Your Roll Number: ....

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examinations, November-December, 2016
Part II Semester III

MATH-302(ii): MATRIX ANALYSIS

Time: 3 hours Maximum Marks: 70

Instructions: • Write your roll number on the space provided at the top of this page immediately on receipt of this question paper. • Attempt **six** questions in all. • **Question no. 1 is compulsory**. • Attempt all parts of one question at one place. • The symbols have their usual meaning unless otherwise stated.

---

(1) True or False? Justify your claim. [$2 \times 5 = 10$ Marks]

(a) Every monotone vector norm is absolute.

(b) A matrix Lie group is a closed subgroup of $M_n(\mathbb{C})$.

(c) A minimal matrix norm on M_n is unital.

(d) The function $A \to \rho(A)$ is a matrix norm on M_n.

(e) A 2-by-2 doubly stochastic matrix has equal diagonal entries.

(2) (a) For $A \in M_n$, prove that the eigenvalues of $A = [a_{ij}]$ lie in the union of its Gershgorin discs. Hence or otherwise prove that [6 Marks]

$$\rho(A) \leq \min_{p_1,\ldots,p_n>0} \max_{1\leq i\leq n} \frac{1}{p_i} \sum_{j=1}^{n} p_j |a_{ij}|.$$

(b) Define a connected matrix Lie group. Prove that $GL_n(\mathbb{C}), n \in \mathbb{N}$ is connected. Is $GL_n(\mathbb{R}), n > 1$ connected? Justify your answer. [6 Marks]

(3) (a) For $A \in M_n$ and $\epsilon > 0$, prove that there is a matrix norm $|||\cdot|||$ such that $\rho(A) \leq |||A||| \leq \rho(A) + \epsilon$. [6 Marks]

(b) Prove that the Schur's product of two positive semidefinite matrices is positive semidefinite. [6 Marks]

(4) (a) State and prove Singular Value Decomposition Theorem for rectangular matrices. [6 Marks]

(b) Let $|||\cdot|||_\alpha$ and $|||\cdot|||_\beta$ be induced matrix norms on M_n such that $|||A|||_\alpha \leq |||A|||_\beta$ for all $A \in M_n$. Prove that $|||A|||_\alpha = |||A|||_\beta$ for all $A \in M_n$. [6 Marks]

(5) (a) Prove that the set of doubly stochastic matrices is compact and convex. If $A = [a_{ij}] \in M_n, n \geq 2$ is a doubly stochastic matrix and $a_{ii} = 0$ for some i. Prove that A is permutation similar to $[1] \oplus B$, B being a doubly stochastic matrix. [6 Marks]

(b) Prove that the maximum row sum matrix norm on M_n is an induced matrix norm. [6 Marks]

(6) (a) For $A \in M_n, A > 0$, prove that $\rho(A)$ is its eigenvalue of strictly largest modulus. [6 Marks]

(b) Prove that a Hermitian matrix H is positive definite if and only if $\det H_i > 0$ for $i = 1, 2, \ldots, n$, where H_i are the principal submatrices of H. [6 Marks]

(7) (a) Let $A \succeq B \succeq 0$. Prove that $\mathrm{tr}(A) \geq \mathrm{tr}(B)$ and $\det A \geq \det B$. [6 Marks]

(b) Prove that for a positive matrix $A \in M_n$, there exist unique positive real vectors x, y such that $Ax = \rho(A)x, y^T A = \rho(A)y^T$ and $x^T y = 1$. [6 Marks]

✠