Your Roll Number: ....

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations, November-December 2018
Part II Semester III

MATH14-301(A): Algebraic Topology

Time: 3 hours | Maximum Marks: 70

Instructions: • Write your roll number on the space provided at the top of this page immediately on receipt of this question paper • Answer **any five** questions • Each question carries 14 marks.

(1) (a) Let X be starlike with respect to origin and compact subsapce of the euclidean sapce $\mathbb{R}^n$. If $f : X \to \mathbb{S}^1$ is a continuous map such that $f(0) = 1$ then show that, for each integer m, there exists $\tilde{f} : X \to \mathbb{R}$ such that $p\tilde{f} = f$ and $\tilde{f}(0) = m$, where p is the exponential map. [10 Marks]

(b) Show that 2-sphere $\mathbb{S}^2$ cannot be embedded in $\mathbb{R}^2$. [4 Marks]

(2) (a) Prove that any continuous map $f : \mathbb{S}^1 \to \mathbb{R}$ can be extended continuously over $\mathbb{D}^2$. Can the identity map $1_{\mathbb{S}^1} : \mathbb{S}^1 \to \mathbb{S}^1$ extended continuously over $\mathbb{D}^2$? Justify [5+3 Marks]

(b) Prove that $\mathbb{R}^n - \{0\}$ is simply connected for $n > 2$. [6 Marks]

(3) (a) Prove that the expotential map $p : \mathbb{R} \to \mathbb{S}^1$ is a covering map. Find all connected covering spaces of $\mathbb{S}^1$. [6+4 Marks]

(b) Let a space X be homotopically equivalent to 2-sphere $\mathbb{S}^2$. Prove that X is path connected. [4 Marks]

(4) (a) Define the degree of a continuous map $f : \mathbb{S}^1 \to \mathbb{S}^1$. Find the degree of f defined by $f(z) = z^n$, for all $z \in \mathbb{S}^1$. [1+3 Marks]

(b) Does there exist a lifting of the identity map $1_{\mathbb{S}^1} : \mathbb{S}^1 \to \mathbb{S}^1$ with respect to covering map $p : \mathbb{R} \to \mathbb{S}^1$, where p is the exponential map ? [3 Marks]

(c) State and prove the fundamental theorem of algebra. [7 Marks]

(5) (a) Give an example of deformation retract that is not a strong deformation retract. [6 Marks]

(b) Let $p : \widetilde{X} \to X$ be a covering map, and $f : I \to X$ be a path with origin x_0. If $\tilde{x}_0 \in p^{-1}(x_0)$ then show that there exists a path $\tilde{f} : I \to \widetilde{X}$ with $p\tilde{f} = f$ and $\tilde{f}(0) = \tilde{x}_0$. [8 Marks]

(6) (a) Define regular covering map. Let $p : \widetilde{X} \to X$ be a covering map. [1+5 Marks]

If for any closed path g in X, either every lifting of g is closed or none is closed then prove that p is regular.

(b) Let $p : \widetilde{X} \to X$ be a covering map. Define the action of $\Gamma = \pi_1(X, x)$ on $p^{-1}(x)$, and show that $\Gamma_{\tilde{x}} = p_{\#}(\pi_1(X, \tilde{x}))$ for every $\tilde{x} \in p^{-1}(x)$. If $\widetilde{X}$ is path connected then deduce that the multiplicity of p is the index of $p_{\#}(\pi_1(X, \tilde{x}))$ in $\pi_1(X, x)$. [5+3 Marks]

(7) (a) Show that $\mathbb{S}^1 \times \mathbb{S}^1$ is not a retract of $\mathbb{S}^1 \times \mathbb{D}^2$. [5 Marks]

(b) Show that any two continuous maps from a simply connected and locally path connected space to $\mathbb{S}^1$ are homotopic. [2 Marks]

(c) State Seifert-van Kampen theorem. Find the fundamental group of wedge of two circles. [1+6 Marks]

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Your Roll Number: ....

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examinations, November-December, 2018
Part II, Semester III

MATH14 301(B): REPRESENTATION OF FINITE GROUPS

Time: 3 hours Maximum Marks: 70

Instructions: • Write your roll number on the space provided at the top of this page immediately on receipt of this question paper. • Attempt any **five** question. • All questions carry equal marks. • G will denote a finite group and F will denote a scalar field.

(1) (a) Let $G = \langle a, b \mid a^4 = 1, a^2 = b^2 = 1, b^{-1}ab = a^{-1}\rangle$. Prove that the map $\rho : G \longrightarrow GL(n, F)$ given by $(a^r b^s)\rho = A^r B^s$ $(1 \le r \le 4, 0 \le s \le 1)$, where $A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, is a faithful representation of G. [6 Marks]

(b) Let $\phi : G \longrightarrow GL(n, F)$ be a representation of a finite group G. Prove that ϕ gives a faithful representation on $G/\ker\phi$. [2 Marks]

(c) For an FG-module V prove that for any $g \in G$, the map $\theta_g : v \longrightarrow vg$ $\forall v \in V$ defines an endomorphism on V. Also deduce that if $\mathcal{B}$ is any basis of V, then the map $\rho : g \longrightarrow [\theta_g]_{\mathcal{B}}$ is a representation of G. [3+3 Marks]

(2) (a) Define a regular FG-module and prove that it is completely reducible. [2+3 Marks]

(b) Let V be an FG-module. Prove that the group algebra FG acts on V and, as a group algebra is a ring with identity element, also show that this action is linear and defines a module structure on V. [2+7 Marks]

(3) (a) Give some necessary and sufficient condition for two FG-modules to be isomorphic. [3 Marks]

(b) State Maschke's Theorem. Does Maschke's Theorem holds for arbitrary groups over all fields? Justify. [2+5 Marks]

(c) Find a group G, a $\mathbb{C}G$-module V and a $\mathbb{C}G$-homomorphism $\theta : V \longrightarrow V$ such that $V \neq \ker\theta \oplus \operatorname{Im}\theta$. [4 Marks]

(4) (a) Prove that a group G is abelian if and only if all its irreducible characters are of degree 1. [6 Marks]

(b) Prove that $\dim(\operatorname{Hom}_{\mathbb{C}}(\mathbb{C}G, \mathbb{C}G)) = |G|$. [8 Marks]

(5) (a) Prove that degree 1 representations of a group G are in bijec- [7 Marks]

tive correspondence with irreducible representations of the group G/G', where G' is the derived subgroup of G.

[4 Marks]

(b) Assuming that the degree of an irreducible character of a group G divides the order of G, prove that for a prime number p a group of order p^2 is abelian.

[3 Marks]

(c) Let V be a $\mathbb{C}G$-module with character χ and let $W = \{v \in V \mid vg = v, \forall g \in G\}$ be a submodule of V. Prove that $\langle \chi, 1_G \rangle = \dim W$.

(6) (a) Find the character table of $D_6 \times C_3$. Deduce whether $Z(D_6 \times C_3)$ is cyclic or not. [6+1 Marks]

(b) Let V be a $\mathbb{C}G$-module. Decompose the $\mathbb{C}G$-module $V \otimes V$ as a direct sum of a symmetric and a anti-symmetric submodule. [4 Marks]

(c) Prove that every character is a class function. Does the converse holds? Justify [3 Marks]

Your Roll Number: ....

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examinations, Dec. 2018
Part II Semester III

MATH14-301(C): COMMUTATIVE ALGEBRA

Time: 3 hours — Maximum Marks: 70

Instructions: • Write your roll number on the space provided at the top of this page immediately on receipt of this question paper. • Throughout the question paper the word "ring" or R shall mean commutative ring with nonzero identity. **Question 1 is compulsory.**

(1) Attempt any six questions.

