Your Roll Number: ....

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examinations, Nov.- Dec. 2016
Part I Semester I

MATH-104: DIFFERENTIAL EQUATIONS

Time: 3 hours | Maximum Marks: 70

Instructions: • Write your roll number on the space provided at the top of this page immediately on receipt of this question paper • Answer **5** questions choosing atleast two questions from each section. • Each question carries 14 marks.• All the symbols have their usual meaning.

Section A

(1) (a) State and prove the uniqueness theorem for the solution of initial value problem [5 Marks]

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0.$$

(b) Discuss the existence and uniqueness of solution of IVP [3 Marks]

$$\frac{dy}{dx} = y^{\frac{1}{3}}, \quad y(0) = 0.$$

(c) State the existence and uniqueness theorem for the IVP [3 Marks]

$$y_1' = f_1(x, y_1, y_2); \quad y_2' = f_2(x, y_1, y_2); \quad y_1(x_0) = y_1, \quad y_2(x_0) = y_2.$$

(d) Does there exists a unique solution of the IVP [3 Marks]

$$(x^2 - 4)y'''' + 2xy'' + (\sin x)y = 0;$$
$$y(0) = 0, \ y'(0) = 1, \ y''(0) = 1, \ y'''(0) = -1.$$

If so, find the interval of existence.

(2) (a) State and prove Sturm-Comparison theorem. [5 Marks]

(b) Prove that zeros of $\cos x$ and $\cos x - \sin x$ occurs alternately on $\mathbb{R}$. [2 Marks]

(c) Define exponential matrix. Prove that if A is $n \times n$ constant matrix,then $\psi(t) = e^{tA}$ is a fundamental matrix of the system $Y' = AY$. Find the solution of the system $Y' = AY$ satisfying the initial condition $\phi(t_0) = E$. [7 Marks]

(3) (a) Define the Green's function $G(x, \xi)$ for a two point BVP. State the fundamental theorem for the Green's function. [2+2 Marks]

(b) Find the Green's function of BVP $y'' = -e^x$, $y(0) = 0, y(1) + y'(1) = 0$ and hence solve it. [5 Marks]

(c) Find the general solution of the system [5 Marks]

$$y_1' = 2y_1 + y_3, \quad y_2' = y_2, \ y_3' = y_1 + 2y_3$$

by the method of eigenvalues.

(4) (a) Define the periodic Sturm Liouville's system and eigenvalues, eigenfunction of the system. Find the eigenvalues and eigenfunction of the system [2+5 Marks]

$$y'' + \lambda y = 0, \quad y(0) = y(\pi), \ y'(0) = y'(\pi).$$

(b) Define the stability of a critical point of a system $\dot{X} = F(X)$ [2 Marks]

(c) Prove that if the critical point $(0,0)$ of the associated linear system is strictly stable, then the critical point of the non-linear system [5 Marks]

$$\dot{x} = ax + by + f_1(x,y), \quad \dot{y} = cx + dy + f_2(x,y)$$

is also strictly stable provided that $f_1 = o(r)$ and $f_2 = o(r)$.

Section B

(5) (a) Define the Lyapunov's function for the non-linear system $\dot{X} = F(X)$. Find the critical point and describe the nature of the critical point of the system $\dot{x} = -3x + 2y$, $\dot{y} = -2x$ by the method of eigenvalues. [2+3 Marks]

(b) Define the Dirichlet's problem for upper half plane and find its solution by the method of Fourier transform. [6 Marks]

(c) Prove that the solution of Neumann's problem is unique up to the addition of a constant. [3 Marks]

(6) (a) State and prove the Maximum principle theorem. [5 Marks]

(b) Show that the solution of the Dirichlet problem $\nabla^2\psi = 0$ in a volume V enclosed by the surface S when the value of ψ is given on S is given by [9 Marks]

$$\psi(r) = \int_S \psi(\bar{r'})\frac{\partial G}{\partial n} dS$$

where G is a Green's function of the problem.

(7) (a) Find the solution of Dirichlet problem $\nabla^2\psi = 0$, $z \geq 0$, $\psi(x,y,0) = f(x,y)$ by the method of Green's function. [5 Marks]

(b) Describe the theory of Green's function for the solution of space form of wave equation in a finite domain V enclosed by a surface S when the value of ψ or $\frac{\partial\psi}{\partial n}$ is given on S. [9 Marks]

(8) (a) State the Helmholtz's first theorem. [2 Marks]

(b) Prove that the solution of heat equation $\theta_t = k\theta_{xx}$, $-\infty < x < \infty$ under the condition $\theta(x,0) = \phi(x)$,where $\phi(x)$ is bounded and continuous in $(-\infty, \infty)$ is [9 Marks]

$$\theta(x,t) = \frac{1}{2\sqrt{\pi kt}}\int_{-\infty}^{\infty} \phi(\xi)e^{-\frac{(x-\xi)^2}{4kt}} d\xi.$$

(c) Define the Green's function for the solution of heat equation $\theta_t = k\nabla^2\theta$, in a volume V enclosed by a surface S under the suitable initial and boundary condition. [3 Marks]

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