Your Roll Number: ...............

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examination, May 2019
Part II Semester IV

# MATH14-401(A): Algebraic Number Theory

Time: 3 hours | Maximum Marks: 70

**Instructions:** • Write your roll number on the top immediately on receipt of question paper • **Section A** is compulsory • Attempt two questions each from **Section B** and **C.** • Throughout $\mathcal{O}_K$ denotes the ring of algebraic integers of a number field $K$ of degree $n$. Each question carries **14** marks.

## Section A

(1) (a) Give an example of a number field $K = \mathbb{Q}(\theta)$ of degree 6 for which $\mathcal{O}_K = \mathbb{Z}[\theta]$. [2]

(b) Let $K = \mathbb{Q}(\zeta)$, where $\zeta$ is a primitive $p$-th root of unity. Prove that $1 + \zeta + \zeta^2 + \cdots + \zeta^{p-2}$ is a unit in $\mathcal{O}_K$. [2]

(c) Is $(2+i)(2-i) = 5 = (1+2i)(1-2i)$ an example of non-unique factorization in $\mathbb{Z}[i]$? Justify. [2]

(d) Let $A, B$ be ideals of $\mathcal{O}_K$. Prove that $(A, B)[A, B] = AB$, where $(A, B)$ and $[A, B]$ denote respectively the gcd and lcm of $A, B$. [3]

(e) Let $\mathcal{P}$ be a prime ideal of $\mathcal{O}_K$ containing a prime number $p$. Prove that $\mathcal{P}$ contains an integer $a$ if and only if $p$ divides $a$. [2]

(f) Find the class number of the number field $\mathbb{Q}(\sqrt{-5})$. [3]

## Section B

(Attempt any two questions)

(2) (a) Define integral basis. Prove that every number field $K$ possesses an integral basis and the additive group of $\mathcal{O}_K$ is a free abelian group of rank equal to the degree of $K$. [1+8]

(b) Let $K = \mathbb{Q}(\sqrt{d})$, where $d$ is a square free integer. Prove that an element $\alpha \in K$ is an algebraic integer if and only if its norm and trace are both integers. Does the result hold for arbitrary number fields? Justify. [3+2]

(3) (a) Find the integral basis and discriminant of the number field $K = \mathbb{Q}(\sqrt{d})$, where $d$ is a square free integer. [8]

(b) Let $K = \mathbb{Q}(\zeta)$, where $\zeta$ is primitive $p$-th root of unity. Prove that [4]

$$\mathrm{disc}(1-\zeta) = \mathrm{disc}(\zeta),$$

where $\mathrm{disc}(\zeta) = \Delta_{K/\mathbb{Q}}(1, \zeta, \zeta^2, \ldots, \zeta^{p-2})$.

(c) Let $K = \mathbb{Q}(\sqrt[3]{m})$, where $m$ is a cube free positive integer. For [2] any $a \in \mathbb{Z}$, find $N_K(\sqrt[3]{m}+a)$.

(4) (a) For which of the number fields, $\mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-2}), \mathbb{Q}(\sqrt{-3}), \mathbb{Q}(\sqrt{-4})$, $\mathbb{Q}(\sqrt{-5})$ the ring of algebraic integers is a PID? Justify. [9]

(b) Let $\alpha \in \mathcal{O}_K$, where $K = \mathbb{Q}(i)$. Show that if $N_K(\alpha)$ is a prime number, then $\alpha$ is irreducible. Show that the same holds if $N_K(\alpha) = p^2$, where $p$ is a prime number of the form $4k+3$. [2+3]

## Section C

(Attempt any two questions)

(5) (a) Prove that every non-zero prime ideal of $\mathcal{O}_K$ is maximal. [5]

(b) Show that for a positive integer $m$, there exist only finitely many ideals of $\mathcal{O}_K$ with norm at most $m$. [4]

(c) Let $\mathcal{P}$ be a prime ideal of $\mathcal{O}_K$. Prove that for all $x \in \mathcal{O}_K$ [5]

$$x^{N(\mathcal{P})} \equiv x \pmod{\mathcal{P}}$$

and the norm $N(\mathcal{P})$ is the smallest positive integer for which the above congruence holds.

(6) (a) Prove that a discrete subgroup of $\mathbb{R}^n$ is a lattice. [5]

(b) State Minkowski's Theorem and use it prove two Squares Theorem. [1+4]

(c) Find all integer solutions of the equation $x^2+1=2y^3$. [4]

(7) (a) Let $L$ be an $n$-dimensional lattice in $\mathbb{R}^n$ and let $\{v_1,\ldots,v_n\}$ and $\{w_1,\ldots,w_n\}$ be any two basis of $L$. Prove that the absolute value of the determinant formed by taking $v_i$ as the rows is equal to the one formed from the $w_i$. [5]

(b) Let $K$ be a number field of degree $n = s+2t$, where $s$ and $2t$ denote respectivley the number of real and complex monomorphisms of $K$, such that for every prime number $p$ with [4]

$$p \le \left(\frac{4}{\pi}\right)^t \frac{n!}{n^n}\sqrt{|d_k|}$$

every prime ideal of $\mathcal{O}_K$ dividing $\langle p \rangle$ is principal. Prove that $\mathcal{O}_K$ is a PID.

(c) State Dedekind's Theorem and find the class number of $\mathbb{Q}(\sqrt{-163})$. [1+4]

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Your Roll Number: ..............

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations, May 2019
Part II Semester IV

**MATH14-401(B): Theory of Non-Commutative Rings**

Time: 3 hours Maximum Marks: 70

**Instructions:** • Write your roll number on the space provided at the top of this page immediately on receipt of this question paper • **Sec A** is compulsory • Attempt **any two** questions each from **Sec A** and **Sec B** • All questions carry equal marks • Through out this question paper the word "ring" means a ring with an identity element 1 which is not necessarily commutative.

## Section A

(1) (a) Show that there exists a non-commutative ring without identity of order $p^2$. Also show that any ring (with identity) of same order is commutative. [2+3]

(b) Prove that any noetherian module $M$ over ring $R$ is Hopfian. Show that the left regular module ${}_RR$ is Hopfian if and only if $R$ is Dedekind-finite. Justify that any left noetherian ring $R$ is Dedekind-finite. [3+3+1]

(c) Show that a semisimple ring is both left and right symmetric. [2]

## Section B

(2) (a) Describe the left, right and two sided ideal structures of a triangular ring. [3+3+3]

(b) Show that the differential polynomial ring $k[x; \delta]$ satisfies Lebnitz rule. Also show that an inner derivation is a derivation but the converse may not hold in general. [2+3]

(3) (a) Let $B_1, \cdots, B_n$ be left ideals in a ring $R$. Show that $R = B_1 \oplus \cdots \oplus B_n$ if and only if there exists idempotents $e_1, \cdots, e_n$ with sum 1 such that $e_i e_j = 0$ whenever $i \neq j$ and $B_i = R_{e_i}$ for all $i$. Prove the above when $B_i$'s are ideals and idempotents are central. Show that in this case when $B_i$'s are ideals, if $R = B_1 \oplus \cdots \oplus B_n$, then each $B_i$ is a ring with identity $e_i$ and $R \cong B_1 \times \cdots \times B_n$. [3+3+1]

(b) In the Hilbert twist ring $R = K[x; \sigma]$ where $K$ is a division ring and $\sigma$ is not an endomorphism, prove that it is left noetherian but not right noetherian. [7]

(4) (a) Let $R$ be a simple ring, then show that the matrix ring $M_n(R)$ is also simple. [5]

(b) For any ring $R$, if all cyclic left $R$-modules are projective, then show that $R$ is left semisimple. [4]

(c) Show that any left semisimple ring $R$ is both left noetherian and left artinian. [5]

## Section C

(5) (a) For semisimple rings, state Wedderburn-Artin structure theorem carefully. [2]

(b) Let $R = M_n(D)$ where $D$ is a division ring. Show that $R$ is simple, left semisimple, left artinian, left noetherian. [6]

(c) Prove that $\mathrm{End}(nM) \cong M_n(E)$, the ring of $n \times n$ matrices over the ring of endomorphisms of an $R$-module $M$. [6]

(6) (a) Prove that the ring $R$ is semisimple if and only if $R$ is semiprimitive and satisfies D.C.C on principal ideals. [8]

(b) State and prove Nakayama's lemma. [6]

(7) (a) Show that in a left artinian ring, any nil left ideal is nilpotent. [5]

(b) Let $R$ be a semiprimary ring. Use Hopkins-Levitzki result to show that A.C.C. implies D.C.C. [9]

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Your Roll Number: ................

Department of Mathematics, University of Delhi, Delhi
M.Sc. (Mathematics) Examinations, May 2019
Part II (Semester IV)
**MATH14 - 401(C): Simplicial Homology Theory**

Time: 3 hours | Maximum Marks: 70

**Instructions:** • Write your roll number on the space provided at the top of this page immediately on receipt of this question paper • Notations used are standard • **Section A is compulsory** • Attempt two questions each from **Section B** and **C**

## Section A

(1) (a) Let $\sigma^n =< v_0, v_1, v_2, \dots, v_n >$, $n \in \mathbb{N}$, be an $n-$simplex. Find a formula giving number of faces of $\sigma^n$. [2 Marks]

(b) Give an example of a continuous map $f : |K| \to |L|$ which has no simplicial approximation $g : K \to L$. [3 Marks]

(c) Find the Euler charactersitic $\chi(X)$ of a compact, convex subspace $X$ of $\mathbb{R}^3$. [3 Marks]

(d) Show that each chain derivation map on a finite simplicial complex $K$ is a chain map. [3 Marks]

(e) Give an example of a continuous map $f : \mathbb{D}^n - \{\hat{0}\} \to \mathbb{D}^n - \{\hat{0}\}$, $n \geq 1$, which fixes every point on $\mathbb{S}^{n-1}$. Justify your answers. [3 Marks]

## Section B

(Do any two questions)

(2) Let $K$ and $L$ be two finite simplicial complexes.