- (a) Let R be a ring. If $f(x)$ and $g(x)$ are primitive polynomials in $R[x]$, then show that $f(x)g(x)$ is also primitive. [3 Marks]
- (b) Let M be a finitely generated R-module and $f : R \to S$ be a ring homomorphism. Show that $M \otimes_R S$ is finitely generated S-module. [3 Marks]
- (c) Let R be a local ring. Find a prime ideal $\mathfrak{p} \in Spec(R)$ such that $R \cong R_{\mathfrak{p}}$. [3 Marks]
- (d) Let $\mathfrak{q}$ be a $\mathfrak{p}$-primary ideal in R and $x \in R \setminus \mathfrak{q}$. Show that $(\mathfrak{q} : x)$ is $\mathfrak{p}$-primary. [3 Marks]
- (e) Let $R \subseteq S$ be a ring extension. Show that the integral closure of R in S is integrally closed in S. [3 Marks]
- (f) Show that any Artin ring is Noetherian. [3 Marks]
- (g) Show that every ideal is principal in a discrete valuation ring R. [3 Marks]

Section A
(Answer any two questions)

(2) (a) Let $\phi : R \to S$ be a ring homomorphism. Define extension and contraction of ideals. Is the extension and contraction of prime ideals necessarily prime? Justify your answer. Show that the set of all contracted ideals in R is precisely the set of ideals $\{I \mid I \text{ is an ideal in } R \text{ and } I^{ec} = I\}$. [4 + 3 Marks]

(b) Let R be a ring and $\mathfrak{p}$ be a prime ideal in R. Show that $\mathfrak{p}^e$ is a prime ideal in $R[x]$. For any maximal ideal $\mathfrak{m}$ of R, is $\mathfrak{m}^e$ a maximal ideal in $R[x]$? [4+2 Marks]

(3) (a) Let R be a Noetherian local domain of dimension one with maximal ideal $\mathfrak{m}$. If R is a discrete valuation ring, then show that $\mathfrak{m}$ is principal. [6 Marks]

(b) Define Dedekind domain with example. Show that a domain R is a Dedekind domain if and only if every nonzero fractional ideal of R is invertible. [7 Marks]

(4) (a) Let R be an Artin ring. Show that if $R \cong \prod_{i=1}^{n} R_i$ for some Artin local rings $R_1, \cdots, R_n$, then R determines $R_1, \cdots, R_n$ uniquely, upto isomorphism. [6 Marks]

(b) Let M be an R-module such that every family of finitely generated submodules of M admits maximal elements. Show that M is Noetherian. [4 Marks]

(c) Show that in a Noetherian ring R the nilradical is nilpotent. [3 Marks]

Section B

(Answer any two questions)

(5) (a) Let M, N be R-modules and let $\phi : M \to N$ be an R-linear map. Show that ϕ is surjective if and only if $\phi_{\mathfrak{m}}$ is surjective for all maximal ideals $\mathfrak{m}$ of R. [3 Marks]

(b) Let $\mathfrak{p}$ be a prime ideal in R. Show that the prime ideals of R contained in $\mathfrak{p}$ are in one to one correspondence with prime ideals of $R_{\mathfrak{p}}$. [5 Marks]

(c) Let R be a ring such that $R_{\mathfrak{p}}$ has no nonzero nilpotent element for all $\mathfrak{p} \in Spec(R)$. Show that R has no nonzero nilpotent element. If $R_{\mathfrak{p}}$ is an integral domain for all $\mathfrak{p} \in Spec(R)$. is R necessarily an integral domain? Justify your answer. [5 Marks]

(6) (a) Let $\mathfrak{q}$ be a $\mathfrak{p}$-primary ideal in R. Show that $\mathfrak{q}[x]$ is $\mathfrak{p}[x]$-primary ideal in $R[x]$. [5 Marks]

(b) State and prove the second uniqueness theorem of primary decomposition. Hence show that any decomposable ideal uniquely determines its isolated primary components. [8 Marks]

(7) (a) Let $R \subseteq S$ be rings and S be integral over R. Show that every prime ideal of R is a contraction of some prime ideal in S. Also, show that if $\mathfrak{p}_1 \subset \mathfrak{p}_2$ are prime ideals in R and $\mathfrak{q}_1$ is a prime ideal of S such that $\mathfrak{q}_1^c = \mathfrak{p}_1$, then there exists a prime ideal $\mathfrak{q}_2$ of S such that $\mathfrak{q}_1 \subset \mathfrak{q}_2$ and $\mathfrak{q}_2^c = \mathfrak{p}_2$. [4+4 Marks]

(b) Let R be a subring of S such that the set $S \setminus R$ is closed under multiplication. Show that R is integrally closed in S. [5 Marks]

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Your Roll Number: ....

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations, November-December 2018
Part II Semester III
MATH14-302(A): Fourier Analysis

Time: 3 hours

Maximum Marks: 70

Instructions: • Write your roll number on the space provided at the top of this page immediately on receipt of this question paper • **Question 1 is compulsory** • **Attempt any other four questions.**

(1) (a) Let $f(t) = e^{-|t|}$. Find $\widehat{f}(y)$, verify that $|\widehat{f}(y)| \to 0$ as $|y| \to \infty$ and $\widehat{f}(y_0) = \lim\limits_{y \to y_0} \widehat{f}(y)$. [4 marks]

(b) If $G = \mathbb{R}^2 \times \mathbb{Z} \times \mathbb{Z}(3)$, find the dual group of G. Justify your answer. [3 marks]

(c) Let $f(t) = \begin{cases} 1, & \text{for } |t| \le 1 \\ 0, & \text{for } |t| > 1 \end{cases}$. Verify $|\widehat{f}(y)| \le \|f\|_1$ for all $y \in \mathbb{R}$. [3 marks]

(d) Let G be a locally compact abelian topological group. Show that the map $\phi : G \times \widehat{G} \to \mathbb{T}$ given by $\phi(x, \gamma) = \gamma(x)$ is continuous. [4 marks]

(2) (a) Show that $L^1(\mathbb{T})$ is a commutative Banach algebra under convolution. [5 marks]

(b) If $\{k_n\}$ is a summability kernel in $L^1(\mathbb{T})$, show that $\{k_n\}$ is a bounded approximate identity for $L^1(\mathbb{T})$. [5 marks]

(c) Let G be a topological group and H be a closed subgroup of G. Show that G/H is a Hausdorff topological space under quotient topology. [4 marks]

(3) (a) Show that the set of trigonometrical polynomials are dense in $L^1(\mathbb{T})$. [4 marks]

(b) If $f \in L^1(\mathbb{T}), t_0 \in \mathbb{T}$ such that $\int\limits_0^{2\pi} \frac{|f(t)-f(t_0)|}{|t-t_0|} dt < \infty$, then show that $S_N f(t_0) \to f(t_0)$ as $N \to \infty$. [5 marks]

(c) Let $f \in C_{00}(\mathbb{R})$, the space of all continuous functions having compact support. Show that $f \to \widehat{f}$ is an isometry in $\|\cdot\|_2$. [5 marks]

(4) (a) Show that $\{f \in L^1(\mathbb{R}), \widehat{f} \text{ has compact support}\}$ is dense in $L^1(\mathbb{R})$. [4 marks]

(b) Let $f \in L^1(\mathbb{R})$, define $F(x) = \int\limits_{-\infty}^{x} f(t)dt$. If $F \in L^1(\mathbb{R})$, show that $\widehat{F}(y) = \frac{1}{iy}\widehat{f}(y), 0 \neq y \in \mathbb{R}$. [5 marks]

