Your Roll Number: ....

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations, November-December 2016
Part-II Semester-III
MATH303(ii): GENERAL MEASURE THEORY

Time: 3 hours Maximum Marks: 70

Instructions: • Write your roll number on the space provided at the top of this page immediately on receipt of this question paper • Question No. 1 is compulsory. Attempt **FIVE** questions in all. • The symbols used have their usual meanings.

---

(1) (a) Give an example to show that a signed measure need not be monotone. [3 Marks]

(b) State Radon-Nikodym theorem. Give an example to show that the hypothesis in the Radon-Nikodym theorem that the measure space is σ-finite, cannot be dropped. [1+3 Marks]

(c) If $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are measure spaces, show that the collection of measurable rectangles of $X \times Y$ forms a semi algebra. [3 Marks]

(d) State Riesz-Markov representation theorem. [2 Marks]

(e) When do you say a Baire measure μ on a locally compact Hausdorff space X to be regular? [2 Marks]

(2) (a) Let ν be a signed measure on a measurable space $(X, \mathcal{A})$. If E is a measurable set such that $0 < mE < \infty$, show that there is a positive set A contained in E with $mA > 0$. [8 Marks]

(b) State and prove Hahn decomposition theorem for a signed measure. [1 +5 Marks]

(3) (a) Prove that the Radon-Nikodym theorem for a finite measure μ implies the theorem for a σ-finite measure μ. [5 Marks]

(b) Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space, $g \in L^\infty(\mu)$, and T be the linear functional on $L^1(\mu)$ defined by

$$T(f) = \int fgd\mu, \quad f \in L^1(\mu).$$

Show that $\|T\| = \|g\|_\infty$. [4 Marks]

(c) Let μ be a measure on an algebra $\mathcal{A}$ and μ^* be the outer measure induced by μ. If $A \in \mathcal{A}$ and (A_j) is any sequence of sets in $\mathcal{A}$ such that $A \subset \cup_{j=1}^{\infty} A_j$, then show that $\mu(A) \leq \sum_{j=1}^{\infty} \mu(A_j)$. Hence conclude that $\mu^*(E) = \mu(E)\ \forall E \in \mathcal{A}$. [5 Marks]

(4) (a) Let μ be a finite Baire measure on $\mathbb{R}$ and F be its cumulative distribution function. Show that [3 Marks]

(i) F is continuous on the right,

(ii) F is continuous at a point $x \in \mathbb{R}$ if and only if $\mu(\{x\}) = 0$.

(b) Find the cumulative distribution function F of $\mu = \delta_{-1} + 2\delta_2$ and sketch its graph, where δ_x is the Dirac measure with mass concentrated at x. [3 Marks]

(c) Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be complete measure spaces and $\mathcal{R}$ be the collection of measurable rectangles in $X \times Y$, $E \in \mathcal{R}_{\sigma\delta}$ and $\mu \times \nu(E) < \infty$. Assuming that $E_x \in \mathcal{B}\ \forall x \in X$, show that the function g defined by $g(x) = \nu(E_x)$, $x \in X$, is measurable and $\int g\, d\mu = \mu \times \nu(E)$. [8 Marks]

(5) (a) Let X be a locally compact Hausdorff space, $K, O \subset X$ be such that K is compact, O is open and $K \subset O$. Show that there is σ-compact open set U and a compact G_δ set H such that $K \subset U \subset H \subset O$. [4 Marks]

(b) Let X be a locally compact Hausdorff space, F be a positive linear functional on $C_c(X)$. For $A, V \subset X$ with V open, define $\mu V = \sup\{F(f) : f \prec V\}$ and $\mu^*(A) = \inf\{\mu W : W \text{ is open and } A \subset W\}$. Show that μ^* is an outer measure on X. [6 Marks]

(c) Let X be a locally compact Hausdorff space and μ and ν be regular Borel measures on X such that

$$\int f d\mu = \int f d\nu$$

for all $f \in C_c(X)$. Then show that $\mu = \nu$. [4 Marks]

(6) (a) Let μ be finite, $1 < p < \infty$ and $g \in L^1(\mu)$ such that for some constant M,

$$\left| \int f\varphi d\mu \right| \leq M\|\varphi\|_p$$

for all simple functions φ. Show that $g \in L^q(\mu)$, where $1/p + 1/q = 1$. [6 Marks]

(b) If X is a locally compact Hausdorff space, show that the class of σ-bounded Baire sets in X is the smallest σ-ring containing the compact G_δ sets. [4 Marks]

(c) Let X be a locally compact Hausdorff space, F be a positive linear functional on $C_c(X)$ and μ^* be the outer measure induced by F. Show that $\mu^*(K) < \infty$ for every compact set K. [4 Marks]

(7) (a) Let $\mathcal{C}$ be a semi-algebra of sets and $\mu : \mathcal{C} \to [0, \infty]$ be a set function. State and prove sufficient conditions under which μ can be extended uniquely to a measure on the algebra $\mathcal{A}$ generated by $\mathcal{C}$. [2+6 Marks]

(b) Let X be a locally compact Hausdorff space. If X is also compact, show that every Baire measure on X is regular. [6 Marks]

✠