1 & 1 & -2 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \end{bmatrix}.$$

Also, explain the term "gradient system" with an example and sketch the surface for the associated example.

(3) Find the solution of forced harmonic oscillator problem

$$\ddot{x} + x = f(t).$$

Also, find the fixed points for the map $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined in polar coordinates $r \geq 0, 0 \leq \theta < 2\pi$ given by

$$f(r, \theta) = (r^2, \theta - sin\theta)$$

and hence classify the nature of these fixed points.

(4) Find the *Poincaré map* of the system

$$\dot{x} = -y + x(1 - x^2 - y^2), \quad \dot{y} = x + y(1 - x^2 - y^2).$$

and discuss the stability of equilibrium points of the system

$$\dot{x} = x(3 - x - 2y), \quad \dot{y} = y(2 - x - y).$$

(5) Determine the nature of the equilibrium points (as sinks, sources or saddles) of the nonlinear system $\dot{x} = f(x)$ with

$$f(x) = \begin{bmatrix} 2x_1 - 2x_1 x_2 \\ 2x_2 - x_1^2 + x_2^2 \end{bmatrix}.$$

Also, determine the center manifold near the origin of the system

$$\dot{x_1} = x_1 y - x_1 x_2^2, \quad \dot{x_2} = x_2 y - x_2 x_1^2,$$
$$\dot{y} = -y + x_1^2 + x_2^2.$$

(6) Determine the stability of the system by considering suitable Lyapunov function

$$\dot{x} = (\epsilon x + 2y)(z+1),$$
$$\dot{y} = (-x + \epsilon y)(z+1),$$
$$\dot{z} = -z^3,$$

where ϵ is a parameter.

(7) Solve the linear system $\dot{x} = Ax$, where

$$A = \begin{pmatrix} a & 0 \\ 0 & -1 \end{pmatrix}.$$

Graph the phase portrait as a varies from $-\infty$ to $+\infty$, showing the qualitatively different cases.

(8) Prove that the total energy $H(x, y)$ of the Hamiltonian system remains constant along trajectories of the same system. Also, determine the flow $\phi_t : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ for the nonlinear system $\dot{x} = f(x)$ with

$$f(x) = \begin{bmatrix} -x_1 \\ 2x_2 + x_1^2 \end{bmatrix}$$

and show that the set $S = \{x \in \mathbb{R}^2 | x_2 = -x_1^2/4\}$ is invariant with respect to the flow ϕ_t.

(9) Sketch the flow for the linear system $\dot{x} = Ax$ with

$$A = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}.$$

and find the stable, unstable and center subspaces E^s, E^u and E^c of the system $\dot{x} = Ax$ with the matrix

$$A = \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix}.$$

Also, write the following system in polar coordinates and determine the nature of origin

$$\dot{x} = -y + x^3 + xy^2, \quad \dot{y} = x + y^3 + x^2 y.$$

(10) Show that the continuous map $H : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by

$$H(x) = \begin{bmatrix} x_1 \\ x_2 + \frac{x_1^2}{3} \end{bmatrix}$$

has a continuous inverse $H^{-1} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ and that nonlinear system $\dot{x} = f(x)$ with

$$f(x) = \begin{bmatrix} -x_1 \\ x_2 + x_1^2 \end{bmatrix}$$

is transformed into the linear system $\dot{x} = Ax$ with $A = Df(0)$ under this map, i.e., if $y = H(x)$, show that $\dot{y} = Ay$. Also, prove that the origin is a spiral, although the linearization predicts a center for the system

$$\dot{x} = -y - x^3, \quad \dot{y} = x.$$

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Final Examinations, 2020, Part II Semester IV
MATH14 404(F): OPTIMIZATION TECHNIQUES AND CONTROL THEORY

Time: 2 hours Maximum Marks: 70

Instructions: • Attempt any seven questions. • All questions carry equal marks. • All symbols have usual meaning.

(1) If C is a convex set in $\mathbb{R}^2$ and f is function defined on $\mathbb{R}$ then evaluate the convex function $f(x) = \inf\{\alpha \in \mathbb{R} : (x, \alpha) \in C\}$ when
(i) $C = \{(x, \alpha) \in \mathbb{R}^2 : 3x^2 + 2\alpha^2 \leq 2\}$;
(ii) $C = \{(x, \alpha) \in \mathbb{R}^2 : x^2 + \alpha^2 < 1\} \cap \{(x, \alpha) \in \mathbb{R}^2 : (x-1)^2 + \alpha^2 \leq 1\}$.
Are the two functions defined above closed?

(2) Let f be a convex function defined on $\mathbb{R}^n$ such that $\text{cl}f \neq f$. Do they have the same conjugate? Justify.
Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper convex function. If x is a point in the effective domain of f and ξ is a vector in the subdifferential of f at x then what is the value of the conjugate of f at ξ? Justify.

(3) Is the subdifferential of a proper convex function at a point in its effective domain a closed convex set? Justify. What if the point is in the interior of its effective domain?
Give an example of a proper convex function $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ such that three different points in the effective domain of f are such that at one point the subdifferntial is empty, at another point the subdifferential is a singleton and at the third point the subdifferential is an infinite set.

(4) Check if the problem (P_ϕ) is stable where $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is defined as

$$\phi(x, w) = \begin{cases} x^2 - w, & \text{if } x \geq w, \\ +\infty, & \text{if } x < w. \end{cases}$$

(5) Is the dual of dual always the primal? If no then under what condition is this true? Find the dual of the problem (P_ϕ) considered in Q.4 Also check the stability and normality of the dual problem.

(6) Use Newton's method, to find the next two iterates for minimizing $f(x, y) = x^3 + y^2 - 2xy - x$ starting from the point $(0, 0)$.

(7) How is the gradient projection method different from gradient descent method? Explain both theoretically and geometrically.

(8) Use dynamic programming approach to solve the linear programming problem

$$\text{Max } z = 2x_1 + 4x_2$$
$$\text{subject to } x_1 \leq 4, 2x_2 \leq 5, 3x_1 + 2x_2 \leq 15,$$
$$x_1 \geq 0, x_2 \geq 0.$$

(9) Solve the following optimal control problem

$$\text{Max } \int_0^1 (x - \dot{x}^2)dt - [x(1)]^2$$
$$\text{subject to } x(0) = 2.$$

(10) What is the difference between the classical approach and modern approach in control theory? What are the necessary and sufficient optimality conditions in the case of modern approach?