Your Roll Number: ....

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examinations, Nov./Dec. 2016
Part II Semester III
MATH304(iii): GRAPH THEORY

Time: 3 hours | Maximum Marks: 70

Instructions: • Write your roll number on the space provided at the top of this page immediately on receipt of this question paper.

Section A

(Answer any one questions)

(1) (a) Show that in any group of $n \geq 2$ persons, there are atleast two persons with the same number of friends. [5 Marks]

(b) Define the following operations: Union of two graphs; Intersection of two graphs; Join of two graphs; Cartesian product of two graphs; Direct product of two graphs. [5 Marks]

(c) Let G be a simple cubic graph. Show that G has a cut vertex if and only if it has a cut edge. [4 Marks]

(2) (a) Is the vertex connectivity and the edge connectivity of a simple cubic graph same? Justify. [5 Marks]

(b) Let G be a simple graph of order n. Show that

(i) if $\delta(G) \geq (n-1)/2$, then G is connected. [3 Marks]

(ii) if G is not connected, then G^c is connected. [3 Marks]

(iii) if G has w components, then the number of edges in G cannot exceed [3 Marks]

$$\frac{(n-w)(n-w+1)}{2}.$$

Section B

(Answer any two questions)

(3) (a) Let $\{v_1, \ldots, v_n\}$, $(n \geq 2)$ be given and let $(d_1, \ldots, d_n)$ be a sequence of positive integers with $\sum_{i=1}^{n} d_i = 2(n-1)$. Show that there exists a tree with vertex set $\{v_1, \ldots, v_n\}$ such that $\deg(v_i) = d_i$ for all $i = 1, \ldots, n$. [5 Marks]

(b) Let $(d_1, \ldots, d_n)$ be a sequence of positive integers with [9 Marks]

$$\sum_{i=1}^{n} d_i = 2(n-1).$$

Show that the number of trees with vertex set $\{v_1, \ldots, v_n\}$ such that $\deg(v_i) = d_i$ for all $i = 1, \ldots, n$ is

$$\frac{(n-2)!}{(d_1-1)! \cdots (d_n-1)!}.$$

(4) (a) Show that a graph G is Eulerian if and only if each edge e of G belongs to an odd number of cycles of G. [9 Marks]

(b) Show that a graph G is Eulerian if and only if it has an odd number of cycle decompositions. [5 Marks]

(5) (a) Show that the line graph of an Eulerian graph is both Eulerian and Hamiltonian. Is the converse true? Justify. [5 Marks]

(b) Let S be a set of vertices of a nontrivial tree T, and let $|S| = 2k,\ k \geq 1$. Show that there exists a set of k pairwise edge disjoint paths in T whose end vertices are all the vertices of S. [5 Marks]

(c) Let G be a connected graph in which every edge belongs to a triangle. If e_1, e_2 are edges of G such that $G \setminus \{e_1, e_2\}$ is connected, then show that there exists a spanning trail of G with e_1 and e_2 as its initial and terminal edges. [4 Marks]

Section C

(Answer any two questions)

(6) (a) Let G be a simple planar graph with $n \geq 3$ vertices and m edges. Show that $m \leq 3n - 6$. Use this to prove that G has a vertex v of degree less than 6. [6+2 Marks]

(b) Let G be a 3-connected graph with at least five vertices. Show that G has an edge e such that $G \circ e$ is 3-connected. [6 Marks]

(7) (a) Let G be a k-crtitcal graph. Show that G is connected, $\delta(G) \geq k - 1$, and there does not exist any pair of subgraphs G_1, G_2 of G such that $G = G_1 \cup G_2$ and $G_1 \cap G_2$ is a complete graph. [2+2+4 Marks]

(b) Show that every k-chromatic graph can be contracted into a con-critical k-chromatic graph. [6 Marks]

(8) (a) Let G be a connected graph which is neither an odd cycle nor a complete graph. Show that [9 Marks]

$$\chi(G) \leq \Delta(G).$$

(b) For any nonempty bipartite graph G, show that [5 Marks]

$$\chi'(G) = \Delta(G).$$

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