Your Roll Number: ....

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations, November-December 2016
Part II Semester III
Math303(i): Introduction to Algebraic Topology

Time: 3 hours | Maximum Marks: 70

Instructions: • Write your roll number on the space provided at the top of this page immediately on receipt of this question paper • Answer **any five** questions • Each question carries 14 marks.

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(1) (a) Prove that the homotopy relation on the set of all continuous maps of a space X into a space Y is an equivalence relation. Deduce that composites of homotopic maps are homotopic. [7 Marks]

(b) Show that a continuous map $f : \mathbb{S}^n \to Y$ is null homotopic if and only if f can be continuously extended to the disc $\mathbb{D}^{n+1}$. [7 Marks]

(2) (a) Let A be deformation retract of X. Show that A is homotopically equivalent to X. Give example of spaces which are homotopically equivalent but not deformation retract. [6 Marks]

(b) Prove that an open interval can never be a retract of real line. [5 Marks]

(c) Show that the contractible spaces are connected. [3 Marks]

(3) (a) If $\phi : X \to Y$ is a homotopy equivalence then show that for any $x_0 \in X$, $\phi_\sharp : \pi_1(X, x_0) \to \pi_1(Y, \phi(x_0))$ is an isomorphism. [5 Marks]

(b) State and prove the fundamental theorem of algebra. [7 Marks]

(c) Give an example that continuous image of a contractible space need not be contractible. [2 Marks]

(4) (a) Define the degree of a loop f in $\mathbb{S}^1$ based at 1. Show that the function $[f] \mapsto \deg f$ is an isomorphism of $\pi_1(\mathbb{S}^1, 1)$ onto the group $\mathbb{Z}$ of integers. [8 Marks]

(b) Define covering map. Show that the exponential map $p : \mathbb{R} \to \mathbb{S}^1$ defined by $p(t) = e^{2\pi i t}$ is a covering map. [6 Marks]

(5) (a) Let X be locally path connected. Show that a continuous map $p : \widetilde{X} \to X$ is a covering map iff for each path component C of X, $p|_{p^{-1}(C)} : p^{-1}(C) \to C$ is a covering map. [7 Marks]

(b) Let $p : \widetilde{X} \to X$ be a covering map, and $f : I \to X$ be a path with origin x_0. If $\tilde{x}_0 \in p^{-1}(x_0)$ then show that there exists unique path $f : I \to \widetilde{X}$ with $p\tilde{f} = f$ and $\tilde{f}(0) = \tilde{x}_0$. [7 Marks]

(6) (a) Prove that there is no continuous map $f : \mathbb{S}^2 \to \mathbb{S}^1$ such that [7 Marks]

$f(-x) = -f(x)$ for all $x \in \mathbb{S}^2$. Hence, or otherwise prove the Borsuk Ulam theorem.

(b) Let $p : \widetilde{X} \to X$ be a 7-sheeted covering map. If $\widetilde{X}$ is simply connected then prove that $\pi_1(X, x_0) \cong \mathbb{Z}_7$. [4 Marks]

(c) Define universal covering space of X. Justify the term universal. [3 Marks]

(7) (a) Let $p_i : \widetilde{X}_i \to X$, $i = 1, 2$ be covering maps, and suppose that $\tilde{x}_1 \in \widetilde{X}_1$, $\tilde{x}_2 \in \widetilde{X}_2$ and $p_1(\tilde{x}_1) = p_2(\tilde{x}_2)$. Show that there exists a homeomorphism $h : \widetilde{X}_1 \to \widetilde{X}_2$ such that $h_1(\tilde{x}_1) = \tilde{x}_2$ if and only if $p_{1_\sharp}(\pi_1(\widetilde{X}_1, \tilde{x}_1))= p_{2_\sharp}(\pi_1(\widetilde{X}_2, \tilde{x}_2))$. (Assume that all spaces are path connected and locally path connected) [7 Marks]

(b) Let $p : \widetilde{X} \to X$ be a covering map and $\tilde{x} \in \widetilde{X}$. Show that the group of deck transformations $\Delta(p)$ is isomorphic to [7 Marks]

$$N(p_{\#}\pi_1(\widetilde{X}, \tilde{x}))/p_{\#}\pi_1(\widetilde{X}, \tilde{x})$$

where $N(p_{\#}\pi_1(\widetilde{X}, \tilde{x}))$ is the normaliser of $p_{\#}\pi_1(\widetilde{X}, \tilde{x}$ in $\pi_1(X, p(\tilde{x}))$.