Your Roll Number: ....

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examinations, November/December 2016
Part I Semester I
MATH101: COMPLEX ANALYSIS

Time: 3 hours | Maximum Marks: 70

Instructions: • Write your roll number on the space provided at the top of this page immediately on receipt of this question paper • Answer **five** questions in all by choosing 3 questions from Section A and 2 questions from Section B • Each question carries equal marks

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Section A
(Answer any **Three** questions)

(1) (a) Show that the zeros of an analytic function are isolated. [3 Marks]

(b) Evaluate $\int_\gamma \frac{\sin z}{z^3} dz$, where $\gamma(t) = e^{it}$; $0 \le t \le 2\pi$. [3 Marks]

(c) Show that a Mobius transformation takes circles onto circles. [3 Marks]

(d) Define a branch of logarithm. If $G \subset \mathbb{C}$ is open and connected and f is a branch of $\log z$ on G then show that the totality of branches of $\log z$ are the functions $f(z) + 2\pi k i$, $k \in \mathbb{Z}$. [5 Marks]

(2) (a) Let G be a region and f an analytic function on G. Suppose there is a constant M such that $\limsup_{z\to a} |f(z)| \le M$ for all $a \in \partial_\infty G$. Show that $|f(z)| \le M$ for all z in G. [5 Marks]

(b) Let $z = a$ be an isolated singularity of f. Prove that $z = a$ is a removable singularity if and only if $\lim_{z\to a}(z - a)f(z) = 0$. [5 Marks]

(c) Let [4 Marks]

$$f(z) = \frac{z^2 - 4}{z^2 + 5z + 4}.$$

Give the Laurent expansion of $f(z)$ in each of the following annuli: (i) $\text{ann}(0; 1, 4)$ (ii) $\text{ann}(0; 4, \infty)$.

(3) (a) State and prove the Residue theorem. [7 Marks]

(b) Evaluate $\int_0^\infty \frac{\sin x}{x} dx$. [7 Marks]

(4) (a) State orientation principle for Mobius transformations. Use it to find an analytic function $f : G \to \mathbb{C}$, where $G = \{z : \text{Re} z < 0\}$, such that $f(G) = D$, and $D = \{z : |z| < 1\}$. [5 Marks]

(b) Show that a Mobius transformation can have at most two fixed point. If z_2, z_3, z_4 are distinct points in $\mathbb{C}_\infty$ and w_2, w_3, w_4 are also distinct points of $\mathbb{C}_\infty$, then show that there is one and only one Mobius transformation S such that $Sz_2 = w_2$, $Sz_3 = w_3$, $Sz_4 = w_4$. [5 Marks]

(c) Let G be a region and $f : G \to \mathbb{C}$ an analytic function. If $\{z \in G : f(z) = 0\}$ has a limit point in G then show that there is a point $a \in G$ such that $f^{(n)}(a) = 0$ for each $n \geq 0$. [4 Marks]

Section B

(Answer any **Two** questions)

(5) (a) Prove that every analytic function in a simply connected region has a primitive. [5 Marks]

(b) State and prove the Casorati-Weierstrass Theorem. [5 Marks]

(c) Let $p(z)$ be a polynomial of degree n and let $R > 0$ be sufficiently large so that p never vanishes in $\{z : |z| \geq R\}$. If $\gamma(t) = Re^{it}$, $0 \leq t \leq 2\pi$, show that [4 Marks]

$$\int_\gamma \frac{p'(z)}{p(z)} dz = 2\pi i n.$$

(6) (a) Let G be an open set and let $f : G \to \mathbb{C}$ be a differentiable function. Show that f is analytic on G. [5 Marks]

(b) Let γ_0 and γ_1 be two rectifiable curves in G from a to b which are fixed-end-point (FEP) homotopic. Show that for any function f analytic in G, $\int_{\gamma_0} f = \int_{\gamma_1} f$. [5 Marks]

(c) Let G be a region and suppose that $f : G \to \mathbb{C}$ is analytic such that $f(G)$ is a subset of a circle. Show that f is constant. [4 Marks]

(7) (a) Let γ be a rectifiable curve and suppose φ is a function defined and continuous on $\{\gamma\}$. For each $m \geq 1$, let [5 Marks]

$$F_m(z) = \int_\gamma \frac{\varphi(w)}{(w - z)^m} dw$$

for $z \notin \{\gamma\}$. Show that each F_m is analytic on $\mathbb{C} - \{\gamma\}$ and $F_m'(z) = mF_{m+1}(z)$.

(b) State and prove Schwarz's Lemma. [5 Marks]

(c) Let $\mathbb{D} = \{z : |z| < 1\}$. Let $f : \mathbb{D} \to \mathbb{D}$ be analytic and $|f(z)| \leq 1$ for $z \in \mathbb{D}$. Let $g : \mathbb{D} \to \mathbb{D}$ is defined by [4 Marks]

$$g(z) = \frac{f(z) - f(0)}{1 - \bar{f(0)} f(z)}.$$

Prove that, for $z \in \mathbb{D}$,

$$\frac{|f(0)| - |z|}{1 - |f(0)||z|} \leq |f(z)| \leq \frac{|f(0)| + |z|}{1 + |f(0)||z|}.$$

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