(c) Show that the connected component of identity in a topological group G is a normal subgroup of G. Illustrate this result for $G = O(n), n \times n$ orthogonal matrices over $\mathbb{R}$. [5 marks]

(5) (a) Give a function in $C_0(\mathbb{R} \times \mathbb{Z})$ which is not in $C_{00}(\mathbb{R} \times \mathbb{Z})$. Justify your answer. [4 marks]

(b) Let G be a locally compact group with left Haar measure λ. For $f \in L^p(G), 1 \leq p < \infty$, show that the map $\phi : G \to L^p(G)$, given by $\phi(x) = {}_x f$ is uniformly continuous. [5 marks]

(c) For G as in part (b) above, let $f \in L^1(G), g \in L^\infty(G)$. Show that $f * g(x)$ exists for all $x \in G$ and $f * g$ is bounded and uniformly continuous. [5 marks]

(6) (a) Let G be an abelian locally compact group. If $\gamma \in \widehat{G}$, the dual group of G, show that the map $f \to \widehat{f}(\gamma)$ is a complex homomorphism of $L^1(G)$ and is not identically zero. Show further that every non-zero complex homomorphism of $L^1(G)$ is obtained in this way. [9 marks]

(b) If $G = \mathbb{T}$, show that $\widehat{G} = \{\chi_n : n \in \mathbb{Z}\}, \chi_n(t) = e^{int}, n \in \mathbb{Z}$. [5 marks]

(7) (a) Define P-topology and Δ-topology on $\widehat{G}$, where G is as in 6(a). Show that both the topologies are same. [7 marks]

(b) Let G be as in $6(a)$. Show that there is a continuous isomorphism of G into $\widehat{\widehat{G}}$. [5 marks]

(c) Let $G = \left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \sim \{0\}, y \in \mathbb{R} \right\}$ with left Haar measure. If $f = \chi_{[2,4] \times (0,1)}$, find $f * f(1,0)$. [2 marks]

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Your Roll Number: ....

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examinations, December 2018
MATH14-302(B): MATRIX ANALYSIS

Time: 3 hours | Maximum Marks: 70

Instructions: • Answer any **Two** questions from **Section A** and **Three** from **Section B**. • Answer all parts at the same place. • All symbols have their usual meaning unless otherwise specified.

Section A
(Answer any **Two** questions)

(1) (a) What do you mean by a matrix Lie group? Does the collection of all n-by-n special unitary matrices form the matrix Lie group? Justify. [4 Marks]

(b) Determine whether the set of all n-by-n doubly substochastic matrices is convex and compact. [4 Marks]

(c) Show that the group $SL(n;\mathbb{C})$ is connected for all $n \geq 1$. [6 Marks]

(2) (a) Define the symplectic group $S_p(n;\mathbb{R})$. Is it bounded? Justify. [3 Marks]

(b) Verify that the maximum column sum matrix norm defined on $M_n(\mathbb{C})$ is induced by the ℓ_1-norm on $\mathbb{C}^n$. [5 Marks]

(c) Give an example of a vector norm on $\mathbb{R}^2$ which is not monotone. Justify your answer. [3 Marks]

(d) If $\lim_{k\to\infty} A^k = 0$ for any $A \in M_n(\mathbb{C})$, then show that $\rho(A) < 1$. [3 Marks]

(3) (a) Let $|||\cdot|||$ be a matrix norm on M_n which is induced by a norm $\|\cdot\|$ on $\mathbb{C}^n$. Show that $|||\cdot|||'$ is induced by the norm $\|\cdot\|^D$. [3 Marks]

(b) State and prove the Gelfand formula. [5 Marks]

(c) Establish an expression for the upper bound on the relative error in a solution to the system $Ax = b$, $A \in M_n(\mathbb{C})$ is nonsingular and $b \in \mathbb{C}^n$ is nonzero, as a function of the relative error in the data and the condition number of the matrix A. [6 Marks]

Section B
(Answer any **Three** questions)

(4) (a) Use the Gerŝgorin disc theorem to verify that the matrix A has distinct eigenvalues, where [3 Marks]

$$A = \begin{bmatrix} 1 & 1/2 & i/2 \\ 1/2 & 3 & 0 \\ 0 & 1 & 5 \end{bmatrix}.$$

(b) State and prove the Fan determinant inequality. [5 Marks]

(c) Let $A = [a_{ij}] \in M_n(\mathbb{C})$ and $0 < \alpha < 1$. If R_i' and C_i' denote the deleted row and column sums of A, then show that the eigenvalues of A are in the union of n discs [6 Marks]

$$\bigcup_{i=1}^{n}\{z \in \mathbb{C} : |z - a_{ii}| \leq R_i'^{\alpha} C_i'^{1-\alpha}\}.$$

(5) (a) Prove that $A \in M_n(\mathbb{C})$ is a Gram matrix if and only if there is a sequence of positive definite matrices $A_1, A_2, \dots$ such that $A_k \to A$ as $k \to \infty$. [4 Marks]

(b) We denote $A \succeq 0$ if A is Hermitian and positive semidefinite. Let $A, B \in M_n$ be Hermitian such that $A \succeq B$ and $C \succeq 0$. Then explain that $A \circ C \succeq B \circ C$. [4 Marks]

(c) Let $A \in M_{n,m}(\mathbb{C})$, $q = \min\{n, m\}$, and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_q$ be the ordered singular values of A. Then show that there exists a matrix [6 Marks]

$$\mathcal{A} = \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix},$$

and the ordered eigenvalues of $\mathcal{A}$ are
$-\sigma_1 \leq -\sigma_2 \leq \cdots \leq -\sigma_q \leq \underbrace{0 = \cdots = 0}_{|n-m|} \leq \sigma_q \leq \cdots \leq \sigma_2 \leq \sigma_1.$

(6) (a) Suppose that the algebraic multiplicity of $\rho(A)$ as an eigenvalue of a positive matrix $A \in M_n(\mathbb{C})$ is 1. If x and y are the right and left Perron vectors of A, then explain that [8 Marks]

$$\lim_{m\to\infty} (\rho(A)^{-1}A)^m = xy^T.$$

(b) Let the ordered singular values of A, $B \in M_n(\mathbb{C})$ be $\sigma_1(A) \geq \sigma_2(A) \geq \cdots \geq \sigma_n(A)$ and $\sigma_1(B) \geq \sigma_2(B) \geq \cdots \geq \sigma_n(B)$. Then verify that $Re\ tr(AB) \leq \sum_{i=1}^{n} \sigma_i(A)\sigma_i(B)$. [6 Marks]

(7) (a) Let $A \in M_n(\mathbb{C})$ be positive definite and suppose that $A = C^*C$ with $C \in M_n(\mathbb{C})$. Show that there is a unitary $V \in M_n(\mathbb{C})$ such that $C = VA^{1/2}$. [3 Marks]

(b) Show that for a doubly stochastic matrix $A \in M_n(\mathbb{C})$ there exist permutation matrices $P_1, P_2, \cdots, P_N \in M_n(\mathbb{C})$ and positive scalars $t_1, t_2, \cdots, t_N$ such that $t_1 + t_2 + \cdots + t_N = 1$ and $A = t_1P_1 + t_2P_2 + \cdots + t_NP_N$. Moreover $N \leq n^2 - n + 1$. [5 Marks]

(c) Let $A \in M_n(\mathbb{C})$ be positive semidefinite, $r = rank(A)$. Then prove that for $k = 2, 3, \cdots$, there is a unique positive semidefinite matrix B such that $B^k = A$. Also show that there is a polynomial p with real coefficients such that $B = p(A)$. [6 Marks]

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Your Roll Number: ....

