Your Roll Number: ....

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examinations, May 2019
Part I Semester II

MATH14-201: MODULE THEORY

Time: 3 hours

Maximum Marks: 70

Instructions: • Write your roll number on the space provided at the top of this page immediately on receipt of this question paper. • This question paper has three sections. • Section A is compulsory. • Attempt any **two** questions from **Section B** and **Section C** each. • Throughout the paper R represents a ring with nonzero unity.

SECTION A (Attempt all) (14 marks)

(1) (i) Give example of a subcategory which is not full. [2]

(ii) Determine the submodule generated by $2\mathbb{Z} \cup 5\mathbb{Z}$ in $\mathbb{Z}$? [2]

(iii) Determine the bound of the left R-module R/I, where $I = Ra$, $0 \neq a \in R$. [2]

(iv) Is submodule of a free module free? Justify your answer. [2]

(v) Define length of a module. What is the length of $\mathbb{Z}$-module $\mathbb{Z}_{10}$? [3]

(vi) Is $2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}_2$ a submodule of $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}_2$? Justify your answer. [3]

SECTION B (Attempt any two) (28 marks)

(2) (a) Let R be a commutative ring; M and N be left R-modules, $k \in \mathbb{N}$. Prove that [6]

$$Hom_R(M, N^k) \cong (Hom_R(M, N))^k.$$

(b) For a short exact sequence $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$, prove that M is Artinian if and only if M' and M'' are both Artinian. [6]

(c) Define a category. [2]

(3) (a) Define an exact contravariant functor. Give example (with justification) of a module M for which the contravariant functor $Hom_R(-, M)$ is exact. [4]

(b) Prove that a short exact sequence $0 \to M' \xrightarrow{\lambda} M \xrightarrow{\mu} M'' \to 0$ splits if and only if μ has a left inverse. [5]

(c) Let M be a semisimple left R-module. Prove that M is semisimple right $End_R(M)$-module. [5]

(4) (a) If N is a proper submodule of a module M of finite length, prove that $l(N) < l(M)$, where $l(M)$ denotes the length of M. [5]

(b) Prove that a module is sum of simple submodules if its every submodule is complemented in it. [4]

(c) Prove that an abelian group M is a right R-module if and only if there exists a ring homomorphism $\phi : R \to End(M)$. [5]

SECTION C (Attempt any two) (28 marks)

(5) (a) For a set I, prove that there exists a free R-module F_I and a mapping $\phi : I \to F_I$ such that for any R-module M and a mapping $f : I \to M$, there exists a unique R-homomorphism $\tilde{f} : F_I \to M$ which satisfies $\phi\tilde{f} = f$. [7]

(b) For commutative ring R, the R-modules M, N and P, prove that [7]

$$(M \otimes N) \otimes P \cong M \otimes (N \otimes P)$$

(6) (a) If a module P is a direct summand of a free module then prove that every homomorphism from P to a quotient of a module B can be lifted to B. Give an example to show that the conclusion need not hold for an arbitrary module. [5+2]

(b) Let R be a principal ideal domain and M be a torsion module over R. Prove that $M = \oplus M_p$ where each M_p is p-primary submodule, p being prime in R. [7]

(7) (a) Let R be a PID and M be an R-module such that $M = \oplus_{i=1}^{n} M_i$, each M_i being cyclic. If M is bounded and cyclic, prove that each M_i is bounded and their bounds are pairwise co-prime. [6]

(b) Construct an example of a free module which has a basis of cardinality n, for every $n \in \mathbb{N}$. [5]

(c) For an abelian group G, prove that $\mathbb{Z} \otimes_{\mathbb{Z}} G \cong G$. [3]

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Your Roll Number: ....

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examinations, May 2019
Part I Semester II
MATH14-202:TOPOLOGY-I

Time: 3 Hours Maximum Marks: 70

Instructions: • Write your roll number on the space provided at the top of this page immediately on receipt of this question paper. •Section A is compulsory. Answer any **TWO** questions from Section B and **TWO** questions from Section C. • Each question carries equal marks.

