Your Roll Number: ....

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examination, November-December 2016
Part II Semester III

MATH-304(ii): MATHEMATICAL PROGRAMMING

Time: 3 hours | Maximum Marks: 70

Instructions: • Write your roll number on the space provided at the top of this page immediately on the receipt of this question paper. • This question paper has three sections. Section A is compulsory. Answer any **Three** questions from Section B. Answer any **Two** questions from Section C. • Use of calculator is permitted.

Section A

(1) (a) Define Fréchet derivative of a function $f : U \to \mathbb{R}$ at a point in U. Is a Fréchet differentiable function always continuous? [2 Marks]

(b) Solve the following problem geometrically [2 Marks]

$$\text{Minimize } x^2 + y^2$$
$$\text{s.t. } x - y = 1.$$

(c) Define Mangasarian-Fromovitz constraint qualification for the problem [2 Marks]

$$\text{Minimize } f(x)$$
$$\text{s.t. } g_i(x) \leq 0, i = 1, 2, ..., r,$$
$$h_j(x) = 0, j = 1, 2, ..., m,$$

where f, g_i and h_j are real valued functions defined on $\mathbb{R}^n$.

(d) Define a positive definite matrix and state Sylvester theorem for symmetric positive definite matrices. [2 Marks]

(e) If $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a convex function then show that epi(f) is a convex set in $\mathbb{R}^{n+1}$. [2 Marks]

Section B (Answer any three questions.)

(2) (a) Prove that a continuous function $f : K \to \mathbb{R}$, defined on a compact subset K of $\mathbb{R}^n$, achieves a global minimum on K. [7 Marks]

(b) Check the nature of critical points of the function $f(x, y) = x^2 + y^2 + \beta xy + x + 2y$ for different values of β. [7 Marks]

(3) (a) Let $C \subseteq \mathbb{R}^n$ be a nonempty closed convex set. If $\Pi_C(x)$ is the projection of x onto C then prove that [7 Marks]

$$\langle x - \Pi_C(x), z - \Pi_C(x) \rangle \leq 0 \text{ for all } z \in C.$$

Also, prove that the projection map $\Pi_C : \mathbb{R}^n \to C$ is a nonexpansive map.

(b) If $f : C \to \mathbb{R}$ is a convex function defined on a convex set C in $\mathbb{R}^n$ then prove that any local minimizer of f on C is a global minimizer of f on C. Also, prove that if f is strictly convex then there exists at the most one global minimizer of f on C. [7 Marks]

(4) (a) Let x^* be a feasible point of the problem considered in Ques.(1)(c) and $(\lambda, \mu) \in \mathbb{R}^r \times \mathbb{R}^m$ be such that (x^*, λ, μ) satisfies KKT condition. If the functions f and g_i are convex and h_j is affine then prove that x^* is a global minimizer. [6 Marks]

(b) Consider the problem [8 Marks]

$$\begin{gathered}\text{Minimize } -xy\\ \text{s.t. } x+y=8\\ x \geq 0, y \geq 0.\end{gathered}$$

Show that KKT conditions are satisfied at all points satisfying Fritz John conditions. Determine all the KKT points of the problem and determine their nature.

(5) (a) Let $L : A \times B \to \mathbb{R}$ be a function where $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ and $(x^*, y^*) \in A \times B$. Prove that the following are equivalent: [7 Marks]
i) (x^*, y^*) is a saddle point of $L(x, y)$;
ii) x^* is a solution of the problem (P) given by

$$\inf_{x \in A} \sup_{y \in B} L(x, y)$$

and y^* is a solution of dual problem (D) and min(P) = max(D).

(b) If A is $m \times n$ matrix, Q is $n \times n$ symmetric positive definite matrix, $c \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$, determine the dual of the problem [7 Marks]

$$\begin{gathered}\text{Minimize } \langle Qx, x\rangle + \langle c, x\rangle\\ \text{s.t. } Ax = b, x \geq 0.\end{gathered}$$

Section C (Answer any two questions.)

(6) Solve the following problem by Wolfe's method [9 Marks]

$$\begin{gathered}\text{Maximize } z = 2x_1 + 2x_2 - x_1^2 - x_2^2\\ \text{s.t. } x_1 + x_2 \leq 1\\ x_1 \geq 0, x_2 \geq 0.\end{gathered}$$

(7) (a) Use convex simplex method to solve the following problem [7 Marks]

$$\begin{gathered}\text{Minimize } z = x_1^2 + x_2^2 - 2x_1 - 4x_2\\ \text{s.t. } x_1 + x_2 \leq 1\\ -x_1 + 2x_2 \leq 2\\ x_1 \geq 0, x_2 \geq 0.\end{gathered}$$

(b) Solve the problem in part (a) geometrically. [2 Marks]

(8) Let $f, g_i, i = 1, 2, ...r, h_j, j = 1, 2, ...m$ be continuous function defined on $\mathbb{R}^n$ and X be a nonempty set in $\mathbb{R}^n$. Let α be a continuous penalty function on $\mathbb{R}^n$ such that for each μ, there is $x_\mu \in X$ such that [9 Marks]

$$\theta(\mu) = \inf \{f(x) + \mu\alpha(x) : x \in X\} = f(x_\mu) + \mu\alpha(x_\mu).$$

Prove the following
i) $\inf \{f(x) : g(x) \leq 0, h(x) = 0, x \in X\} \geq \sup_{\mu \geq 0} \theta(\mu)$;
ii) $f(x_\mu)$ is a nondecreasing function of $\mu \geq 0$;
iii) $\theta(\mu)$ is a nondecreasing function of μ.

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