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations, November-December 2018
Part II Semester III

MATH14-302(C): Theory of Bounded Operators

Time: 3 hours Maximum Marks: 70

Instructions: • Write your roll number on the space provided at the top of this page immediately on receipt of this question paper • Answer **five** questions in all • All symbols have their usual meaning.

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(1) (a) State the Spectral Mapping Theorem for Polynomials for operators on Banach spaces. Use it to determine the spectrum of an orthogonal projection P defined on a complex Hilbert space H. Further, compute $\sigma_p(P)$ and $\sigma_{comp}(P)$. (1+3)

(b) Prove that range of a compact operator is always separable. (4)

(c) Show that if T is a compact operator on a Banach space X then TS and ST are comapct operators for every bounded operator S defined on X. Show further that if X is a Hilbert space, and T^* is the Hilbert adjoint of T then the compactness of T^*T implies that of T. (4+2)

(2) (a) Determine whether or not the operator $T : l^2 \to l^2$ given by $Tx = (x_1, 0, x_3, 0, \ldots)$ where $x = (x_1, x_2, \ldots) \in l^2$ is compact. Justify your answer. (4)

(b) Show that if the operator T defined on a Banach space X is compact and $\{x_n\} \subset X$ is a weakly convergent sequence then $\{Tx_n\}$ is (norm) convergent. (5)

(c) Show that $x \in X$ is a solution of the equation $Tx - \lambda x = y$ if and only if $y \in X$ is such that $f(y) = 0$, for all $f \in X'$ satisfying $T^*f - \lambda f = 0$. (5)

(3) Let $T : H \to H$, be a bounded self adjoint operator on a complex Hilbert space H.

(a) Show that $\sigma(T) \subset [m, M]$ where $m = \inf_{\|x\|=1}\langle Tx, x\rangle$ and $M = \sup_{\|x\|=1}\langle Tx, x\rangle$. (4)

(b) Show further that $\sigma(T) \subset [0, \infty)$ if and only if T is positive. Hence or otherwise prove that $(I + T^*T)^{-1}$ exists as a bounded operator on H. (4)

(c) Show that if T is positive and S is a positive operator commuting with T then ST is also positive. (6)

(4) (a) State and prove a necessary and sufficient condition for the sum of two projections to be a projection. Further, determine the range of the sum projection in terms of the ranges of the two given projections. (5)

(b) Let $Q_1, Q_2, \ldots$ be projections on a Hilbert space H such that $Q_i(H) \perp Q_j(H)$ for $i \neq j$. Show that for every $x \in H$ the series $Qx = \sum_{j=1}^{\infty} Q_j x$ converges (in norm on H) and Q is a projection. Find the range of the projection Q. (4)

(c) Let Y be a closed subspace of a Hilbert space H, P be the orthogonal projection of H onto Y and T be a bounded linear operator on H. Show that $TP = PT$ if and only if $T(Y) \subset Y$ and $T(Y^{\perp}) \subset Y^{\perp}$. (5)

(5) (a) What is meant by a spectral family on a Hilbert space? Let H be an n dimensional Hilbert space and T be a self adjoint linear operator on H with distinct eigenvalues $\{\lambda_1, \lambda_2, \ldots \lambda_n\}$. Let P_j be the orthogonal projection of H onto the eigenspace of T corresponding to λ_j. Set $E_\lambda = \sum_{\lambda_j \leq \lambda} P_j$. Show that $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ forms a spectral family on H. (1 + 3)

(b) Let T be a bounded self adjoint operator on a Hilbert space H. Show that there exist bounded self adjoint operators T^+, T^- and E which commute with T and with each other and satisfy (i) $T = T^+ - T^-, T^+T^- = 0$ (ii) $T^+E = 0$ (iii) $T^-E = T^-$ (iv) $TE = -T^-, T(I-E) = T^+$. (7)

(c) Find the spectral family associated with the self adjoint operator T given by $Tx = 4x, x \in H$, where H is a Hilbert space. (3)

(6) (a) What is meant by a partial isometry? Give an example, with justification, of a partial isometry which is not an isometry nor a projection. (1+2)

(b) Let W be a bounded linear operator on a Hilbert space H. Show that then the following are equivalent: (i) W is a partial isometry (ii) W^*W is a projection (iii) $WW^*W = W$ (6)

(c) Show that every operator T on an infinite dimensional Hilbert space H can be written as $A = W|A|$, where W is a partial isometry. (5)

(7) (a) Show that every compact operator on a Hilbert space admits a singular value decomposition. (4)

(b) What is meant by a *trace class operator*? Define the trace $tr A$ of a trace class operator A on a Hilbert space H and show that it is independent of the choice of orthonormal basis of H. Find the trace of the operator $T : l^2 \to l^2$ where $T(x_1, x_2, \ldots) = (x_1, x_2, x_3, x_4, 0, \ldots)$. (1+3+1)

(c) Let C_1 be the set of all trace class operators on a Hilbert space H. Show that C_1 is a normed space with respect to an appropriate norm. (5)

Your Roll Number: ....

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations, November-December 2018
Part II Semester III

MATH14-303(A): Advanced Complex Analysis

Time: 3 hours | Maximum Marks: 70

Instructions: • Write your roll number on the space provided at the top of this page immediately on receipt of this question paper.
• **Question no. 1 is compulsory.**
• Attempt any **five(5)** parts from question no. 1 and any other **five(5)** questions from questions 2 - 8.
• The figure in the margin indicates full mark for the question.

(1) (a) Show that if a differentiable function is convex then its derivative is an increasing function. [3 Marks]

(b) Show that if a sequence $\{f_n\}$ in $(C(G,\Omega),\rho)$ converges to f then $\{f_n\}$ converges to f uniformly on all compact subsets of G. [3 Marks]

(c) If $|z| < \frac{1}{2}$ show that $\frac{1}{2} \leq |log(1+z)| \leq \frac{3}{2}|z|$. [3 Marks]

(d) Define *simply connected region* in six different ways. [3 Marks]

(e) State *Harnack's inequality*. Show that if u is harmonic, then $f = u_x - iu_y$ is analytic. [3 Marks]

(f) If f is analytic on the disk $B(a;r)$ such that $|f'(z) - f'(a)| < |f'(a)|$ far all z in $B(a;r), z \neq a$, then show that f is one-to-one. [3 Marks]

(2) (a) Let $a < b$ and let G be the vertical strip $G = \{x+iy : a < x < b\}$. Suppose that $f : G \to \mathbb{C}$ is continuous, f is analytic in G, $|f(z)| < B$ for all z in G and $|f(w)| \leq 1$ for w in ∂G. Show that $|f(z)| \leq 1$ for all z in G. [5 Marks]

(b) State and prove *Phragmen-Lindelöf Theorem*. [6 Marks]

(3) (a) Let $\{f_n\} \subset H(G)$ be a sequence of one-to-one functions which converges to f. Show that either f is one-to-one or f is a constant. Give an example of a sequence of one-to-one functions converging to a constant function. [5 Marks]

(b) If $\mathcal{F} \subset C(G,\Omega)$ is *normal*, show that $\mathcal{F}$ is *equicontinuous* at each point of G and, for each $z \in G$, the set $\{f(z) : f \in \mathcal{F}\}$ has *compact closure* in Ω. [6 Marks]

(4) (a) Show that a *locally bounded* family in $\mathcal{H}(G)$ is normal. [5 Marks]

(b) Show that $cos\pi z = \prod_{n=1}^{\infty}[1 - \frac{4z^2}{(2n-1)^2}]$. [6 Marks]