Section A

(1) (a) For the subset $A = \{1, \frac{1}{2}, \frac{1}{3}, \cdots\}$ of the real line $\mathbb{R}$ find ∂A. [3.5 Marks]

(b) Let $Y = \{(x, y) \in \mathbb{R}^2 : xy = 0\}$ be the subspace of the Euclidean space $\mathbb{R}^2$ and $p : \mathbb{R}^2 \to \mathbb{R}^1$ be the map defined by $p(x, y) = x$. Show that $p|_Y$ is closed but not open. [3.5 Marks]

(c) Find limit(s) of the sequence $\langle 1/n \rangle$ if $\mathbb{R}$ is assigned the co-finite topology. [3.5 Marks]

(d) Show that a closed subspace of a Lindelöf space is Lindelöf. [3.5 Marks]

Section B

(Attempt any TWO questions)

(2) (a) Let X be a topological space and $A \subseteq X$. Then show that $\overline{A} = A \cup A'$. [5 Marks]

(b) Prove that G is open in a topological space X if and only if $\overline{G \cap \overline{A}} = \overline{G \cap A}$, for every subset A of X. [5 Marks]

(c) Let $\mathfrak{B}$ be a family of subsets of X such that $X = \cup\{B | \; B \in \mathfrak{B}\}$, and for every two members $B_1, B_2 \in \mathfrak{B}$ and for each point $x \in B_1 \cap B_2$, there exists $B_3 \in \mathfrak{B}$ with $x \in B_3 \subseteq B_1 \cap B_2$. Show that there is a topology on X for which $\mathfrak{B}$ is a basis. [4 Marks]

(3) (a) Prove that $A \cup B$ is a connected subset of the Euclidean space $\mathbb{R}^2$, where [5 Marks]

$$A = \{(x, y) | \; x \text{ is irrational and } 0 \leq y \leq 1\}, \quad \text{and}$$
$$B = \{(x, y) | \; x \text{ is rational and } -1 \leq y \leq 0\}.$$

(b) Let $\{X_\alpha, \; \alpha \in A\}$ be a family of topological spaces. Show that the component of a point $x = (x_\alpha)$ in the product space ΠX_α is $\Pi C(x_\alpha)$, where $C(x_\alpha)$ is the component of x_α in X_α. [5 Marks]

(c) Prove that a space X is locally path-connected if for each point $x \in X$, and each open neighbourhood (nbd) U of x, there is a nbd V of x such that $V \subseteq U$ and any point of V can be joined to x by a path in U. [4 Marks]

(4) (a) If A is a compact subset of a Hausdorff space X and $x \in X - A$, then show that there are disjoint open sets U and V such that $x \in U$ and $A \subseteq V$. [5 Marks]

(b) Let X be any topological space and Y be a compact space. If N is an open nbd of the slice $\{x\} \times Y$ in $X \times Y$, then prove that there is an open nbd U of x such that $U \times Y \subseteq N$. [5 Marks]

(c) Show that every universal net in a compact space converges. [4 Marks]

Section C

(Attempt any TWO questions)

(5) (a) Let X and Y be topological spaces. Prove that a function $f : X \to Y$ is continuous $\Leftrightarrow f^{-1}(B^{\circ}) \subseteq (f^{-1}(B))^{\circ}$ for every subset B of $Y \Leftrightarrow \partial(f^{-1}(B)) \subseteq f^{-1}(\partial B)$ for every subset B of Y. [7 Marks]

(b) If $\{X_n : n \in \mathbb{N}\}$ is a countable family of metrisable spaces, then show that the product space $\prod X_n$ is metrisable. [7 Marks]

(6) (a) Let X be a topological space and $x \in X$ a cluster point of a net ϕ. Show that there is a subnet of ϕ which converges to x. [6 Marks]

(b) Let X and Y be topological spaces. Then show that a function $f : X \to Y$ is continuous if the net $f \circ \phi$ in Y converges to $f(x)$ for every net ϕ in X converging to a point $x \in X$. [4 Marks]

(c) Let X be a T_1-space and $A \subseteq X$. Prove that A' is a closed set. [4 Marks]

(7) (a) Show that $\mathbb{R}_l$ ($\mathbb{R}$ with the lower limit topology) is first countable but not second countable. [5 Marks]

(b) Does every closed subspace of a separable space separable? Justify. [5 Marks]

(c) Let X be a first countable space. Show that X is Hausdorff if each convergent sequence in X has a unique limit. [4 Marks]

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M.A./M.Sc. Mathematics Examinations, May 2019

Part I Semester II

MATH14-203: Functional Analysis

Time: 3 hours Maximum Marks: 70

Instructions: • Write your roll number on the space provided at the top of this page immediately on receipt of this question paper • **Question No. 1 is compulsory** • Answer **Two questions from Section II** and **Two questions from Section III** • Each question carries 14 marks. • Symbols have their usual meaning.

