Your Roll Number: ....

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examinations, November-December 2016
Part II Semester III
MATH302(iii): THEORY OF OPERATORS

Time: 3 hours Maximum Marks: 70

Instructions: • Write your roll number on the space provided at the top of this page immediately on receipt of this question paper • Attempt **seven** questions in all. **Question no. 1 is compulsory.** • By operator, we mean a bounded linear transformation.

(1) [5x2=10 Marks]
(a) Justify the strict inclusion of the point spectrum of an operator in its approximate point spectrum through an example.
(b) Find the spectrum of the operator S on a complex Hilbert space given by $S = T^2 + 3T + 2$, where T is an operator on the Hilbert space having spectrum $\{-2, -1\}$.
(c) Prove that a surjective operator $T : \ell_\infty \to \ell_\infty$ can not be compact.
(d) Prove or disprove that every partial isometry is a projection.
(e) If T is a positive operator on a Hilbert space then prove that the operator $I + T^*T$ is invertible.

(2) (a) Prove that the difference of two compact operators on a normed linear space is compact. What about the converse? Justify your answer. [6 Marks]
(b) Define approximate point spectrum of an operator on a Banach space and prove that it is a closed set. [4 Marks]

(3) (a) Show that identity operator on an infinite dimensional Hilbert space cannot be compact. [5 Marks]
(b) Show that the operator $T : \ell_2 \mapsto \ell_2$ given by [5 Marks]

$$T(x_1, x_2, x_3, \cdots) = (x_1, \frac{x_2}{2}, \frac{x_3}{3}, \dots)$$

is compact.

(4) (a) If T is a compact operator on a normed linear space and λ is a non-zero scalar, then show that for each integer $m \geq 0$, the null space $N(T - \lambda I)^m$ of the operator $(T - \lambda I)^m$ is finite dimensional. What happens if $\lambda = 0$. Justify your answer. [6 Marks]
(b) Does there exist a compact operator with non-closed range? Justify your answer. [4 Marks]

(5) (a) Prove the following for a self adjoint operator T on a Hilbert space: [3+4 Marks]

(i) Point spectrum of T is a subset of the set of real numbers.

(ii) Residual spectrum of T is empty.

(b) Prove that every positive operator is self adjoint. What about the converse? Justify your answer. [3 Marks]

(6) (a) Show that a self adjoint operator on a Hilbert space can always be expressed as the difference of two positive operators whose product is zero. [6 Marks]

(b) If S and T are two operators on a Hilbert space such that $S^*S \leq T^*T$ and T is compact, then show that S is compact. [4 Marks]

(7) (a) Show that the product PQ of two projections P and Q on a Hilbert space is a projection if and only if $PQ = QP$. Determine the range of PQ in that case. [5 Marks]

(b) Prove that a closed subspace M is invariant under an operator T on a Hilbert space if and only if $M^\perp$, the orthogonal complement of M, is invariant under the adjoint of T. [5 Marks]

(8) (a) Let T be a self adjoint operator on a Hilbert space H with spectral family of projections $\{E(\lambda)\}_{\lambda\in\mathbb{R}}$. Then prove that $N(T-\lambda_0 I) = (E(\lambda_0) - E(\lambda_0-))(H)$. Hence deduce that the point spectrum of T contains λ_0 if and only if the function $f(\lambda) = E(\lambda)$ is discontinuous at λ_0. [7 Marks]

(b) Can we always express an operator T on a Hilbert space as $T = UP$, where U is an unitary and P is a positive operator? Justify your answer. [3 Marks]

(9) (a) Is the operator $T = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ positive? Justify your answer. [4 Marks]

(b) Prove the following: [3+3 Marks]

(i) If T is a compact operator on a Hilbert space H with singular values $\{\lambda_n\}$, then

$$\sum_n |\lambda_n|^2 = \sum_n^\infty \|Te_n\|^2 = \sum_{n,m=1}^\infty |\langle Te_n, e_m\rangle|^2$$

for any orthonormal basis $\{e_n\}$ of H.

(ii) Let y be a fixed element of a Hilbert space H and $T : H \to H$ be defined by $Tx = \langle x, y\rangle$ for $x \in H$. Show that T is a Hilbert-Schmidt operator.