(5) (a) Let γ be a rectifiable curve and K be a compact set such that $K \cap \{\gamma\} = \phi$. If f is a continuous function on $\{\gamma\}$ and $\epsilon > 0$, show that there is a rational function R having all it's poles on $\{\gamma\}$ such that $|\int_\gamma \frac{f(w)}{w-z}dw - R(z)| < \epsilon$ for all $z \in K$. [5 Marks]

(b) State and prove *Mittag-Leffler's theorem.* [6 Marks]

(6) (a) Define the *Mean Value Property*(MVP) for a real-valued continuous function f on G. Prove *Maximum Principle* for a continuous function with the MVP on a region G. [5 Marks]

(b) Let $D = \{z : |z| < 1\}$ be the unit disk and $f : \partial D \longrightarrow \mathbb{R}$ be continuous. It is known that the function u defined by $u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)f(e^{it})dt$, where $0 \leq r < 1$, and $u(e^{i\theta}) = f(e^{i\theta})$ for $r = 1$ is harmonic in D. Show that u is continuous on $\overline{D}$. [6 Marks]

(7) (a) Show that the *Poisson kernel* $P_r(\theta)$ satisfies $\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta)d\theta = 1$ and that $P_r(\theta) \leq P_r(\delta)$ if $0 < \delta < |\theta| \leq \pi$. [5 Marks]

(b) Let $D = \{z : |z| < 1\}$ be the unit disk. Show that if f is analytic on a region containing $\overline{D}$ with $f(0) = 0$ and $f'(0) = 1$, then $f(D)$ contains a disk of radius L,where L is the *Landau's constant*. Deduce that if f is analytic on a region containing $\overline{B}(0; R)$, then $f(B(0 : R))$ contains a disk of radius $R|f'(0)|L$ [6 Marks]

(8) (a) Write a note on *Bloch's constant* B. Give an example of analytic function f which is one-to-one on a disk $S \subset D$ so that $f(S)$ contains a disk of radius $> B$. [5 Marks]

(b) If f is analytic on a simply connected region G and $f(z) \neq 0, 1$, show that there is an analytic function g on G such that [6 Marks]

$$f(z) = -\exp(i\pi \cosh[2g(z)])$$

and $g(G)$ contains no disk of radius 1.

Your Roll Number: ....

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examinations, November 2018
Part II Semester III
MATH14 303(B): MEASURE THEORY

Time: 3 hours

Maximum Marks: 70

Instructions: • Write your roll number on the space provided at the top of this page immediately on receipt of this question paper • Answer any **three** questions from Section I, **two** from Section II and **two** from Section III • The symbols used have their usual meanings. In section III, X will denote a locally compact Hausdorff space.

Section I

(Answer any three questions)

(1) (a) Let f be an integrable function on a measure space $(X, \mathcal{A}, \mu)$. Show that $\nu(E) = \int_E f d\mu$, $E \in \mathcal{A}$, defines a signed measure on $(X, \mathcal{A})$ and determine a Hahn decomposition for ν. [3+2 Marks]

(b) State Lebesgue decomposition theorem and prove only the uniqueness part. [1+4 Marks]

(2) (a) Let ν be a signed measure on $(X, \mathcal{A})$ and $E \in \mathcal{A}$ such that $0 < \nu E < \infty$. Show that there is a positive set $A \subset E$ with $\nu A > 0$. [6 Marks]

(b) Define absolute continuity and mutual singularity of measures and give one example each with justification. [4 Marks]

(3) (a) Let μ be a σ-finite measure and T be a bounded linear functional on $L^p(\mu)$ with $1 < p < \infty$. Show that there is a g in $L^q(\mu)$ with $1/p + 1/q = 1$ such that $T(f) = \int fg d\mu$, $\forall f \in L^p(\mu)$, by assuming that the result holds for any finite measure μ. [6 Marks]

(b) Show that for any finite signed measures ν, ν_1, ν_2 on $(X, \mathcal{A})$, and $\alpha \in \mathbb{R}$, $|\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|$ and that $|\alpha\nu| = |\alpha||\nu|$. [4 Marks]

(4) (a) Let $(X, \mathcal{A}, \mu)$ be a finite measure space and f, g nonnegative measurable functions on X such that $\int_E f d\mu = \int_E g d\mu \; \forall E \in \mathcal{A}$. Then show that $f = g$ a.e.. [5 Marks]

(b) Let $1 \leq p < \infty, \mu$ a σ-finite measure and $g \in L^q(\mu)$, $1/p + 1/q = 1$. Show that the bounded linear functional T on $L^p(\mu)$ defined by $T(f) = \int fg d\mu$, $f \in L^p(\mu)$, has norm $\|T\| = \|g\|_q$. [5 Marks]

Section II

(Answer any two questions)

(5) (a) Let μ be a σ-finite measure on an algebra $\mathcal{A}$, and μ^* be the induced outer measure. Show that a set E is μ^*- measurable if and only if

E is the proper difference $A \setminus B$ of a set $A \in \mathcal{A}_{\sigma\delta}$ and a set B with $\mu^*(B) = 0$. [5 Marks]

(b) Let $X = \mathbb{Q}$ and $\mathcal{A}$ be the algebra of finite unions of intervals of the form $(a, b]$ or (a, ∞), where $a \in \mathbb{R} \cup \{-\infty\}$. Consider the measure μ on $\mathcal{A}$ defined by $\mu(E) = \infty$ if $E \neq \emptyset$ and $\mu(\emptyset) = 0$. Show that the σ-algebra $\mathcal{A}'$ generated by $\mathcal{A}$ is the power set of $\mathbb{Q}$ and that the extension of μ to $\mathcal{A}'$ is not unique. [5 Marks]

(6) (a) Let μ^* be an outer measure. Then show that the collection $\mathcal{A}$ of all μ^*-measurable sets is a σ-algebra [6 Marks]

(b) Define the Lebesgue outer measure m_n^* on $\mathbb{R}^n$ and show that it is translation invariant. [4 Marks]

(7) (a) Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be complete measure spaces. Give the detailed construction of the product measure $\mu \times \nu$ on $X \times Y$. [6 Marks]

(b) State Fubini's theorem and illustrate it with an example. [4 Marks]

Section III

(Answer any two questions)

(8) (a) Define Baire sets of X and show that the class $\mathcal{B}a(X)$ of Baire sets is the σ-algebra generated by the compact G_δ sets of X. [1+5 Marks]

(b) Let F be a closed subset of X. Show that the Borel sets of F are those Borel sets of X which are contained in F. [4 Marks]

(9) (a) Let F be a positive linear functional on $C_c(X)$ and μ' be the regular Borel measure on X induced by F. Show that $F(f) = \int_X f d\mu'$, $f \in C_c(X)$. [7 Marks]

(b) Prove that every σ-compact open set in X is the union of a countable collection of compact G_δ sets. [3 Marks]

(10) (a) Let F be a positive linear functional on $C_c(X)$ and μ^* be its induced outer measure. Show that $\mu^*(K) < \infty$ for every compact set $K \subset X$, and that $\mu^*(K_1 \sqcup K_2) = \mu^*(K_1) + \mu^*(K_2)$ for disjoint compact subsets K_1, K_2 of X. [6 Marks]

(b) Define regularity of Baire measures on X and show that the Dirac measure δ_0 on $(\mathbb{R}^n, \mathcal{P}(\mathbb{R}^n))$ is regular, where $\mathcal{P}(\mathbb{R}^n)$ is the power set of $\mathbb{R}^n$. [1+3 Marks]

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Your Roll Number: ....