(a) If $f : |K| \to |L|$ is a continuous map, then show that $f$ has a simplicial approximation if and only if $K$ is star related to $L$. [7 Marks]

(b) Let $|K|$ be the geometric carrier $K$ such that the space $|K|$ is path connected. Prove that the 0-dimensional homology group [7 Marks]

$$H_0(K; \mathbb{Z}) \cong (\mathbb{Z}, +).$$

(3) Let $\mathbb{S}^n$ denote the $n-$dimensional sphere in $\mathbb{R}^{n+1}$, $n \geq 1$.

(a) Compute all the homology groups $H_p(K; \mathbb{Z})$, $p \geq 0$, where $K$ is a triangulation of the $n-$sphere $\mathbb{S}^n$. [9 Marks]

(b) Define the degree of a continuous map $f : \mathbb{S}^n \to \mathbb{S}^n$. Prove that if $f$ is a homeomorphism, then degree of $f$ is either $+1$ or $-1$. [5 Marks]

(4) (a) State and prove Euler-Poincaré theorem. Hence show that if $S$ is any simple polyhedron with $V$ vertices, $E$ edges, and $F$ faces, then [9 Marks]

$$V - E + F = 2.$$

(b) Let $K$ denote a 2-pseudomanifold. Derive a formula for finding minimal triangulation of $K$. Hence, find a minimal triangulation of the 2-sphere $\mathbb{S}^2$. [5 Marks]

## Section C

(Do any two questions)

(5) (a) Let $K$ and $L$ be two finite simplicial complexes, and $f : |K| \to |L|$ be a continuous map. State and prove the functorial properties of the induced homomorphisms $f_p^* : H_p(K) \to H_p(L)$ for each $p \geq 0$. [9 Marks]

(b) Show that if $m \neq n$, then [5 Marks]

(i) $\mathbb{R}^m$ cannot be homeomorphic to $\mathbb{R}^n$.

(ii) $\mathbb{S}^m$ cannot be homeomorphic to $\mathbb{S}^n$.

(6) Let $K$ and $L$ be two finite simplicial complexes, and $f, g : |K| \to |L|$ be continuous maps. Prove that if $f$ and $g$ are homotopic, then for each $p \geq 0$, the induced homomorphisms [14 Marks]

$$f_p^*, g_p^* : H_p(K; \mathbb{Z}) \to H_p(L; \mathbb{Z})$$

are equal.

(7) (a) Prove that every simplicial map $\phi : K \to L$ induces a homomorphism $\phi_p^* : H_p(K; \mathbb{Z}) \to H_p(L; \mathbb{Z})$ for each $p \geq 0$. [5 Marks]

(b) State and prove Lefschetz fixed point theorem. [9 Marks]

Your Roll Number: ................

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations, May-June 2019
Part II Semester IV
**MATH14 401 (D): Advanced Group Theory:**

Time: 3 hours | Maximum Marks: 70

**Instructions:** • Write your roll number on the space provided at the top of this page immediately on receipt of this question paper • **Section-A** is compulsory • Answer **any two** questions from **Section-B** and **Section-C** • Each question carries 14 marks.

## Section-A

(1) Answer any *four* questions:

(a) Let $H \leq Z(G)$ and $G/H$ be nilpotent. Show that $G$ is nilpotent. [$3\frac{1}{2}$ Marks]

(b) Let $N$ be a nontrivial proper normal and $H$ be a Hall $p$-complement of a finite group $G$. Show that $HN/N$ is a Hall $p$-complement of the quotient group $G/N$. [$3\frac{1}{2}$ Marks]

(c) Let $N \triangleleft G$. Show that $N \cap F(G) = F(N)$, where $F(G)$ denotes the Fitting subgroup of $G$. [$3\frac{1}{2}$ Marks]

(d) Let $H$ char $G$ and $H \leq K \leq G$. Show that $K/H$ char $G/H$ implies that $K$ char $G$. [$3\frac{1}{2}$ Marks]

(e) Define presentation of a group $G$. Give a presentation of a dihedral group. [$3\frac{1}{2}$ Marks]

## Section-B

(2) (a) Let $A \triangleleft A^*$ and $B \triangleleft B^*$ be four subgroups of a group $G$. Show that $B(B^* \cap A) \triangleleft B(B^* \cap A^*)$, $A(A^* \cap B) \triangleleft A(B^* \cap A^*)$, and there is an isomorphism $B(B^* \cap A^*)/B(B^* \cap A) \cong A(B^* \cap A^*)/A(A^* \cap B)$. [7 Marks]

(b) Let $G$ be a finite group. Show that $G$ is nilpotent if and only if every maximal subgroup of $G$ is normal. [7 Marks]

(3) (a) Let $G$ be a group such that the higher center $\zeta^2(G) = G$. For $a \in G$, show that the map $\phi : G \to G$, defined by $\phi(x) = [a, x]$, is a homomorphism. Deduce that the centralizer $C_G(a)$ of $a$ is normal in $G$. [4+2 Marks]

(b) Prove that a characteristically simple finite group is either simple or direct product of isomorphic simple groups. [8 Marks]

(4) (a) Let $H$ be a solvable subgroup of a finite group $G$ and $P$ be a unique sylow subgroup of $G$. Show that $HP$ is a solvable subgroup of $G$. [4 Marks]

(b) Let a group $G$ has a composition series and $H \triangleleft G$. Show that $H$ has a composition series. [4 Marks]

(c) Let $G$ be a finite solvable group of order $mn$, where $(m, n) = 1$. Assume that any group $G$ satisfying this condition has a subgroup of order $m$. Show that any subgroup of $G$ of order $k$, where $k|m$, is contained in a subgroup of order $m$. [6 Marks]

## Section-C

(5) (a) Prove that every finite group has a unique maximal normal nilpotent subgroup. [9 Marks]

(b) Define semidirect product. Is the quaternion group $Q_8$ semidirect product of any of its two subgroups ? Justify. [1+4 Marks]

(6) (a) Let $\phi$ and $\psi$ be normal endomorphism on alternating group $A_{31}$. If $\phi + \psi$ is normal endomorphism on $A_{31}$ then show that it is nilpotent. [8 Marks]

(b) Let $H \triangleleft G$ be such that $H$ and $G/H$ have descending chain conditions (DCC). Show that $G$ has DCC. [6 Marks]

(7) (a) Define reduced word and binary operation juxtaposition on the collection of reduced words for a given set $X$. Show that the collection of reduced words under the operation juxtaposition is a free group with basis $X$. [2+2+5 Marks]

(b) Define Frattini subgroup $\Phi(G)$ of a group $G$. Show that [1+2+2 Marks]
(i) $\Phi(G)$ char $G$.
(ii) if $g$ is nongenerator of $G$, then $g \in \Phi(G)$.

Roll Number: ............................

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics End-semester Examinations, May 2019
Part-II Semester-IV

# MATH14-402(A) : ABSTRACT HARMONIC ANALYSIS

Time: 3 hours Maximum Marks: 70

**Instructions:**

- Write your roll number on the space provided at the top of this page immediately after receiving this question paper.
- Section - A is compulsory.
- Attempt any two questions each from Section - B and Section - C.
- All questions carry equal marks.

## Section - A

Question - 1 Marks distribution : 4+3+3+4

(a) Let $\mathcal{A} = C_0(\mathbb{R})$, and let $p$ be defined on $\mathcal{A}$ by $p(f) = \int_{-\infty}^{\infty} f(t)e^{-t^2}\,dt$. Show that $p$ is a bounded positive linear functional.

(b) Give a 2-dimensional unitary irreducible representaion of $SU(2) \times \mathbb{T}$.

(c) Describe the Heisenberg group as a semidirect product of two groups.

(d) Let $G = \{(b, a) \in \mathbb{R}^2 \ : \ a > 0\}$ with multiplication $(b, a)(b', a') := (b + ab', aa')$. If $H = L^2[0, \infty)$, show that $G \ni u \mapsto V_u \in B(H)$ given by $V_u\varphi(\lambda) = e^{i\lambda b}\varphi(\lambda a)a^{\frac{1}{2}}$, where $u = (b, a)$, $\varphi \in L^2[0, \infty)$ and $\lambda \in [0, \infty)$, is a unitary representation of $G$.

## Section - B (Attempt any two questions)

Question - 2 Marks distribution : 4+10

(a) Let $\mathcal{A}$ be a Banach $*$-algebra with unit $u$. If $x \in \mathcal{A}$ with $\|x\| \leq 1$, then show that there is $y \in \mathcal{A}$ such that $y^2 = u - x$.