Section I (Question No. 1 is Compulsory: 14 marks)

(1) (a) Define $f : X = (c_{00}, \|.\|_\infty) \to \mathbb{C}$ by $f(x) = \sum_{k=1}^{\infty} \xi_k$, $x = <\xi_k> \in c_{00}$. Is f bounded? Justify your answer. [2 Marks]

(b) Give an example of a total subset in ℓ^1. Justify your answer. [3 Marks]

(c) Give an example, with justification, of a bijective isometry which is not linear. [3 Marks]

(d) Find the spectrum $\sigma(T)$ and spectral radius $r_\sigma(T)$ of $T : (\mathbb{R}^3, \|.\|_2) \to (\mathbb{R}^3, \|.\|_2)$ given by $Tx = (\xi_2, -\xi_1, 0)$, $x = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$. [2 Marks]

(e) State the Hahn-Banach theorem for normed spaces. Show that if X is a normed space and $f(x) = f(y)$ for all $f \in X'$, then $x = y$. [1+1== 2 Marks]

(f) Show that if H_1 and H_2 are Hilbert spaces and $T \in B(H_1, H_2)$, then $\|T^*T\| = \|T\|^2$, where T^* is the Hilbert-adjoint operator of T. [2 Marks]

Section II (Answer any Two questions: 2×14 marks = 28 marks)

(2) (a) Show that if X and Y are normed spaces, then a linear operator $T : X \to Y$ is bounded if and only if it is continuous at 0 of X. [5 Marks]

(b) Show that the dual space X' of a normed space X is a Banach space. [5 Marks]

(c) Show that a reflexive normed space is complete. [4 Marks]

(3) (a) Show that any two norms on a finite dimensional vector space X are equivalent. What happens if X is infinite dimensional? Justify your answer. [3+3= 6 Marks]

(b) Give an example, with justification, of a normal operator which is not unitary. [2 Marks]

(c) State and prove the projection theorem. [1+5= 6 Marks]

(4) (a) Can every norm on a vector space be obtained from an inner product? Justify your answer. [4 Marks]

(b) Let T be a bounded linear operator on a complex inner product space X such that $\langle Tx, x\rangle = 0$ for all $x \in X$. Show that $T = O$. What happens if X is real inner product space? Justify your answer. [3+1= 4 Marks]

(c) State and prove the Riesz's representation theorem for bounded sesquilinear forms. [1+5= 6 Marks]

Section III (Answer any Two questions: 2×14 marks $= 28$ marks)

(5) (a) Show that the resolvent operator $R_\lambda(T)$ of a bounded linear operator T on a complex Banach space is locally holomorphic on the resolvent set $\rho(T)$. [5 Marks]

(b) Show that if Y is a proper closed subspace of a normed space X and $x_o \in X \setminus Y$, then there exists a $\tilde{f} \in X'$ such that $\tilde{f}(x_o) = 1$, $\tilde{f}(Y) = \{0\}$, and $\|\tilde{f}\| = \frac{1}{\text{dist}(x_o,Y)}$. [5 Marks]

(c) Define the canonical embedding. Show that the canonical embedding is a linear isometry. [1+1+2= 4 Marks]

(6) (a) Show that if $X \neq \{0\}$ is a complex Banach space and $T \in B(X)$, then $\sigma(T)$ is compact. [5 Marks]

(b) Show that if X and Y are Banach spaces and $T \in B(X, Y)$ is surjective, then the image $T(B_o)$ of an open unit ball $B_o = B(0,1) \subset X$ contains an open ball about 0 of Y. [6 Marks]

(c) Show that if X and Y are normed spaces and $T \in B(X, Y)$, then the adjoint operator $T^\times$ of T is linear and bounded. [1+2= 3 Marks]

(7) (a) Define weak convergence of a sequence $< x_n >$ in a normed space X. Show that $< x_n > \subset X$ converges weakly to $x \in X$ if and only if $< \|x_n\| >$ is bounded and $f(x_n) \to f(x)$ for every f in a total subset $M \subset X'$. [1+4= 5 Marks]

(b) Give an example of a linear operator T such that $\sigma_p(T)$ is contained properly in $\sigma(T)$. Justify your answer. [1+2= 3 Marks]

(c) State and prove the closed graph theorem. [1+5=6 Marks]

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Your Roll Number: ....