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examinations, December 2018
Part -II Semester- III

MATH 14-303(c): TOPOLOGY II

Time: 3 hours | Maximum Marks: 70

Instructions: • Write your roll number on the space provided at the top of this page immediately on receipt of this question paper • Question 1 is compulsory and answer any 5 questions from Question 2 to Question 8. • All notations are standard.

Q. 1 State, giving reasons, whether following statements are TRUE or FALSE. (Try any FIVE).

(a) Every locally compact subspace of a space X is locally closed. [2 Marks]

(b) The one-point compactification of a second countable, locally compact, Hausdorff space is metrizable. [2 Marks]

(c) If $f : X \to Y$ and $g : Y \to Z$ are continuous onto maps and $g \circ f$ is proper, then g is proper. [2 Marks]

(d) The cone CS^1 over the unit circle S^1 is homeomorphic to the closed disc D^2. [2 Marks]

(e) Every completely regular space is paracompact. [2 Marks]

(f) If X is a normal space and $A \subseteq X$ is closed, then X/A is a normal space. [2 Marks]

Q.2 (a) Let X be a space with an equivalence relation $'\sim'$. If the quotient space $X/_\sim$ is Hausdorff, then prove that $C = \{(x, x') \in X \times X | x \sim x'\}$ is closed in $X \times X$. [3 Marks]

(b) Let $f : X \to Y$ be a continuous surjection. Prove that f is an identification map if and only if for any space Z, the continuity of a function $g : Y \to Z$ follows from that of $g \circ f : X \to Z$. [4 Marks]

(c) Let $f : X \to Y$ be an identification map such that $f^{-1}(y)$ is connected for each $y \in Y$. Prove that X is connected if and only if Y is connected. [5 Marks]

Q. 3 (a) Let $\{X_\alpha\}_{\alpha \in A}$ be a family of spaces, then prove that $\prod X_\alpha$ is locally compact if and only if each X_α is locally compact and all but finitely many X_α are compact. [5 Marks]

(b) Prove that the one-point compactification X^* of a space X is Hausdorff if and only if X is Hausdorff and locally compact and in this case, any homeomorphism between X and the complement of a single point of a compact Hausdorff space Y extends to a homeomorphism between X^* and Y. [7 Marks]

Q. 4 (a) Let $f : X \to Y$ be a proper surjection. Prove that if Y is Lindelöf, then X is also Lindelöf. [3 Marks]

(b) State and prove Urysohn Lemma for normal spaces. [9 Marks]

Q. 5(a) Prove that every metrizable space is normal. [4 Marks]

(b) Let X be a normal space and $A \subseteq X$ be closed then prove that any continuous map $f : A \to \mathbb{R}$ can be extended to a continuous map $g : X \to \mathbb{R}$. [5 Marks]

(c) Let X be a completely regular space and $F \subseteq X$ be closed. Show that for each $x \in (X \setminus F)$, there exists a continuous function $f : X \to [0, 1]$ such that f is 0 on F and 1 on a neighbourhood of x. [3 Marks]

Q. 6(a) Prove that a completely regular space X is connected if and only if its Stone-Čech compactification $\beta(X)$ is connected. [4 Marks]

(b) Prove that a completely regular, second countable space is metrizable. [8 Marks]

Q. 7 (a) Prove that every paracompact Hausdorff space is a regular space. [3 Marks]

(b) Prove Nagata-Smirnov metrization theorem. [9 Marks]

Q. 8 (a) Let X be a paracompact Hausdorff space and $\{U_\alpha | \alpha \in \mathcal{A}\}$ be an open covering of X. Then prove that, there is a locally finite open refinement $\{V_\alpha | \alpha \in \mathcal{A}\}$ of U_α such that $\overline{V_\alpha} \subseteq U_\alpha$ for every $\alpha \in \mathcal{A}$. [5 Marks]

(b) Use (a) to prove that every open covering of a paracompact Hausdorff space has a partition of unity subordinate to it. [7 Marks]

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Your Roll Number: ....

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examinations, November-December 2018
Part II Semester III
MATH14-304(B): COMPUTATIONAL FLUID DYNAMICS

Time: 3 hours

Maximum Marks: 70

Instructions: • Write your roll number on the space provided at the top of this page immediately on receipt of this question paper • Answer any **five** of the following questions • Each question carries 14 marks as mentioned within brackets.• Symbols have their usual meaning. • Use of scientific calculator only for doing numerical calculations is allowed in this examination.

(1) Write the Beam-Warming and the Lax-Wendroff multistep schemes for $u_t + au_x = 0$. Find the local truncation error, order of accuracy and stability condition of the Lax-Wendroff multistep scheme. [3+3+4+4]

(2) Find the solutions of

$$u_t + u_x = 0$$

subject to the initial conditions

$$u(x,0) = \begin{cases} 0, & x < 0 \\ \frac{x}{2}, & 0 \le x \le 2 \\ 2 - \frac{x}{2}, & 2 \le x \le 4 \\ 0, & x > 4 \end{cases}$$

using the (i) Lax-Wendroff scheme and (ii) leap-frog scheme with $h=\frac{1}{2}$ and $\lambda=\frac{1}{2}$ where $\lambda=\frac{k}{h}$. Find the solutions upto two time levels. [8+6]

(3) Find the local truncation error, order of accuracy and stability condition of the Crank-Nicolson scheme for $u_t = bu_{xx}$. [3+1+3]
Solve the equation $u_t = u_{xx}$,
satisfying the initial condition, $u = 1$ for $0 \le x \le 1$ when $t = 0$
and the boundary conditions

$$u_x = u \quad \text{at} \quad x = 0, \forall t,$$
$$u_x = -u \quad \text{at} \quad x = 1, \forall t,$$

using the Crank-Nicolson scheme by employing forward difference for the boundary condition at $x = 0$ and central difference for boundary condition at $x = 1$ with $h=\frac{1}{3}$ and $\lambda=\frac{1}{3}$. Compute the solutions upto two time levels. [7]

(4) Consider the conservation law for the transport of a scalar in an unsteady flow in most general form as:
$(\rho\phi)_t + div(\rho\mathbf{u}\phi) = div(\Gamma grad\phi) + (S_\phi)$. Explain physical significance of each term of this equation. Perform the finite volume integration of this equation. By choosing suitable value of θ in the general discretised 1-D unsteady heat conduction equation, derive the fully implicit scheme. Compare the accuracy and efficiency of this scheme with the explicit and the Crank-Nicolson schemes for 1-D unsteady heat conduction equation. [2+3+4+3+2]

(5) What is the essence of a staggered grid? Elucidate with the help of a neat diagram. Compute the convective flux/unit mass F and the diffusive conductance D at cell faces of the u and v control volume faces. [2+2+4]
Consider the steady, one-dimensional flow of a constant density fluid through a duct with constant cross-sectional area. Draw and use a suitable staggered grid, wherein the pressure p is evaluated at the main nodes $I = A, B, C$ and D, whilst velocity u is calculated at the backward staggered nodes i=1,2,3 and 4. The problem data are as follows: (i) Density $\rho = 1.0kg/m^3$ is constant. (ii) Duct area A is constant. (iii) Multiplier d in $u' = d(p'_I + p'_{I+1})$ is assumed to be constant. Take d=1.0. (iv) Boundary conditions: $u_1 = 10m/s$, $P_D = 0Pa$. (v) Initial guessed velocity field: say $u_2^* = 8.0m/s$, $u_3^* = 11.0m/s$, $u_4^* = 7.0m/s$. Use the SIMPLE algorithm and these problem data to calculate pressure corrections at nodes I=A to D and obtain the corrected velocity fields at nodes i=2 to 4. [6]

(6) What do you understand by Conservativeness, Boundedness and Transportiveness with regard to the finite volume discretization scheme? Give an assessment of the Upwind and Hybrid difference schemes for 1-D convection-Diffusion problem. [2+4]
Derive by using QUICK scheme, the finite volume discretization for one-dimensional convection-diffusion problem. Using the QUICK scheme, solve the following problem for $u = 0.2m/s$ on a three point grid. Consider a suitable geometry of a 1-D domain in which a property ϕ is transported by means of convection and diffusion. Use the required governing equations; the boundary conditions are $\Phi_0 = 1$ at $x = 0$ and $\Phi_L = 0$ at $x = L$. The data of this problem is $F = F_e = F_w = 0.2$, $D = D_e = D_w = 0.5$, $Pe_w = Pe_e = \frac{\rho u \delta x}{\Gamma} = 0.4$. [4+4]

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Your Roll Number: . . . .