(b) Let $\mathcal{A}$ be a Banach $*$-algebra and let $p$ be a positive linear functional on $\mathcal{A}$ satisfying $p(x^*) = \overline{p(x)}$ and $|p(x)|^2 \leq ap(x^*x)$ for every $x \in \mathcal{A}$ and some constant $a$. Show that there is a cyclic $*$-representation $T$ of $\mathcal{A}$ on a Hilbert space $H$ with cyclic vector $\xi$ such that $p(x) = \langle T_x\xi, \xi\rangle$, for every $x \in \mathcal{A}$.

Question - 3 Marks distribution : 6+8

(a) State and prove Schur's Lemma for irreducible representations of a Banach $*$-algebra.

(b) Let $G$ be a locally compact Hausdorff topological group with a left Haar measure, and let $\rho$ be a cyclic $*$-representation of $L^1(G)$ on a Hilbert space $H$, such that, for every $0 \neq v \in H$, there is $f \in L^1(G)$ satisfying $\rho(f)v \neq 0$. Show that there is a unitary representation $\pi$ of $G$ such that

$$\langle \rho(g)u, v\rangle = \int g(x)\langle \pi(x)u, v\rangle \, dx$$

for all $u, v \in H$, and $g \in L^1(G)$.

Question - 4 Marks distribution : 3+6+5

(a) Let $G$ be an infinite compact group, and let $\pi$ be the left regular representation of $G$ on $L^2(G)$. Show that $\pi$ is a representation of $G$, which is unitary. Is it irreducible? Justify your answer.

(b) State and prove the Gelfand-Raikov theorem.

(c) Show that a compact group has sufficiently many finite dimensional irreducible representations to separate points.

**Section - C** (Attempt any two questions)

Question - 5 Marks distribution : 5+7+2

(a) Let $G$ be a compact group and let $\sigma_j \in \hat{G}$, $f_j \in I_{\sigma_j}(G)$ for $j = 1, 2$. Show that $\int_G f_1(x)\overline{f_2(x)}\, dx = 0$.

(b) State and prove the Peter-Weyl theorem.

(c) Is $Q(x) := e^{x^2}$ a positive definite function on $\mathbb{R}$? Justify your answer.

Question - 6 Marks distribution : 3+6+5

(a) Show that for a non-negative integrable function $g$ on $\mathbb{R}$, the function $f(x) := \int_{-\infty}^{\infty} e^{itx} g(t)\, dt$ is positive definite on $\mathbb{R}$.

(b) Describe irreducible unitary representations of $SU(2)$. Show the irreducibility.

(c) Let $G$ be a compact group, and let $0 \neq f \in L^1(G)$. Suppose that $h$ is a function on $G$ satisfying $h(x)f(y) = \int f(txt^{-1}y)\, dt$ for all $x, y \in G$. Show that $h = \frac{1}{d_\sigma}\chi_\sigma$ for some irreducible representation of $G$.

Question - 7 Marks distribution : 4+4+6

(a) Let $G$ be a compact group. Show that $\{\chi_\pi : \pi \in \hat{G}\}$ is an orthonormal basis for $Z(L^2(G))$.

(b) Let $G$ be a locally compact group and $H$ be a closed subgroup of $G$. Show that there is a surjective map from $C_{00}(G)$ into $C_{00}(G/H)$.

(c) State and prove the Bochner's theorem for locally compact abelian topological groups.

--- End of the Questions ---

Your Roll Number: ................ No. of Printed Pages 2

M.A./M.Sc. Mathematics Examinations, May 2019
Part II Semester IV
**MATH14-402(B): Frames and Wavelets**

Time: 3 hours Maximum Marks: 70

**Instructions:** • Write your roll number on the space provided at the top of this page immediately on receipt of this question paper • **Question No. 1** of **Section A** is compulsory • Answer **Two questions from Section B** and **Two questions from Section C** • Each question carries 14 marks. • Symbols have their usual meaning.

## Section A (14 marks)

(1) (a) Give an example, with justification, of a complete infinite family of vectors in $\mathcal{V}$ which is not a frame for $\mathcal{V}$. [2 Marks]

(b) Give an example, with justification, of a Parseval non-exact frame for $\mathcal{H}$. [2 Marks]

(c) Define Riesz frame. Give an example, with justification, of a frame for $\mathcal{H}$ which is not a Riesz frame. [1+2= 3 Marks]

(d) Give an example of a Gabor Parseval frame for $L^2(\mathbb{R})$. Justify your answer. [3 Marks]

(e) Show that $\{e_k + e_{k+1}\}_{k=1}^{\infty}$ is a minimal Bessel sequence in $\mathcal{H}$, where $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}$. [2 Marks]

(f) Define Haar wavelet? State any one property of the Haar wavelet. [1+1= 2 Marks]

## Section B (Answer any Two questions: 2 × 14 marks = 28 marks)

(2) (a) Show that the frame coefficients have minimal $\ell^2$-norm among all scalars in the representation of a signal. [3 Marks]

(b) Show that if $\dim\mathcal{V} = n$ and $\{f_k\}_{k=1}^{m}$ is a frame for $\mathcal{V}$, then there exist $n-1$ vectors $\{h_j\}_{j=1}^{n-1} \subset \mathcal{V}$ such that $\{f_k\}_{k=1}^{m} \bigcup \{h_j\}_{j=1}^{n-1}$ forms a tight frame for $\mathcal{V}$. [5 Marks]

(c) State and prove the Haar reconstruction theorem. [2+4 = 6 Marks]

(3) (a) Show that functions in $V_j$ associated with the Haar scaling function forms a Parseval frame for $V_j$, $j \in \mathbb{Z}$. [3 Marks]

(b) Show that if $\{f_k\}_{k=1}^{m}$ is a frame for $\mathcal{V}$ with pre-frame operator $T$ and frame operator $S$, then $T^{\dagger} f = \{\langle f, S^{-1} f_k\rangle\}_{k=1}^{m}$, where $T^{\dagger}$ is the pseudo-inverse of $T$. [4 Marks]

(c) Prove that a sequence of non-zero vectors $\{e_k\}_{k=1}^{\infty}$ in a Banach space $X$ is a Schauder basis for $X$ if and only if $[e_k] = X$ and there is a [7 Marks] constant $K > 0$ such that for all $n, m \in \mathbb{N}$ with $n \geq m$,

$$\left\|\sum_{k=1}^{m} c_k e_k\right\| \leq K \left\|\sum_{k=1}^{n} c_k e_k\right\|$$

for all scalar-valued sequences $\{c_k\}_{k=1}^{\infty}$. [5 Marks]

(4) (a) How frames are used in signal processing? Explain. [5 Marks]

(b) Prove that $L^2(\mathbb{R}) = V_0 \bigoplus W_0 \bigoplus W_1 \bigoplus \ldots$, where $W_j$ and $V_0$ are subspaces of $L^2(\mathbb{R})$ associated with the Haar wavelet and Haar scaling function, respectively. [1+3= 4 Marks]

(c) Define similar frames. Show that any two similar Parseval frames for $\mathcal{V}$ are unitarily equivalent.

**Section C (Answer any Two questions: $2 \times 14$ marks $= 28$ marks)**

(5) (a) State and prove the frame minus one theorem. [1+ 5 = 6 Marks]

(b) Prove that every frame for $\mathcal{H}$ is a multiple of a sum of three orthonormal bases for $\mathcal{H}$. [5 Marks]

(c) Show that if $\{f_k\}_{k=1}^{\infty}$ is a frame for $\mathcal{H}$ with frame bounds $A$, $B$; $\{g_k\}_{k=1}^{\infty} \subset \mathcal{H}$ and $\{f_k - g_k\}_{k=1}^{\infty}$ is a Bessel sequence with Bessel bound $B_g < A$, then $\{g_k\}_{k=1}^{\infty}$ is a frame for $\mathcal{H}$ with frame bounds $(\sqrt{A} - \sqrt{B_g})^2$ and $(\sqrt{B} + \sqrt{B_g})^2$. [3 Marks]

(6) (a) Show that a sequence $\{f_k\}_{k=1}^{\infty} \subset \mathcal{H}$ is a Riesz basis for $\mathcal{H}$ if and only if $[f_k]_{k=1}^{\infty} = \mathcal{H}$ and the Gram matrix associated with $\{f_k\}_{k=1}^{\infty}$ defines a bounded, linear and invertible operator on $\ell^2(\mathbb{N})$. [6 Marks]

(b) Show that if $\{f_k\}_{k=1}^{\infty}$ is a frame for $\mathcal{H}$ with frame bounds $A, B$ and $U \in \mathcal{B}(H)$ with closed range, then $\{Uf_k\}_{k=1}^{\infty}$ is a frame sequence in $\mathcal{H}$ with frame bounds $A\|U^{\dagger}\|^{-2}$, $B\|U\|^2$. [4 Marks]

(c) Show that a frame for $\mathcal{H}$ is $\omega$-independent if and only if it is exact. [4 Marks]

(7) (a) Define dual of a frame. Prove that if $\{f_k\}_{k=1}^{\infty}$ be a frame for $\mathcal{H}$ with frame operator $S$, then the dual frames of $\{f_k\}_{k=1}^{\infty}$ are precisely the families [1+7= 8 Marks]

$$\{g_k\}_{k=1}^{\infty} = \left\{S^{-1}f_k + h_k - \sum_{j=1}^{\infty}\langle S^{-1}f_k, f_j\rangle h_j\right\}_{k=1}^{\infty},$$

where $\{h_k\}_{k=1}^{\infty}$ is a Bessel sequence in $\mathcal{H}$.

