Your Roll Number: ................

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations, November-December 2016
Part II Semester III
**MATH303(iii): Representation Theory of Finite Groups**

Time: 3 hours | Maximum Marks: 70

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**Instructions:** • Write your roll number on the space provided at the top of this page immediately on receipt of this question paper • Attempt any **five** questions. All questions carry equal marks.• $G$ will denote a finite group and $F$ will denote a scalar field throughout the question paper.

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**1** **a)** Define an $FG$-module. Make $V = F^n$ into an $FG$-module when a representation $\rho : G \longrightarrow GL(n, F)$ is given. Also prove that the mapping $g \longmapsto [g]_\beta$, $g \in G$ is a representation of $G$ over $F$ where $\beta$ is a basis of an given $FG$-module $V$. [1+3+3 marks]

**b)** Given $G = S_n$ and $V$ a vector space over $F$, show that $V$ becomes an $FG$-module with the following definition,

$$vg = \begin{cases} v & \text{if } g \text{ is an even permutation,} \\ -v & \text{if } g \text{ is an odd permutation.} \end{cases}$$

[4 marks]

**c)** Give an example of an irreducible 2-dimensional $FG$-module when $G = D_8$. [3 marks]

**2** **a)** If $V$ is an $FG$-module with basis $\beta$ and $W$ is an $FG$-module with basis $\beta'$,then show that $V \cong W$ if and only if the representations $\rho : g \longmapsto [g]_\beta$ and $\sigma : g \longmapsto [g]_{\beta'}$ are equivalent. [6 marks]

**b)** State and prove Maschke's theorem. [1+7 marks]

**3** **a)** Prove that every irreducible $\mathbb{C}G$-module has dimension one when $G$ is abelian. [3 marks]

**b)** If $\mathbb{C}G = U_1 \oplus \cdots \oplus U_r$ is a direct sum of irreducible $\mathbb{C}G$-submodules, then prove that every irreducible $\mathbb{C}G$-module is isomorphic to one of the $\mathbb{C}G$-submodules $U_i$. [5 marks]

**c)** Show that the dimensions of the space of $\mathbb{C}G$-homomorphisms from regular $\mathbb{C}G$-module $\mathbb{C}G$ to any other $\mathbb{C}G$-module $U$ is equal to $\dim U$. [6 marks]

**4** Prove the following:

**a)** Isomorphic $\mathbb{C}G$-modules have the same character. [3 marks]

**b**) Any character $\chi$ of $G$ takes same value on conjugacy classes of $G$. [3 marks]

**c**) For any character $\chi$ of a $\mathbb{C}G$-module $V$ $\chi(g^{-1}) = \overline{\chi(g)}$, $g \in G$. [3 marks]

**d**) Show that $|\chi(g)| = \chi(1)$ if and only if $g\rho = \lambda I_n$ for some $\lambda \in \mathbb{C}$, when $\chi$ is a character of a representation $\rho$ of $G$. [5 marks]

**5** **a**) Suppose $V$ and $W$ be $\mathbb{C}G$-modules with characters $\chi$ and $\psi$ respectively, then prove that $\dim(\mathrm{Hom}_{\mathbb{C}\mathrm{G}}(\mathrm{V}, \mathrm{W})) = \langle \chi, \psi \rangle$. [4 marks]

**b**) If $\chi_1, \cdots \chi_k$ be the irreducible characters of $G$ and $g_1, \cdots g_k$ be the representatives of the conjugacy classes of $G$, then prove that

$$\sum_{i=1}^{k} \chi_i(g_r)\overline{\chi_i(g_s)} = \delta_{rs}|C_G(g_r)|$$

where $\delta_{rs}$ is the Kroneker delta function and $C_G(x)$ is the centralizer of $x$ in $G$. [6 marks]

**c**) Determine the character table of $C_3$. [4 marks]

**6** **a**) Prove that the linear character of $G$ are precisely the lifts to $G$ of the irreducible character of $G/G'$ where $G'$ is the derived subgroup of $G$. Find the linear characters of $S_n$. [4+3 marks]

**b**) Determine the complete character table of the symmetric group $S_4$. [7 marks]