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations, December 2018
Part II Semester III

MATH14-304(C): COMPUTATIONAL METHODS FOR ODEs

Time: 3 hours | Maximum Marks: 70

Instructions: • Write your roll number on the space provided at the top of this page immediately on receipt of this question paper • Question 1 is compulsory • Answer **any four** questions from remaining questions. • Each question carries 14 marks.

(1) (a) Give an example to show that the linear multistep method or k-step method is convergent if and only if the method is consistent and satisfies the root condition. [3]

(b) Define the absolutely stable, relatively stable, periodically stable and A-stable multistep method. [3]

(c) Consider $u' = f(t, u)$. Prove that if $f(t, u)$ is independent of u, then the classical fourth order Runge-Kutta method reduces to the Simpson's rule for integration. [4]

(d) Given $\rho(\xi) = (23\xi^2 - 16\xi + 5)/12$, find $\rho(\xi)$ and write down an explicit linear multistep method. [4]

(2) Discretize the initial value problem (IVP) [14]

$$u' = 1 + a(1 + t - u), u(0) = 2$$

using explicit Euler method with step height h.

(i) Solve the IVP to estimate $u(1)$ using $h = 0.5$ and $h = 0.25$ with $a = 0.25$. Find the errors in each case.

(ii) Find the stability condition.

(3) (a) The solution of the system of equations [10]

$$y' = u, \qquad y(0) = 1$$
$$u' = -4y - 2u, \quad u(0) = 1$$

is to be obtained by Classical Runge-Kutta fourth order method. Can a step length $h = 0.1$ be used for integration? If so, find $y(0.2)$ and $u(0.2)$.

(b) Find the interval of absolute stability of Runge-Kutta second order method. [4]

(4) The method [14]

$$u_{j+1} = u_j + \frac{h}{2}(u'_{j+1} + u'_j) + \frac{h^2}{12}(u''_j - u''_{j+1})$$

is written for the solution of the IVP $u' = f(t, u); u(t_0) = u_0$.

(i) Find the order of the method.

(ii) Find the interval of absolute stability.

(iii) Is it A-stable?

(iv) Use the above method to solve IVP $u' = 3t + 2u, u(0) = 1, h = 0.1$. Determine an approximation to $u(0.2)$.

(5) Use the shooting method to solve the boundary value problem [14]

$$u'' = 2uu', 0 < x < 1,$$
$$u(0) = 0.5, u(1) = 1.$$

Apply the two stage Runge-Kutta method

$$k_1 = \frac{h^2}{2} f(x_j, u_j, u'_j)$$
$$k_2 = \frac{h^2}{2} f\left(x_j + \frac{2}{3}h, u_j + \frac{2}{3}hu'_j + \frac{2}{3}k_1, u'_j + \frac{4}{3h}k_1\right)$$
$$u_{j+1} = u_j + hu'_j + \frac{1}{2}(k_1 + k_2)$$
$$u'_{j+1} = u'_j + \frac{1}{2h}(k_1 + 3k_2)$$

with $h = 0.25$, to solve the corresponding initial value problem. Use Newton's method, assuming the starting value of the slope at $x = 0$ as $s^{(0)} = u'(0) = 0.3$. Perform two iterations and compare with the exact solution $u(x) = \frac{1}{(2-x)}$.

(6) (a) Solve the initial value problem [7]

$$y' = t^2 - y^2, y(0) = 1, t \in [0, 0.6].$$

Use the third order Adams-Bashforth method with $h = 0.1$. Obtain the starting values using the third order Taylor series method.

(b) For the initial value problem [7]

$$u' = t + u, u(0) = 1$$

estimate $u(0.5)$ using the Milne-Simpson fourth order method with $h = 0.1$ and compare the results with exact solution.

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Your Roll Number: ....

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examination, November-December 2018
Part II Semester III

MATH14-304(D): MATHEMATICAL PROGRAMMING

Time: 3 hours | Maximum Marks: 70

Instructions: • Write your roll number on the space provided at the top of this page immediately on receipt of this question paper. • This question paper has three sections. • Section A is compulsory. • Answer any **Three** questions from Section B. • Answer any **Two** questions from Section C.• Use of calculator is allowed.

Section A

(1) (a) If $f : C \to \mathbb{R}$ is a Gâteaux differentiable function on a convex open set $C \subseteq \mathbb{R}^n$ then prove that f is convex if and only if [2]

$$f(y) \geq f(x) - \langle \nabla f(x), y - x \rangle, \text{ for all } x, y \in C$$

(b) Let C be a nonempty set in $\mathbb{R}^n$. Prove that C is convex if and only if $C + C = 2C$. [2]

(c) State a necessary optimality condition for an unconstrained minimization problem in terms of directional derivative. [2]

(d) What is convex transposition theorem? [2]

(e) State a second order sufficient conditions for the problem considered in Q.4(b), assuming that the functions f, g_i and h_j have continuous second order partial derivatives on $\mathbb{R}^n$. [2]

Section B (Answer any three questions.)

(2) (a) Let C be a convex set in $\mathbb{R}^n$ and f be a twice Fréchet differentiable function defined on an open set containing C. Prove that f is convex on C if and only if the Hessian matrix $Hf(x)$ is positive semidefinite for every $x \in C$. If f is strictly convex then is $Hf(x)$ positive definite for every $x \in C$? Is the converse implication true? Justify. [8]

(b) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined as [6]

$$f(x,y) = \begin{cases} \dfrac{x^3}{x^2+y^2}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

Is the function f Gâteaux differentiable at $(0,0)$? Is it Fréchet differentiable at $(0,0)$? Justify.

(3) (a) If $f : D \to \mathbb{R}$ is a continuous coercive function defined on a closed set D in $\mathbb{R}^n$ then prove that f achieves a global minimum on D. [8]

(b) Find the critical points of the function $f(x,y) = x^3 - y^3 + 9xy$ defined on $\mathbb{R}^2$ and determine their nature. [6]

(4) (a) Show that Karush-Kuhn-Tucker conditions fail for the problem [4]

$$\text{Minimize } -x$$
$$\text{s.t. } (x-1)^3 + y \leq 0, x \geq 0, y \geq 0.$$

(b) Derive Fritz John optimality conditions for the problem [10]

$$\text{Minimize } f(x)$$
$$\text{s.t. } g_i(x) \leq 0, i = 1, 2, ..., r,$$
$$h_j(x) = 0, j = 1, 2, ..., m,$$

where f, g_i and h_j are real valued functions defined on $\mathbb{R}^n$.