(b) Show that if $\{f_k\}_{k=1}^{\infty}$ is a frame for $\mathcal{H}$ with frame operator $S$, then $\{S^{-1/2}f_k\}_{k=1}^{\infty}$ is a Parseval frame for $\mathcal{H}$. [3 Marks]

(c) What is the relation between Schauder bases and Riesz bases for $\mathcal{H}$? Justify your answer. [3 Marks]

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Your Roll Number: ................

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examinations, May 2019
Part II Semester IV

**MATH14 402(C): OPERATORS ON HARDY HILBERT SPACES**

Time: 3 hours Maximum Marks: 70

**Instructions:** • Write your roll number on the space provided at the top of this page immediately on receipt of this question paper. • This question paper has three sections. • Section A is compulsory. • Answer **two** questions from Section B and **two** questions from Section C. • All symbols have usual meaning.

**Section A (Attempt all parts; $2 \times 7$ marks = 14 marks)**

(1) (a) Give an example of a closed subspace $M$ of $l^2$ such that $M$ and $M^{\perp}$ are infinite dimensional. [2 Marks]

(b) Assuming that $l^2$ is isometrically isomorphic to $\mathbf{H}^2(\mathbb{D})$ identify the image of the element $(1, \alpha, \alpha^2, \ldots, \alpha^n, \ldots)$ of $l^2$ in $\mathbf{H}^2(\mathbb{D})$ and show that the function has no zeroes in $\mathbb{D}$. [2 Marks]

(c) Give an example of an operator on $\mathbf{H}^2(\mathbb{D})$ that is invertible but not a scalar multiple of an isometry. [2 Marks]

(d) Give an example of a function in $L^2$ which is not in $L^{\infty}$. [2 Marks]

(e) Give an example of a non-constant inner function and a non-constant outer function in $\mathbf{H}^2(\mathbb{D})$. [2 Marks]

(f) Give an example of a Toeplitz matrix that is not a bounded operator. [2 Marks]

(g) Give an example of a bounded Hankel matrix. [2 Marks]

**Section B (Answer any two questions; $2 \times 14$ marks = 28 marks)**

(2) (a) In the space $\mathbf{H}^2(\mathbb{D})$ prove that every invariant subspace of the unilateral shift $S$ is of the form $\phi\mathbf{H}^2(\mathbb{D})$, where $\phi$ is an inner function. Give an example of two inner functions $\phi, \psi$ in $\mathbf{H}^2(\mathbb{D})$ such that [7 Marks]

$$\phi\mathbf{H}^2(\mathbb{D}) \subsetneqq \psi\mathbf{H}^2(\mathbb{D}).$$

(b) (i) Show that $S^*$ has no reducing subspaces in $\mathbf{H}^2(\mathbb{D})$ but $I - SS^*$ has reducing subspaces in $\mathbf{H}^2(\mathbb{D})$, where $S$ is the unilateral shift. [7 Marks]

(ii) Let $O_1, O_2$ be outer functions in $\mathbf{H}^2(\mathbb{D})$. Show that $O_1O_2$ is an outer function in $\mathbf{H}^2(\mathbb{D})$.

(3) (a) (i) Show that the spectrum of the unilateral shift $S$ on $\mathbf{H}^2(\mathbb{D})$ is the closed disk $\overline{\mathbb{D}}$. [8 Marks]

(ii) Let $f \in \mathbf{H}^{\infty}(\mathbb{T})$ such that $f^{-1} \in \mathbf{H}^{\infty}(\mathbb{T})$. Show that $f$ is an outer function.

(b) Prove that if $T$ is a bounded operator in $\mathbf{H}^2(\mathbb{D})$ such that $TS = ST$, where $S$ is the unilateral shift on $\mathbf{H}^2(\mathbb{D})$. Then $T = M_{\phi}$ (multiplication by $\phi$) where $\phi \in \mathbf{H}^{\infty}(\mathbb{D})$. [6 Marks]

(4) (a) Prove that $S^2 + S$ is irreducible on $\mathbf{H}^2(\mathbb{D})$, where $S$ is the unilateral shift. [6 Marks]

(b) If $\alpha \in \mathbb{D}$ (open unit disk) and $M_{\alpha} = \{f \in \mathbf{H}^2(\mathbb{D}) : f(\alpha) = 0\}$. Then show that $M_{\alpha}$ is a closed subspace of $\mathbf{H}^2(\mathbb{D})$ invariant under the unilateral shift $S$. Identify the inner function $\phi$ such that [8 Marks]

$$M_{\alpha} = \phi\mathbf{H}^2(\mathbb{D}).$$

**Section C (Answer any two questions; $2 \times 14$ marks = 28 marks)**

(5) (a) Prove that the operator $T$ in $B(\mathbf{H}^2(\mathbb{D}))$ is a Toeplitz operator if and only if $S^*TS = T$, where $S$ is the unilateral shift on $\mathbf{H}^2(\mathbb{D})$. [6 Marks]

(b) If $\phi.\psi \in L^\infty$ and $T_\phi, T_\psi$ are the corresponding Toeplitz operators on $\mathbf{H}^2$ and $S$ is the unilateral shift on $\mathbf{H}^2$, then [8 Marks]

$$S^*T_\psi T_\phi S - T_\psi T_\phi = P(e^{-i\theta}\psi) \otimes P(e^{-i\theta}\bar{\phi}),$$

where $P$ is the orthogonal projection (analytic) from $L^2$ to $H^2$.

(6) (a) If $H$ is a Hankel operator on $\mathbf{H}^2$, then show that $H^*(f^*) = (Hf)^*$. [7 Marks]

(b) Show that for all $\phi \in L^\infty$ [7 Marks]

$$\text{essential range}\, \phi = \Pi(M_\phi) = \sigma(M_\phi).$$

(7) (a) Let $\phi$ and $\psi$ be in $L^\infty$ and suppose that $H_\psi \neq 0$. If $H_\phi H_\psi = H_\psi H_\phi$, then prove that there exists a complex number $c$ such that $H_\phi = cH_\psi$. [6 Marks]

(b) For every $\phi \in L^\infty$, show that $H_\phi^* = H_{\phi^*}$. Furthermore show if $H_\phi$ is self-adjoint, then there exists $\psi \in L^\infty$ such that $\psi = \psi^*$ a.e. and $H_\phi = H_\psi$. [8 Marks]

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Your Roll Number: ...............

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examinations, May 2019
Part II Semester IV
**MATH14 403(A): CALCULUS ON $\mathbb{R}^n$**

Time: 3 hours | Maximum Marks: 70

**Instructions:** • Write your roll number on the space provided at the top of this page immediately on receipt of this question paper • Question No. **1** of **Section A** is compulsory. Attempt any **Two** questions from **Section B** and any **Two** questions from **Section C**. Unless otherwise mentioned, $U$ will be an open subset of $\mathbb{R}^n$ .

## Section A

(1) (a) Let $f : U \to \mathbb{R}, a \in U$ such that $\lim_{h\to 0} \frac{f(a+h)-f(a)}{\|h\|}$ exists. Then is $f$ differentiable at $a$ ? Justify. [2 Marks]

(b) Give an example to show that the analog of Lagrange's mean value theorem does not hold for vector valued functions. [2 Marks]

(c) If $a \in \mathbb{R}^n$ and $f(x) = x + a$, $x \in \mathbb{R}^n$ , show that $f$ is a composition of primitive mappings. [2 Marks]

(d) If $I = \{1, 4, 6\}$ and $J = \{2, 3, 5, 8\}$ are increasing indices, find $\alpha$ so that $dx_I \wedge dx_J = (-1)^\alpha dx_{[I,J]}$. [1 Marks]

(e) Suppose $\alpha$ and $\beta$ are $k$- and $m$-forms, respectively, of class $C^1$ in $U$. If $\alpha$ is closed and $\beta$ is exact, show that $\alpha \wedge \beta$ is also exact. [2 Marks]

(f) State Stokes' theorem and verify it for the case where $\omega = f(x, y)dy$ is a 1-form in $\mathbb{R}^2$ with $f : \mathbb{R}^2 \to \mathbb{R}$ being a function, of class $C^\infty$ and $\Phi = \sigma_1 + \sigma_2$ is a 2-chain in $\mathbb{R}^2$ with $\sigma_1 = [0, e_1, e_2]$ and $e_2 = [e_1 + e_2, e_2, e_1]$. [1+4 Marks]

## Section B

(Answer any TWO questions)

(2) (a) Suppose $a \in U$ and $f : U \to \mathbb{R}$ is such that for $1 \le j \le n$, $D_j f$ exist and are continuous in a neighborhood of $a$. Show that $f$ is differentiable at $a$. [6 Marks]

(b) Give an example with justification to show that the existence of all directional derivatives at a point $a$ is not enough to guarantee the differentiability at $a$. [4 Marks]

(c) On $\mathbb{R}^2$ define $g(x, y) = (xe^{2y}, 2ye^x)$ and $f(x, y) = (2x - y^3, 3x + 2y, xy + y^2)$. Show that $g$ has a local inverse of class $C^\infty$ in a neighbourhood of $(0, 1)$ and compute $D(f \circ g^{-1})$ at $(0, 2)$. [4 Marks]

