Your Roll Number: ....

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examinations, November-December 2016
Part I Semester I

MATH103: FIELD THEORY

Time: 3 hours | Maximum Marks: 70

Instructions: • Write your roll number on the space provided at the top of this page immediately on receipt of this question paper.

Section A

(Answer any two questions)

(1) (a) Show that the polynomial ring in $\sqrt[7]{2}$ over $\mathbb{Q}$, that is, $\mathbb{Q}[\sqrt[7]{2}]$ is the set of all polynomials in $\sqrt[7]{2}$ over $\mathbb{Q}$ of degree atmost 6. Is it a field? Justify. [5 Marks]

(b) For any two roots α, β of $x^2 - 4$ over $\mathbb{Q}$, is $\mathbb{Q}(\alpha) \cong \mathbb{Q}(\beta)$? Justify. If α, β are the roots of an irreducible polynomial $f(x)$ over $\mathbb{Q}$, is $\mathbb{Q}(\alpha) \cong \mathbb{Q}(\beta)$? Justify. [6 Marks]

(c) Find the degree of minimal splitting field of $x^6 + 1$ over $\mathbb{Z}_2$. [3 Marks]

(2) (a) Let $\mathcal{F}$ be the family of all polynomials over a field k. Does there exist a minimal splitting field of $\mathcal{F}$ over k? Justify. [5 Marks]

(b) Show that an algebraic extension of a perfect field is perfect. [4 Marks]

(c) Show that an algebraic closure of a countable field is again countable. [5 Marks]

(3) (a) Show that a finite extension $E|k$ is normal if and only if E is the minimal splitting field of a polynomial over k. [5 Marks]

(b) Let E be the minimal splitting field of the family of all polynomials over a field k. Show that E is an algebraic closure of k. [4 Marks]

(c) Show that all the zeros of an irreducible polynomial over a field k have the same multiplicity. [5 Marks]

Section B

(Answer any two questions)

(4) (a) Let $F|k$ be an extension of degree n and let $\phi : F \to F', \phi_0 : k \to k', \phi|_k = \phi_0$ be isomorphisms. If E' is a normal extension of k' containing F', then show that there are exactly n homomorphisms from F to E' extending ϕ_0 if and only if F is generated by separable elements over k. [5+4 Marks]

(b) For any finite extension $E|F$, show that $E|F$ is a Galois extension if and only if it is normal and separable. [5 Marks]

(5) (a) Show that every finite group occurs as a Galois group of some extension. [5 Marks]

(b) Show that the cyclotomic polynomial $\phi_n(x)$ is irreducible over $\mathbb{Q}$. [6 Marks]

(c) Show that $n = \sum_{d/n} \phi(d)$, for all $n \in \mathbb{N}$, where ϕ is Euler's phi function. [3 Marks]

(6) (a) Show that the only finite fields are the minimal splitting field of $x^{p^n} - x$ over F_p, for all prime number p and $n \in \mathbb{N}$. [6 Marks]

(b) Show that a finite extension $F|k$ is simple if and only if F has only finitely many subfields containing k. [5 Marks]

(c) Determine all $\mathbb{Q}$-automorphisms of $\mathbb{Q}(\alpha)$, where α is the real root of $x^3 = 2$. [3 Marks]

Section C

(Answer any one questions)

(7) (a) Let $f(x)$ be a polynomial over a field k of characteristic zero. If $f(x)$ is solvable by radicals over k, then show that the Galois group of $f(x)$ over k is solvable. [8 Marks]

(b) Let $f(x)$ be a separable polynomial over a field k with Galois group G of prime order p. Show that the Galois group of $f(x)$ over any extension of k is either G or of order 1. [6 Marks]

(8) (a) Let $f(x)$ be a polynomial over a field k of characteristic zero. If the Galois group of $f(x)$ over k is solvable, then show that $f(x)$ is solvable by radicals over k. [9 Marks]

(b) Let $f(x)$ be a separable polynomial over a field k. Show that $f(x)$ is irreducible over k if and only if the Galois group of $f(x)$ acts on the roots of $f(x)$ transitively. [5 Marks]

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