(5) (a) If $C \subseteq \mathbb{R}^n$ is a nonempty convex set and $\bar{x} \notin \text{int}(C)$, then prove that there exists a hyperplane $H_{(c,\alpha)}$ such that $\bar{x} \in H_{(c,\alpha)}$ and $\langle c, x\rangle \geq \langle c, \bar{x}\rangle$ for all $x \in C$. [7]

(b) Find the dual of the problem [7]

$$\text{Minimize } \tfrac{1}{2}x_1^2 + \tfrac{1}{2}(x_2 - 3)^2$$
$$\text{s.t. } x_1^2 - x_2 \leq 0, -x_1 + x_2 \leq 2.$$

Find the optimal solutions of both the primal and dual problems.

Section C (Answer any two questions.)

(6) (a) Consider the problem [6]

$$\text{Minimize } f(x)$$
$$\text{s.t. } Ax = b, x \geq 0$$

where f is continuously differentiable function defined on $\mathbb{R}^n$, A is an $m \times n$ matrix $(m \leq n)$ and b is $m \times 1$ vector. If x is a feasible solution of the problem such that $x^T = (x_B^T, x_N^T)$, $x_B > 0$ and $A = (B, N)$ where B is $m \times m$ invertible matrix, then prove that x is a Karush-Kuhn-Tucker point if and only if $\alpha = 0$ and $\beta = 0$ where $\alpha = \max\{-r_j : r_j \leq 0\}$ and $\beta = \max\{x_j r_j : r_j \geq 0\}$ and $r^T = \nabla f(x)^T - \nabla_B f(x)^T B^{-1} A$.

(b) If either α or β is positive in Q.6(a) then find an improving direction with justification. [3]

(7) (a) Use Wolfe's method to solve the following problem [7]

$$\text{Maximize } z = 2x_1 - 4x_2 - x_1^2 - x_2^2$$
$$\text{s.t. } x_1 + x_2 \leq 2$$
$$x_1 \geq 0, x_2 \geq 0.$$

(b) Solve the problem in part (a) geometrically. [2]

(8) (a) Consider the problem considered in Q.4(b) without the equality constraints where each $f, g_i, i = 1, 2, ...r$, are continuous function defined on $\mathbb{R}^n$. Let X be a nonempty closed set in $\mathbb{R}^n$. Suppose that the set $\{x \in X : g(x) < 0\}$ is not empty and the barrier function B is continuous on $\{x : g(x) < 0\}$. Furthermore, assume that for any given $\mu > 0$, if $\{x_k\}$ in X satisfies $g(x_k) < 0$ and $f(x_k) + \mu B(x_k) \to \theta(\mu)$, then $\{x_k\}$ has a convergent subsequence. Prove that for each $\mu > 0$, there exists an $x_\mu \in X$ with $g(x_\mu) < 0$ such that [6]

$$\theta(\mu) = f(x_\mu) + \mu B(x_\mu) = \inf\{f(x) + \mu B(x) : g(x) < 0, x \in X\}.$$

Also prove that for each $\mu > 0$, $f(x_\mu)$ and $(\theta\mu)$ are nondecreasing functions of μ, and $B(x_\mu)$ is a nonincreasing function of μ.

(b) Use penalty function method to solve the problem [3]

$$\text{Minimize } x_1^2 + x_2^2$$
$$\text{subject to } x_1 + 3x_2 = 0$$

starting from the point $(0,0)$ with $\mu = 1$, $\beta = 10$ and $\epsilon = 0.01$.

Roll Number: ....

Department of Mathematics. University of Delhi
M.A./M.Sc. Mathematics Examinations, December 2018
Part II Semester III

MATH14-304(E): METHODS OF APPLIED MATHEMATICS

Time: 3 hours | Maximum Marks: 70

Instructions: • Answer **five** questions. • Each question carries 14 marks. • All the symbols have their usual meaning unless otherwise mentioned.

(1) (a) If $\alpha(x)$ is continuous in $[a,b]$, and if $\int_a^b \alpha(x)h(x)dx = 0$ for every function $h(x) \in C(a,b)$ such that $h(a) = h(b) = 0$, then $\alpha(x) = 0$ for all x in [a,b]. [5 Marks]

(b) Obtain the necessary condition for the functional [5 Marks]

$$\mathcal{J}[z] = \int\int_R F(x, y, z, z_x, z_y)dxdy,$$

where R is some closed region, to have an extremum for a given function $z = z(x,y)$.

(c) Solve the integral equation $\phi(x) = cosx - x - 2 + \int_0^x (t-x)\phi(t)dt$. [4 Marks]

(2) (a) Find the solution in terms of the sinh function for the WKB approximation to the IVP: $\epsilon^2 y'' - q(x)y = 0,\ y(0) = 0,\ y'(0) = 1;\ q(x) > 0$. [6 Marks]

(b) Find the extremals of the fixed end point problem corresponding to the functional $\int_{x0}^{x1} (2yz - 2y^2 + y'^2 - z'^2)dx$. [4 Marks]

(c) Solve the following integro-differential equation: [4 Marks]

$$\phi''(x) - 2\phi'(x) + \phi(x) + 2\int_0^x \cos(x-t)\phi''(t)dt + \int_0^x \sin(x-t)\phi'(t)dt = \cos x;\ \phi(0) = \phi'(0) = 0.$$

(3) (a) Consider the boundary value problem: [6 Marks]

$$\mathcal{L}y \equiv t^2\ddot{y} + \epsilon t^2\dot{y} + \frac{1}{4}y = 0,\ \ 1 \le t \le e,$$
$$y(1) = 1,\ \ y(e) = 0,\ \ 0 < \epsilon << 1.$$

Using regular perturbation, find the profiles of $y_0(t)$ and $y_1(t)$. Compute an upper bound for $|\mathcal{L}y_0|$ on $1 \le t \le e$ when $\epsilon = 0.01$. Can you conclude that y_0 is a good approximation to the exact solution?

(b) Derive the *Abel's* generalised equation. Prove that solution of *Abel's* problem when $f(x) \equiv$ constant is the cycloid. [5 Marks]

(c) Define inner and outer approximations for singular perturbation method with suitable example. [3 Marks]

(4) (a) Use the WKB method to find the solution of *Schrödinger* equation for oscillatory and non-oscillatory case. [8 Marks]

(b) Define equivalent functionals. Find the extremals of the functional $\int_a^b (y^2 - y'^2 - 2y \cosh x)dx$. [2+4 Marks]

(5) (a) State the Hilbert-Schmidt theorem. Investigate the solvability of the integral equation $\phi(x) - \lambda \int_{-1}^{1} xe^t\phi(t)dt = x$ for different values of parameter λ. [1+3 Marks]

(b) Use the Green's function to reduce the BVP: [6 Marks]

$$y'' + \frac{\pi^2}{4}y = \lambda y + \cos\frac{\pi x}{2};\ y(-1) = y(1),\ y'(-1) = y'(1);$$

into an integral equation.

(c) Prove that the characteristic numbers of a symmetric kernel are real. [4 Marks]

(6) (a) Using the Hilbert-Schmidt method, solve the equation [8 Marks]

$$\phi(x) = e^x + \lambda \int_0^1 \kappa(x,t)\phi(t)dt,$$

where

$$\kappa(x,t) = \begin{cases} \frac{\sinh x\ \sinh(t-1)}{\sinh 1}, & 0 \le x \le t, \\ \frac{\sinh t\ \sinh(x-1)}{\sinh 1}, & t \le x \le 1. \end{cases}$$

(b) The thrust (P) of a propeller depends upon the diameter (D), speed (V), mass density (ρ), revolutions per minute (N) and coefficient of viscosity (μ). Show that [6 Marks]

$$P = \rho D^2 V^2 f\left(\frac{\mu}{\rho D V}, \frac{DN}{V}\right).$$

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