(3) (a) Let $\Omega$ be open in $\mathbb{R}^{k+n}$, $f : \Omega \to \mathbb{R}^n$ be of class $C^r$, where any element of $\Omega$ is expressed as $(x, y)$, with $x \in \mathbb{R}^k$ and $y \in \mathbb{R}^n$. If $(a, b) \in \Omega$ is such that $f(a, b) = 0$ and det $\frac{\partial f}{\partial y}(a, b) \neq 0$, then show

that there is a neighbourhood $V$ of $a$ in $\mathbb{R}^k$ and a unique function $\varphi : V \to \mathbb{R}^n$ of class $C^r$ such that $\varphi(a) = b$ and $f(x, \varphi(x)) = 0$ for all $x \in V$. [8 Marks]

(b) Suppose $I^k$ is a $k$-cell and $f \in C(I^k; \mathbb{R})$. Assuming that $\int_{I^k} f$ exists as an iterated integral, show that the definition does not depend on the order in which the $k$ integrals are carried out. [6 Marks]

(4) (a) Let $C(a, r) = \{x \in \mathbb{R}^n : \|a - x\|_\infty < r\}$ and $f \in C^k(C(a; r); \mathbb{R})$. State and prove the Taylor's formula for $f$ of order $k$ with Lagrange's remainder. [1+5 Marks]

(b) State and prove the existence theorem for partition of unity. [1+5 Marks]

(c) If $a \in U$ and $G$ is a differentiable primitive map on $U$, derive condition under which $G'(a)$ is invertible. [2 Marks]

## Section C

(Answer any TWO questions)

(5) (a) State the change of variables theorem and prove it only for the case where the transformation $T$ is a primitive $C^1$-mapping. [1+5 Marks]

(b) What do you mean by a $k$-form in $U$? What happens if $n < k$ ? Justify. [3 Marks]

(c) Let $T : U \to \mathbb{R}^m$ be a $C^2$-map and $\omega$ be a $k$-form of class $C^1$ in $\mathbb{R}^m$. Then prove that $d(\omega_T) = (d\omega)_T$. [5 Marks]

(6) (a) Explain the concept of positively oriented boundary of a set $E = T(Q^n)$, where $T \in C^1(Q^n; \mathbb{R}^n)$ is one-to-one and $\det J_T > 0$. [4 Marks]

(b) Suppose $V \subset \mathbb{R}^m$ and $W \subset \mathbb{R}^p$ are open sets, $T : U \to V$ and $S : V \to W$ are $C^1$ maps and $\omega$ is a $k$-form in $W$. Show that $(\omega_S)_T = \omega_{ST}$. [6 Marks]

(c) Evaluate $\int_\Phi \omega$, where $\omega = z\,dx \wedge dy \wedge dz$ is a 3-form in $\mathbb{R}^3$ and $\Phi(r, u, v) = (r \sin u \cos v, r \sin u \sin v, r \cos u)$ for $(r, u, v) \in D = \{(r, u, v) : 0 \le r \le a, 0 \le u \le \pi, 0 \le v \le 2\pi\}$. [4 Marks]

(7) (a) Suppose $k \ge 2$ and $\sigma = [P_0, P_1, \cdots, P_k]$ is an oriented affine $k$-simplex. Show that $\partial(\partial\sigma) = 0$. [4 Marks]

(b) Let $E$ be an open subset of $\mathbb{R}^k$ containing $Q^k, k > 1$ and $\sigma = [0, e_1, e_2, \cdots, e_k]$. For any $(k-1)$-form $\lambda$ of class $C^1$ in $E$, prove that $\int_\sigma d\lambda = \int_{\partial\sigma} \lambda$. [8 Marks]

(c) For the 1-form $\omega = \frac{x}{x^2+y^2}dy - \frac{y}{x^2+y^2}dx$ in $\mathbb{R}^2 \setminus \{(0,0)\}$ compute the pull back $\omega_T$, where $T(r, \theta) = (r\cos\theta, r\sin\theta)$ is the polar coordinate mapping. [2 Marks]

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Your Roll Number: ...............

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examinations, May 2019
Part II Semester IV
**MATH14 403(B): Differential Geometry**

Time: 3 hours                                                    Maximum Marks: 70

**Instructions:** • Write your roll number on the space provided at the top of this page immediately on receipt of this question paper • Question No. **1** of **Section A** is compulsory. Attempt any **Two** from **Section B** and any **Two** from **Section C**.

## Section A

(1) (a) Is the parametrized curve $\alpha(t) = (cost, sint, t)$ a *geodesic* on the cylinder $x^2 + y^2 = 1$ in $R^3$ ? Why ? [3 Marks]

(b) Prove that the graph of a smooth function $f : U \to R$, where $U$ is open in $R^n$, is an n-surface in $R^{n+1}$. [2 Marks]

(c) Show that if $\alpha : I \to R^{n+1}$ is a parametrized curve with constant speed then $\ddot{\alpha}(t) \perp \dot{\alpha}(t)$ for all $t \in I$. [2 Marks]

(d) Compute $\nabla_{\mathbf{v}} f$ where $f(x_1, x_2) = x_1^2 - x_2^3$, $\mathbf{v} = (1, 1, cos\theta, sin\theta), n = 2$. [2 Marks]

(e) Compute the following line integral $\int_\alpha (x_2 dx_1 - x_1 dx_2)$, where $\alpha(t) = (2cost, 2sint), 0 \le t \le 2\pi$. [2 Marks]

(f) Define cylinder over a given parametrized n-surface in $R^{n+k}$. Show that the cylinder over an n-surface is a parametrized $(n+1)$-surface in $R^{n+k+1}$. [3 Marks]

## Section B

(Answer any TWO questions)

(2) (a) Let $S = f^{-1}(c)$ be an n-surface in $R^{n+1}$ where $f : U \to R$ is such that $\nabla f(q) \neq 0$ for all $q \in S$. Suppose $g : U \to R$ is a smooth function and $p \in S$ is an extreme point of $g$ on $S$. Show that there is a real number $\lambda$ such that $\nabla g(p) = \lambda \nabla f(p)$. What happen if $S$ is compact ? Determine the extreme values on the unit circle $S^1$ in the plane of the function $g : R^2 \to R$ given by $g(x_1, x_2) = 4x_1^2 + 4x_1x_2 + x_2^2$. [8 Marks]

(b) Determine the integral curve at $p = (0, 1)$ of the vector field whose associated function is given by $X(x_1, x_2) = (x_2, -x_1)$ [4 Marks]

(c) Show that the covariant differentiation of a tangent vector field $\mathbf{D_v X}$ has the property $\nabla_{\mathbf{v}}(\mathbf{X} \cdot \mathbf{Y}) = (\mathbf{D_v})\mathbf{X} \cdot \mathbf{Y}(p) + \mathbf{X}(p) \cdot (\mathbf{D_v Y}(p))$, where $p$ is a point in an n-surface S and $\mathbf{v} \in S_p$. [2 Marks]

(3) (a) Show that in a connected n-surface in $R^{n+1}$ there are exactly two unit normal vector fields. [5 Marks]

(b) Prove that for a compact connected oriented n-surface in $R^{n+1}$ spherical image is the whole of the unit n-sphere $S^n$ in $R^{n+1}$. [9 Marks]

(4) (a) Show that a parametrized curve $\alpha$ is a geodesic in the unit n-sphere in $R^{n+1}$ if and only if it is of the form $\alpha(t) = (cosat)\mathbf{e}_1 + (sinat)\mathbf{e}_2$ for some orthonormal vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ in $R^{n+1}$. [6 Marks]

(b) Let $\alpha : I \to S$ be a geodesic in a 2-surface $S$ in $R^3$. Prove that a vector field $\mathbf{X}$ tangent to $S$ along $\alpha$ is parallel along $\alpha$ if and only if both $\| \mathbf{X} \|$ and the angle between $\mathbf{X}$ and $\dot{\alpha}$ are constant. [8 Marks]

## Section C

(5) (a) Find global parametrization of the plane curve $x_2 - ax_1^2 = c$ and determine its curvature. [6 Marks]

(b) Define a 1-form $\omega$ on an open set $U \subset R^{n+1}$. Prove that for each 1-form $\omega$ on $U$ there exist unique functions $f_i : U \to R$, $i \in \{1,..,n+1\}$ such that $\omega = \sum_{i=1}^{n+1} f_i dx_i$. [6 Marks]

(c) What are principal curvatures and principal curvature directions at $p$ on an n-surface $S$ in $R^{n+1}$. [2 Marks]

(6) (a) Let $S$ be an n-surface in $R^{n+1}$, oriented by the unit normal vector field $\mathbf{N}$. Let $p \in S$ and $\mathbf{v} \in S_p$. Show that for every parametrized curve $\alpha : I \to S$ with $\dot{\alpha}(t_0) = \mathbf{v}$ for some $t_0 \in I$, we have $\ddot{\alpha} \cdot \mathbf{N}(p) = L_p(\mathbf{v}) \cdot \mathbf{v}$. Elaborate the result for different parametrized curves with similar properties. [6 Marks]

(b) Find the Gaussian curvature $K : S \to R$, where $S$ is elliptic hyperboloid $\frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} - x_3 = 0$. [4 Marks]

(c) Show that the 1-form $\omega = \frac{x}{x^2+y^2}dy - \frac{y}{x^2+y^2}dx$ on $R^2 \setminus \{0\}$ is not exact. [4 Marks]

(7) (a) Define second fundamental form at a point $p$ on an n-surface in $R^{n+1}$. Prove that on a compact oriented n-surface $S$ in $R^{n+1}$ there exists a point $p$ such that the second fundamental form at $p$ is definite. [8 Marks]

(b) Let $a > b > 0$, for the parametrized torus in $R^3$, $\varphi : R^2 \to R^3$ defined by $\varphi(\theta, \phi) = ((a + bcos\phi)cos\theta, (a + bcos\phi)sin\theta, bsin\phi)$ determine the principal curvatures and the Gaussian curvature $K$. Discuss the conditions $K > 0, K < 0$ and $K = 0$. [6 Marks]

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Your Roll Number: ..................