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examinations, May 2019
Part I Semester II
MATH14-204: FLUID DYNAMICS

Time: 3 hours | Maximum Marks: 70

Instructions: • Question No. 1 is compulsory. • Answer **Two** questions each from Sections B and C. • All the symbols have their usual meaning unless otherwise mentioned.

Section A

(1) Answer all parts:

(a) Prove that vortex lines and tubes can not originate or terminate at internal points in a fluid. [3]

(b) For an incompressible inviscid fluid moving under an arbitrary body force $\mathbf{F}$ per unit mass, show that the generation of vorticity is determined by the equation $d\zeta/dt = (\zeta.\nabla)\mathbf{q} + \nabla \times \mathbf{F}$. [3]

(c) Show that the velocity potential ϕ at a point P for a simple source of strength m at O is given by $\phi(r) = m/r$, where $r = |\mathbf{OP}|$. [3]

(d) Determine the image of a line source in a circular cylinder. [3]

(e) Find the dimensions of coefficient of viscosity and kinematic coefficient of viscosity. [2]

Section B (Answer any two questions)

(2) (a) Derive the equation of continuity for a region of fluid, which contains neither sources nor sinks. Using equation of continuity and suitable hypotheses, show that velocity potential is harmonic. [5+2]

(b) Determine the acceleration of a fluid particle, if the velocity of flow is $(r^2 z cos\theta, rzsin\theta, z^2 t)$. [3]

(c) Show that $(x^2/a^2)tan^2 t + (y^2/b^2)cot^2 t = 0$ is a possible form for the bounding surface of a liquid, and find the normal velocity. [4]

(3) (a) Show that at any point P of a moving inviscid fluid the pressure is same in all directions. [7]

(b) If Σ is the solid boundary of a large spherical surface of radius R, containing fluid in motion and also enclosing one or more closed surfaces, If ϕ_P denotes the potential at any point P of the fluid, the $\phi_P \to C$ as $P \to \infty$, where C is a constant, provided that the fluid extends to infinity and is at rest there. [7]

(4) (a) In a two dimensional motion, a source of strength m is placed at each of points $(-1, 0)$ and $(1, 0)$, and a sink of strength $2m$ is placed at the origin.

Show that the streamlines are the curves: $(x^2 + y^2)^2 = x^2 - y^2 + kxy$, where k is a parameter. [6]

(b) State and prove Milne-Thomson's circle theorem. [5]

(c) Discuss the flow for which $w = z^2$. [3]

Section C (Answer any two questions)

(5) (a) Prove that the force exerted on a circular cylinder $|z| = a$ in the irrotational flow produced by a line source m at $z = 3a$ is $X = \rho m^2/(48\pi a)$, $Y = 0$. [6]

(b) A three-dimensional doublet of strength μ whose axis is in the direction **OX** is distant a from the rigid plane $x = 0$ which is the sole boundary of liquid of density ρ, infinite in extent. Find the pressure at a point on the boundary distant r from the doublet given that pressure at infinity is p_∞. Show that the pressure on the plane is least at a distance $a\sqrt{5}/2$ from the doublet. [6]

(c) Find the image of a simple source of strength m at $(c, 0, 0)$ in a solid sphere. [2]

(6) (a) Define axi-symmetric flows. Show that Stokes's Stream function is not harmonic. [1+3]

(b) State and prove Kelvin's inversion theorem. [6]

(c) Determine the Stokes's Stream function for a doublet of strength M situated at origin whose axis is along **OZ**. [4]

(7) (a) Show that the stress matrix is diagonally symmetric and contains only six unknowns. [6]

(b) Write the relations between stress and rate of strain in tensor form. [2]

(c) Consider the steady flow between two concentric infinite cylinders of radii a, b $(b > a)$ with viscous liquid in between. The inner cylinder is held at rest whilst the outer is rotated with constant angular velocity Ω. Show that the fluid velocity at radial distance R has the magnitude $q = \Omega b^2(R - a^2R^{-1})(b^2 - a^2)^{-1}$. [6]