Your Roll Number: ....

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations, November-December 2016
Part II Semester III

Math-301(i): Advanced Complex Analysis(Old Course)

Time: 3 hours Maximum Marks: 70

Instructions: • Write your roll number on the space provided at the top of this page immediately on receipt of this question paper • Answer **any five** questions • Each question carries 14 marks.

---

(1) (a) Define *convex function*. Show that a function $f : [a, b] \to \mathbb{R}$ is convex if and only if, for $a \leq x < u < y \leq b$, [5 Marks]

$$\frac{f(u) - f(x)}{u - x} \leq \frac{f(y) - f(u)}{y - u}.$$

(b) Let $a < b$ and let G be the vertical strip $G = \{x+iy : a < x < b\}$. Suppose that $f : \overline{G} \to \mathbb{C}$ is continuous, f is analytic in G and $|f(z)| < B$ for all $z \in G$. If $|f(z)| \leq 1$ for all $z \in \partial G$ then show that $|f(z)| \leq 1$ for all $z \in G$ and that the function [7 Marks]

$$M(x) := \sup\{|f(x + iy)| : -\infty < y < \infty\}$$

defined on $[a, b]$ is *logarithmically convex.*

(c) State *Hadamard three circles theorem.* [2 Marks]

(2) (a) State and prove *Phragmen-Lindelöf Theorem.* [6 Marks]

(b) Show that a sequence $\{f_n\}$ in $(\mathcal{C}(G, \Omega), \rho)$ converges to f if and only if $\{f_n\}$ converges to f uniformly on all compact subsets of G. [4 Marks]

(c) If $\mathcal{F} \subset \mathcal{C}(G, \Omega)$ is normal, show that it is *equicontinuous* at each point of G. [4 Marks]

(3) (a) If a sequence $\{f_n\}$ in $\mathcal{H}(G)$ converges to a function f in $(\mathcal{C}(G, \Omega), \rho)$, show that f is analytic and that $f_n^{(k)} \longrightarrow f^{(k)}$ for each integer $k \geq 1$. Deduce that the mapping from $\mathcal{H}(G)$ into $\mathcal{H}(G)$ given by $f \longrightarrow f^{'}$ is continuous. [7 Marks]

(b) Define *locally bounded* set in $H(G)$. Show that a set $\mathcal{F} \subset H(G)$ is normal if and only if it is locally bounded. [7 Marks]

(4) (a) State and prove the *Riemann Mapping Theorem.* [14 Marks]

(5) (a) Define the *gamma function* $\Gamma(z)$. Suppose f is a function defined on the interval $(0, \infty)$ such that $f(x) > 0$ for all $x > 0$ with the following properties: [6 Marks]

(i) $\log f(x)$ is a convex function,

(ii) $f(x+1) = xf(x)$ for all x,

(iii) $f(1) = 1$.

Show that $f(x) = \Gamma(x)$ for all x.

(b) Write a note on *Riemann Hypothesis*. [3 Marks]

(c) Define the *Riemann zeta function* $\xi(z)$. Prove *Euler's Theorem*: if $\text{Re}z > 1$ then [5 Marks]

$$\xi(z) = \prod_{n=1}^{\infty} \frac{1}{1 - p_n^{-z}}$$

where $\{p_n\}$ is the sequence of prime numbers.

(6) (a) Let γ be a rectifiable curve and let K be a compact set such that $K \cap \{\gamma\} = \phi$. If f is continuous on $\{\gamma\}$ and $\epsilon > 0$, show that there is a rational function $R(z)$ having all its poles on $\{\gamma\}$ such that [5 Marks]

$$\left| \int_\gamma \frac{f(\omega)}{\omega - z} d\omega - R(z) \right| < \epsilon$$

for all $z \in K$.

(b) Show that the complex plane $\mathbb{C}$ and the unit disk $\mathbb{D}$ are homeomorphic. [2 Marks]

(c) State and prove *Mittag-Leffler's Theorem*. [7 Marks]

(7) (a) Show that a harmonic function on an open set G has the *mean value property* (MVP). [3 Marks]

(b) Show that if u is harmonic function then so are [3 Marks]

$$u_x = \frac{\partial u}{\partial x} \quad \textit{and} \quad u_y = \frac{\partial u}{\partial y}$$

(c) What do you mean by *Dirichlet Problem*? Show that Dirichlet Problem can be solved for the unit disk. [8 Marks]

(8) (a) Define the *Poisson kernel* $P_r(\theta)$ for $0 \le r \le 1$ and $-\infty < \theta < \infty$. Show that [3 Marks]

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta) = 1$$

(b) Define *Green's function* of a region G with *singularity* at $a \in G$. Prove the following: [11 Marks]

(i) If it exists, Green's function is positive and unique.

(ii) There is no Green's function in the complex plane with singularity at zero.

(iii) If G is a bounded Dirichlet Region, then there is a Green's function on G with singularity at $a \in G$ for each a in G.

✠