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examinations, May 2019
Part II Semester IV

# MATH14 403(C): TOPOLOGICAL DYNAMICS

Time: 3 hours                                                                 Maximum Marks: 70

**Instructions:** • Write your roll number on the space provided at the top of this page immediately on receipt of this question paper. • This question paper has three sections. • Section A is compulsory. • Answer any **two** questions from Section B and any **two** questions from Section C.• All notations are standard.

## Section A

(1) (a) Is the shift map $\sigma$ on $X_A$ transitive for $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$? Justify. [3 Marks]

(b) Find suitable values of $\alpha$, $\beta$, $c$ so that the logistic map $f(x) = \mu x(1-x)$ is topologicaly conjugate to $g(x) = x^2 + c$ via homeomorphism $h(x) = \alpha x + \beta$. [3 Marks]

(c) For $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$, find stable and unstable sets of the fixed point $'0'$. [3 Marks]

(d) Construct a generator for the right shift homeomorphism on $X = \{\frac{1}{n}, 1 - \frac{1}{n} | n \in \mathbb{N}\}$ under usual metric. [3 Marks]

(e) Prove that $f(x) = \frac{1}{\sqrt{7}}x$ on $[0, 1]$ has POTP. Is $f$ minimal also? Justify. [2 Marks]

## Section B (Answer any two questions)

(2) For a $k \times k$ matrix $A$ with enteries in $\{0, 1\}$, prove the following:

(a) $X_A$ is a closed subset of $\Sigma_k$. [5 Marks]

(b) $A$ is irreducible if and only if $A \vee (A * A) \ldots\ldots \vee \underbrace{(A * \ldots * A)}_{n-times} = J$, for some $n \in \mathbb{N}$. [5 Marks]

(c) If $A$ is irreducible, then its associated digraph is strongly connected. [4 Marks]

(3) (a) If $f : \mathbb{R} \to \mathbb{R}$ is a $C^1$- map and $p$ is an attracting fixed point of $f$, then prove that there exists an interval $U$ about $p$ such that if $x \in U, x \neq p$, then $lim_{n\to\infty} f^n(x) = p$. [5 Marks]

(b) Stating (only) necessary lemmas, prove Sarkovskii's Theorem. [9 Marks]

(4) (a) Do the graphical analysis of the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = -(x + x^3)$ and draw the phase portrait of the orbit of $1/2$. [4 Marks]

(b) Let $X$ be a compact metric space and $f : X \to X$ be minimal, then prove that $\omega(x) = X$, for each $x \in X$. [5 Marks]

(c) If $f : X \to X$ is an expansive homeomorphism with expansive constant $e$, then prove that for all $\gamma > 0$, there exists $N > 0$ such that [5 Marks]

(i) $f^n(W^s_e(x, d)) \subseteq W^s_\gamma(f^n(x), d)$

(ii) $f^{-n}(W^u_e(x, d)) \subseteq W^u_\gamma(f^{-n}(x), d)$.

**Section C (Answer any two questions)**

(5) (a) If $(X, d)$ is a compact metric space and $f : X \to X$ is an expansive homeomorphism, then prove that the set of points having converging semi-orbits under $f$ is a countable set. [5 Marks]

(b) Prove that no simple closed curve admits an expansive homeomorphism. [9 Marks]

(6) (a) Let $(X, d)$ be a compact metric space and $f : X \to X$ be a homeomorphism. If for every $\epsilon > 0$, there exists $\delta > 0$ such that for each $k > 0$, every finite $\delta$-pseudo orbit $\{x_i : 0 \leq i \leq k\}$ is $\epsilon$-traced by some point in $X$, then prove that $f$ has POTP. [6 Marks]

(b) Prove that the shift map $\sigma$ on $X^{\mathbb{Z}}$, where $X$ is a compact metric space, has POTP. [8 Marks]

(7) (a) Let $(X, d)$ be a compact metric space and $f : X \to X$ be an expansive homeomorphism having POTP. Then prove that for every $\epsilon > 0$, there exists $\delta > 0$ such that for any homeomorphism $g : X \to X$ satisfying $d(f(x), g(x)) < \delta$, for each $x \in X$, there exists a continuous map $h : X \to X$ satisfying $d(h(x), x) < \epsilon$, for each $x \in X$ and $h \circ g = f \circ h$ [9 Marks]

(b) If $f$ is a topologically Anosov homeomorphism on a compact metric space $(X, d)$, then prove that $f^{-1}$ is also a topologically Anosov homeomorphism on $(X, d)$. [5 Marks]

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examinations, May 2019
Part II Semester IV

**MATH14-404(B): ADVANCED FLUID DYNAMICS**

Time: 3 Hour

Maximum Marks: 70

**Instructions:** • Q.No.1 in Section **A** is compulsory. • Attempt any **TWO** questions from Section **B** and **C** each. • Attempt FIVE questions in all. Each question carries 14 marks. • All the symbols have their usual meaning.

## Section A
(Answer all parts, 14 Marks)

(1) (a) Describe the internal energies of a perfect gas and a real gas. [3 Marks]

(b) Find the dimension of medium porosity $\epsilon$ in the equation of continuity for the porous medium $\epsilon\frac{\partial\rho}{\partial t} = div(\rho\bar{v})$. [2 Marks]

(c) Write the Maxwell's electromagnetic field equation for the conducting medium in rest and motion. [4 Marks]

(d) Derive the estimation for the dimensionless boundary layer thickness for the laminar flow of a viscous fluid over a flat plate of length $l$. [3 Marks]

(e) Explain geometrically the difference between the subsonic and the supersonic flows. [2 Marks]

## Section B
(Answer any TWO questions, 28 Marks)

(2) (a) Write the mass, momentum and energy equation in integral and differential equation form for one dimensional motion of an inviscid gas and hence derive the corresponding shock conditions. [9 Marks]

(b) Show that a small disturbance is propagated in an isentropic and irrotational flow of a gas with a speed $a_0 = (\frac{\gamma p_0}{\rho_0})^{\frac{1}{2}}$. [5 Marks]

(3) (a) Define normal and oblique shock wave. Prove that the shock wave is compressive in nature. [1+5 Marks]

(b) Define the compressibility. Calculate the isentropic compressibility of an ideal gas in terms of speed of sound. [2+2 Marks]

(c) Reduce the equation [2+2 Marks]
$-\nabla p - \frac{\mu}{\gamma}\vec{v} + \mu\nabla^2\vec{v} + \rho\vec{g} = 0$,
where $\gamma$ and $\vec{g}$ are medium permeability and acceleration due to gravity, to non-dimensional form and obtain the relevant dimensionless numbers.

(4) (a) Check whether the heat added $Q$ and entropy $S$ per unit mass of an ideal gas are functions of state or not. Derive entropy equation for non-ideal gas with equation of state $p = \frac{RT}{(v-b)}$. [3+3 Marks]

(b) State the principle of conservation of energy for a fluid flow. Derive the internal energy equation $\rho\frac{De}{Dt} = \frac{p}{\rho}\frac{D\rho}{Dt} - \nabla.\vec{q} + \overleftrightarrow{T} : (\nabla\vec{w})^t$. [1+4 Marks]

(c) Show that $c_p - c_v = -T(\frac{\partial v}{\partial T})_p^2(\frac{\partial p}{\partial v})_T$. Also find the corresponding [3 Marks] relation for the non-ideal gas with equation of state $p(v - b) = RT$.

## Section C

(Answer any TWO questions, 28 Marks)

(5) (a) Show that the MHD wave propagates with the speed $\sqrt{a^2 + V_A^2}$, where $V_A$ is Alfven's velocity. [4 Marks]

(b) Derive the electric and magnetic field equation in a conducting fluid under the Pre-Maxwell's equation and explain them physically. [5 Marks]

(c) Define the Magnetic Reynolds number and explain it physically. [2 Marks]

(d) Show that under MHD approximation the magnetic field energy is negligible in comparison to electric field energy. [3 Marks]

(6) (a) Derive the equation of motion of a non-viscous conducting fluid $\frac{\partial(\rho\vec{v})}{\partial t} + \rho(\vec{v}.\nabla)\vec{v} = \overrightarrow{f(ex)} + \frac{\mu}{4\pi}(\vec{H}.\nabla)\vec{H} - \nabla(p + \frac{\mu\vec{H}^2}{8\pi}) - \vec{v}\nabla.(\rho\vec{v})$. Write the corresponding equation for viscous and incompressible fluid. [3+2 Marks]

(b) State and prove Alfven's theorem. [5 Marks]

(c) Define the displacement and momentum thickness of a boundary layer and find the expression for each of them. [4 Marks]

(7) (a) Derive the Prandtl's boundary layer equations along with the boundary conditions for two dimensional viscous incompressible fluid flow over a slender body. Also write the corresponding equations for the steady flow. [8 Marks]

(b) Derive the von Karman's momentum integral equation for steady, two dimensional boundary layer flow of incompressible fluid. [6 Marks]

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Your Roll Number: ................

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examinations, May 2019
Part II Semester IV

**MATH14-404 (C): COMPUTATIONAL METHODS FOR PDES**

Time: 3 hours

Maximum Marks: 70

**Instructions:** • Write your roll number on the space provided at the top of this page immediately on receipt of this question paper. • This question paper has three sections. • Section A is compulsory. • Attempt any **two** questions from **Section B** and **Section C** each. • Non-programmable scientific calculators are allowed

## SECTION A (Attempt all) (14 marks)

(1) (i) Consider the following problem posed in a domain $\Omega$, an open bounded subset of $\mathbb{R}^3$. [ 3 ]

$$-\Delta u(x) = f(x)$$
$$u(x) = 0, \ x \in \partial\Omega$$

with $f \in C^0(\bar{\Omega})$ and $\Delta = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$. Derive weak formulation of the above problem.

(ii) State true or false and justify: Leap Frog scheme for the numerical solution of one dimensional parabolic PDE is unconditionally stable. [ 3 ]

(iii) Explain with an example the fact that satisfying CFL condition is not a sufficient condition for the convergence of a finite difference scheme associated with a PDE. [ 3 ]

(iv) Derive point iteration matrix of Gauss Seidel iterative scheme. [ 2 ]

(v) Construct Crank-Nicolson scheme for the numerical solution of three dimensional parabolic PDE with Dirichlet boundary conditions. [ 3 ]

## SECTION B (Attempt any two) (28 marks)

(2) (a) Consider the problem [ 7 ]

$$u_t(x,t) - u_{xx}(x,t) = f(x,t), \quad 0 < x < 1$$
$$u(x,0) = 0, \quad u(0,t) = u(1,t) = 0, \quad t \geq 0.$$

Apply finite element method with linear basis elements for space discretization and backward Euler scheme for time discretization. Find the global assembly matrix.

(b) Consider the problem [ 7 ]

$$u_{xx} + u_{yy} = 1, \text{ in } \Omega = (0,1) \times (0,1)$$
$$u = 0 \text{ on } \partial\Omega.$$

Apply finite element technique using triangular elements and piecewise linear polynomial functions as basis to obtain the resulting system of algebraic equations.

(3) (a) Consider the initial value problem [7]

$$u_t = \nu(u_{xx} + u_{yy}), \quad x,\ y \in \mathbb{R} = (0,1)\times(0,1),\ t \geq 0$$
$$u(x,y,0) = f(x,y),\ x,y \in \partial\mathbb{R}.$$

Derive Douglas- Rachford scheme for the numerical solution of the above problem and discuss its consistency.

(b) Show that the BTCS scheme for the numerical solution of the partial differential equation [7]

$$u_t = \nu(u_{xx} + u_{yy} + u_{zz})$$

with appropriate dirichlet boundary conditions is of the $O(\Delta t + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2)$.

(4) (a) Consider an application of $\theta$ method to approximate the equation $u_t = u_{xx}$ with choice $\theta = \frac{1}{2} + \frac{(\Delta x)^2}{12\Delta t}$. Show that the resulting scheme is unconditionally stable and has a truncation error which is $O((\Delta t)^2 + (\Delta x)^2)$. [7]

(b) Consider the problem [7]

$$u_t = u_{xx};\ 0 < x < 1,\ t > 0$$

with boundary and initial conditions given by

$$u(0,t) = 0, \quad u_x(1,t) = -\frac{u(1,t)}{2},\ t > 0, \quad u(x,0) = x(1-x),\ 0 \leq x \leq 1.$$

Solve the above problem by Crank-Nicolson Scheme, employing central-difference for the boundary conditions and taking $\Delta x = 0.25$, $\Delta t = 0.2$ for one time level.

## SECTION C (Attempt any two) (28 marks)

(5) (a) Solve the mixed boundary value problem [8]

$$u_{xx} + u_{yy} = 0, \quad 0 \leq x, y \leq 1$$
$$u(x,0) = 2x,\ u(x,1) = 2x - 1,\ 0 \leq x \leq 1$$
$$\text{and } (u_x + u)(0,y) = 2 - y,\ u(1,y) = 2 - y,\ 0 \leq y \leq 1,$$

using Laplace five point formula with $\Delta x = \Delta y = 1/3$.

(b) Analyze the difference scheme [6]

$$u_{j,k}^{n+1} = u_{j,k}^n - \frac{R_x}{2}\delta_{x0}u_{jk}^n + \frac{R_x^2}{2}\delta_x^2 u_{jk}^n - \frac{R_y}{2}\delta_{y0}u_{jk}^n + \frac{R_y^2}{2}\delta_y^2 u_{jk}^n,\ R_x = \frac{a\Delta t}{\Delta x},\ R_y = \frac{a\Delta t}{\Delta y}$$

for consistency when applied to the problem

$$u_t + au_x + bu_y = 0,\ v(x,y,0) = f(x,y).$$

(6) (a) Derive necessary and sufficient conditions for the convergence of an iterative method for the solution of the system of algebraic equations of the form [ 6 ]

$$AX = F$$

where $A$ is an $n \times n$ matrix and $X$, $F$ are $n$ vectors.

(b) Solve $u_t + u_x = 0$, $x \in [0, 2]$ with boundary condition $u(0, x) = \begin{cases} 0, & x < 0 \\ x, & 0 \leq x \leq 1 \\ 2 - x, & 1 \leq x \leq 2 \\ 0, & x > 2, \end{cases}$ using FTFS scheme with $\Delta x = 1/2$, $\Delta t = 1/4$ upto two time level. [ 8 ]

(7) (a) When do we say that a scheme satisfy CFL condition? For the linear advection equation $u_t + au_x = 0$, where $a$ is a positive constant, discuss CFL condition and stability results for the scheme $u_k^{n+1} = \alpha u_{k+1}^n + \beta u_k^n + \gamma u_{k-1}^n$, where $\alpha$, $\beta$, $\gamma$ are positive constants. [ 4 ]

(b) Consider [ 7 ]

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = f(r, z, u)$$

in the region $[0 \leq r \leq R] \times [0 \leq z \leq c]$ subject to the boundary conditions

$$\frac{\partial u}{\partial r}(0, z) = 0,\ u(R, z) = g(z),\ u(r, 0) = f(r),\ u(r, c) = h(r).$$

Derive a five point difference scheme for the numerical solution of the above problem.

(c) Use method of characteristics to solve the following PDE [ 3 ]

$$x^2\frac{\partial u}{\partial x} + t^2\frac{\partial u}{\partial t} = (x + t)u.$$

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Your Roll Number: ...........

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations, May 2019
Part II Semester IV
**MATH14-404 (E): Dynamical Systems**

Time: 3 hours

Maximum Marks: 70

**Instructions:** • Write your roll number on the space provided at the top of this page immediately on receipt of this question paper • Section A is compulsory • Attempt **any two** questions from Section B • Attempt **any two** questions from Section C.

## Section A

(1) (a) Describe the separatrices for the linear system [3 Marks]

$$\dot{x}_1 = x_1 + 2x_2$$
$$\dot{x}_2 = 3x_1 + 4x_2.$$

(b) (i) Compute the derivative of the following function [3 Marks]

$$f(x) = \begin{bmatrix} x_1 + x_1 x_2^2 \\ -x_2 + x_2^2 + x_1^2 \end{bmatrix}.$$

(ii) Find the zeros of the above function and evaluate $Df(x)$ at these points.

(iii) For the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined in part (i) above, compute $D^2 f(x_0)(x, y)$, where $x_0 = (0, 1)$ is a zero of $f$.

(c) State Sotomayor Theorem. [4 Marks]

(d) Define the following terms: nondegenerate critical point, Hamiltonian system with $n$ degrees of freedom, gradient system, and an attracting set. [4 Marks]

## Section B

*(Attempt any two questions)*

(2) (a) State and prove the Hartman-Grobman Theorem. [10 Marks]

(b) Solve the forced harmonic oscillator problem [4 Marks]

$$\ddot{x} + x = f(t).$$

(3) (a) Classify the equilibrium points of the Lorenz equation $\dot{x} = f(x)$ with [6 Marks]

$$f(x) = \begin{bmatrix} x_2 - x_1 \\ \mu x_1 - x_2 - x_1 x_3 \\ x_1 x_2 - x_3 \end{bmatrix}$$

for $\mu > 0$. At what value of the parameter $\mu$ do two new equilibrium points "bifurcate" from the equilibrium point at the origin?

(b) Solve the initial value problem $\dot{x} = Ax$, $x(0) = x_0$ with the matrix [5 Marks]

$$A = \begin{bmatrix} -1 & 1 & -2 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \end{bmatrix}.$$

[3 Marks]

(c) Construct the phase portrait for the undamped pendulum

$$\ddot{x} + sin(x) = 0.$$

[6 Marks]

(4) (a) Find the *Poincaré map* of the system

$$\dot{x} = -y + x(1 - x^2 - y^2)$$
$$\dot{y} = x + y(1 - x^2 - y^2).$$

[5 Marks]

(b) Determine the stability of the system using Lyapunov function

$$\dot{x_1} = -2x_2 + x_2 x_2$$
$$\dot{x_2} = x_1 - x_1 x_3$$
$$\dot{x_3} = x_1 x_2.$$

[3 Marks]

(c) Classify the equilibrium points (as sinks, sources or saddles) of the nonlinear system $\dot{x} = f(x)$ with

$$f(x) = \begin{bmatrix} x_1 - x_1 x_2 \\ x_2 - x_1^2 \end{bmatrix}.$$

## Section C

*(Attempt any two questions)*

[3 Marks]

(5) (a) State and prove Conservation of Energy.

[4 Marks]

(b) If the origin is a focus of the Hamiltonian system

$$\dot{x} = H_y(x, y)$$
$$\dot{y} = -H_x(x, y),$$

then the origin is not a strict local maximum or minimum of the Hamiltonian function $H(x, y)$.

[7 Marks]

(c) Determine the center manifold near the origin of the system

$$\dot{x_1} = x_2 + y$$
$$\dot{x_2} = y + x_1^2$$
$$\dot{y} = -y + x_2^2 + x_1 y.$$

[6 Marks]

(6) (a) Define bifurcation value. Consider the one-dimensional system

$$\dot{x} = \mu x - x^3.$$

Determine the critical points and the bifurcation value for this differential equation. Draw the phase portraits for various values of the parameter $\mu$ and draw the bifurcation diagram.

(b) Show that the continuous map $H : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by [5 Marks]

$$H(x) = \begin{bmatrix} x_1 \\ x_2 + \frac{x_1^2}{3} \end{bmatrix}$$

has a continuous inverse $H^{-1} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ and that nonlinear system $\dot{x} = f(x)$ with

$$f(x) = \begin{bmatrix} -x_1 \\ x_2 + x_1^2 \end{bmatrix}$$

is transformed into the linear system $\dot{x} = Ax$ with $A = Df(0)$ under this map, i.e., if $y = H(x)$, show that $\dot{y} = Ay$.

(c) Determine the flow and invariant set of the nonlinear system $\dot{x} = f(x)$, where [3 Marks]

$$f(x) = \begin{bmatrix} -x_1 \\ x_2 + x_1^2 \end{bmatrix}.$$

(7) (a) Sketch the flow for the linear system $\dot{x} = Ax$ with [3 Marks]

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}.$$

(b) Write the following system in polar coordinates and determine the nature of origin [3 Marks]

$$\dot{x} = -y - x^3 - xy^2$$
$$\dot{y} = x - y^3 - x^2 y.$$

(c) Find the stable, unstable and center subspaces $E^s, E^u$ and $E^c$ of the system $\dot{x} = Ax$ with the matrix [3 Marks]

$$A = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}.$$

(d) Determine the stable manifold $S$ and unstable manifold $U$ for the nonlinear system [5 Marks]

$$\dot{x}_1 = -x_1 - x_2^2, \quad \dot{x}_2 = x_2 + x_1^2.$$

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Your Roll Number: ..................

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examinations, May 2019
Part II Semester IV
**MATH14 404(F): OPTIMIZATION TECHNIQUES AND CONTROL THEORY**

Time: 3 hours | Maximum Marks: 70

**Instructions:** • Write your roll number on the space provided at the top of this page immediately on receipt of this question paper. • This question paper has three sections. • Section A is compulsory. • Answer **two** questions from Section B and **two** questions from Section C.• All symbols have usual meaning.

**Section A (Attempt all parts; $2 \times 7$ marks = 14 marks)**

(1) (a) Give an economic interpretation of conjugate functions. [2 Marks]
(b) Find the subdifferential of the function $f(x) = e^{|x|}$ on $\mathbb{R}$ at $x = 0$. [2 Marks]
(c) Is the sum of two proper functions also a proper function? Justify. [2 Marks]
(d) Use Newton's method to minimize the function $f(x, y) = x^2 - 2xy + 2y^2 - x + 2$ over $\mathbb{R}^2$ starting from the point $x = (0, 0)$. [2 Marks]
(e) If $\phi(x, w)$ is a closed proper convex function defined on $\mathbb{R}^{n+k}$, then prove that $\Psi(0) = -\ \mathrm{cl}\Phi(0)$. [2 Marks]
(f) State Bellman's principle of optimality. [2 Marks]
(g) When is a control variable said to be admissible? [2 Marks]

**Section B (Answer any two questions; $2 \times 14$ marks = 28 marks)**

(2) (a) Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a convex function and $x \in \mathbb{R}^n$ be a point such that $f(x)$ is finite. Prove that a vector $\xi \in \mathbb{R}^n$ is a subgradient of $f$ at $x$ if and only if $D^+f(x, y) \geq \xi^T y$, for every $y \in \mathbb{R}^n$. Also, prove that $f(y) - f(x) \geq D^+f(x; y - x)$ for every $y \in \mathbb{R}^n$. [7 Marks]
(b) Let $f$ be a real valued differentiable function defined on an open interval $D \subseteq \mathbb{R}$. Prove that the first derivative $f'$ is a nondecreasing function on $D$ if and only if $f$ is convex on $D$. What will be the corresponding statement for a differentiable function defined on $\mathbb{R}^n$? [4 Marks]
(c) Find the support function of the set $C = [-1, 1] \times [-1, 1] \subseteq \mathbb{R}^2$. [3 Marks]

(3) (a) Prove that the conjugate of a convex function $f$ defined on $\mathbb{R}^n$ is a convex function and $f^{**} = \mathrm{cl} f$. Also, find the conjugate of the function $f(x) = x^4$ defined on $\mathbb{R}$. [7 Marks]
(b) Find the dual problem ($D_\phi{}^*$) where $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is [7 Marks]

$$\phi(x, w) = \begin{cases} x + w, & \text{if } x^2 = w, \\ +\infty, & \text{if } x^2 \neq w. \end{cases}$$

Is the problem ($P_\phi$) stable and normal? Justify.

(4) (a) Prove that the Lagrangian function $L(x, \lambda)$ corresponding to the primal problem ($P_\phi$) is convex in $x \in \mathbb{R}^n$ for any fixed $\lambda$ and concave in $\lambda \in \mathbb{R}^m$ for fixed $x$. [7 Marks]

(b) If $x^* \in \mathbb{R}^n$ is a solution of the following problem (SP) [7 Marks]

$$\text{Min } f(x)$$
$$\text{subject to } g_i(x) \geq 0, i = 1, 2, ...m,$$

where $f$, $-g_i$ are proper convex functions on $\mathbb{R}^n$, then prove that (SP) is stable if and only if there exists $\lambda^* \in \mathbb{R}^m$ such that $\lambda_i^* \geq 0, \lambda_i^* g_i(x^*) = 0, i = 1, ...m$ and $(x^*, \lambda^*)$ is a saddle point of Lagrangian.

**Section C (Answer any two questions; $2 \times 14$ marks = 28 marks)**

(5) (a) In the steepest-descent method for minimizing a convex quadratic function $q(x) = \frac{1}{2}\langle Qx, x\rangle - \langle b, x\rangle + a$ where $Q$ is $n \times n$ symmetric positive definite matrix, $b \in \mathbb{R}^n$, $a \in \mathbb{R}$, prove that the optimality gap $E(x) = q(x) - \min_{\mathbb{R}^n} q$ decreases at a geometric rate $E(x_{k+1}) \leq (\frac{\tau-1}{\tau+1})^2 E(x_k)$ where $\tau$ is the condition number of the matrix $Q$. [6 Marks]

(b) Use improved conjugate gradient method to find the critical points of [6 Marks]

$$q(x_1, x_2) = 3x_1^2 + 2x_1x_2 + 2x_2^2 - 2x_1 + x_2$$

starting from the point $(0, 0)$.

(c) Give the steps of the algorithm used in the gradient projection method. [2 Marks]

(6) (a) There are $n$ machines which can do two jobs. If $x$ of them do the first job, then they produce goods worth $g(x) = 4x$ and if $y$ of them do the second job then they produce goods worth $h(y) = 5y$. Machines are subject to depreciation so that after doing the first job only $a(x) = \frac{2x}{3}$ machines remain available and after doing the second job only $b(y) = \frac{y}{2}$ machines remain available in the beginning of second year. If the process is repeated with the remaining machines, obtain the maximum total return after 3 years and find the optimal policy in each year. [7 Marks]

(b) A vessel is to be loaded with stocks of 3 items. Each unit of item $i$ has a weight $w_i$ and value $v_i$. The maximum cargo weight the vessel can load is 5 and details of the three items are tabulate below. Find the most valuable cargo load without exceeding the maximum cargo weight by using dynamic programming approach. [7 Marks]

| $i$ | $w_i$ | $v_i$ |
|---|---|---|
| 1 | 1 | 35 |
| 2 | 3 | 105 |
| 3 | 2 | 75 |

(7) (a) Solve the following control problem [5 Marks]

$$\text{Max } \int_0^{t_1} (x_1^2 + 4x_2^2)dt$$
$$\text{subject to } x_1(0) = 0, x_2(0) = 1, \dot{x}_1(t) = x_2(t).$$

(b) Consider the optimal control problem [7 Marks]

$$\text{Max } J(u) = \int_{t_0}^{t_1} I(t, x, u)dt + F(t_1, x_1)$$
$$\text{subject to } \dot{x} = f(t, x, u)$$

where $t_0, x(t_0) = x_0$ are given and $(t, x(t)) \in \Gamma$ at $t = t_1$. Using principle of imbedding derive Bellman's equation under appropriate conditions.

(c) State Pontryagin's maximum principle for the control problem considered in Q 7.(b). [2 Marks]

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