Your Roll Number: ....

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations, November-December 2016
Part II Semester III

MATH-304(iv): Methods of Applied Mathematics

Time: 3 hours | Maximum Marks: 70

Instructions: • Write your roll number on the space provided at the top of this page immediately on receipt of this question paper • Question No. 1 is compulsory • Answer **any four** questions from the remaining questions • Each question carries 14 marks.

(1) (a) Define a linear functional and the variation of a functional. [3 Marks]

(b) Solve the integro-differential equation [3 Marks]

$$\phi''(x) + \int_0^x e^{2(x-t)}\phi'(t)dt = e^{2x}, \phi(0) = \phi'(0) = 0.$$

(c) Find the iterated kernel of $K(x,t) = x - t, a = 0, b = 1$. [3 Marks]

(d) Solve $\int_0^x \phi(t)/(x-t)^{\frac{1}{3}}dt = x + x^2$, by using method of Laplace transformation. [3 Marks]

(e) Solve $\phi(x) = \sin x + \lambda \int_0^{\frac{\pi}{2}} \sin x \cos t\phi(t)dt$. [2 Marks]

(2) (a) Prove that for boundary value problem [7 Marks]

$$\begin{aligned} \varepsilon y'' + p(t)y' + q(t)y &= 0, \quad 0 < t < 1, \ 0 < \varepsilon << 1 \\ y(0) = a, \ \ y(1) &= b \end{aligned}$$

where p and q are continuous functions on $0 \le t \le 1$ and $p(t) > 0$ for $0 \le t \le 1$, there exist a boundary layer at $t = 0$ with inner and outer approximations given by

$$y_i(t) = C_1 + (a - C_1)e^{-\frac{p(0)t}{\varepsilon}},$$
$$y_0(t) = b\exp\left(\int_t^1 \frac{q(s)}{p(s)}ds\right),$$

where

$$C_1 = b\exp\left(\int_t^1 \frac{q(s)}{p(s)}ds\right).$$

(b) Discuss the regular perturbation method using spring-mass oscillator where a mass m is connected to a spring whose restoring force has magnitude $ky+ay^3$, y is the replacement of the mass, measured positively from equilibrium, k and a are positive constants characterizing the stiffness properties of the spring. [7 Marks]

(3) (a) Show that the integral equation $\phi(x) = f(x)+\frac{1}{\pi}\int_0^{2\pi} \sin(x+t)\phi(t)dt$ possess no solution for $f(x) = x$, but it possesses infinitely many solutions when $f(x) = 1$. [5 Marks]

(b) Show that if $\alpha(x)$ and $\beta(x)$ are continuous in $[a, b]$ and if [5 Marks]

$$\int_a^b [\alpha(x)h(x) + \beta(x)h'(x)]dx = 0$$

for every function $h(x) \in D_1(a,b)$ such that $h(a) = h(b) = 0$, then $\beta(x)$is differentiable and $\beta'(x) = \alpha(x), \forall x \in [a,b]$.

(c) Prove that the differential of a differentiable functional is unique. [4 Marks]

(4) (a) Show that the solution of the non-homogeneous volterra's integral equation of second kind is [5 Marks]

$$\phi(x) = f(x) + \lambda \int_0^x R(x,t;\lambda)f(t)dt$$

where $R(x,t;\lambda) = \sum_{n=0}^{\infty} \lambda^n K_{n+1}(x,t)$.

(b) With the help of the resolvent kernel, solve [5 Marks]

$$\phi(x) = e^x \sin x + \int_0^x \left(\frac{2+\cos x}{2+\cos t}\right)\phi(t)dt.$$

(c) Construct the resolvent kernel of $K(x,t) = e^{x+t}, a = 0, b = 1$. [4 Marks]

(5) (a) If $F(x,y,z)$ is a function with continuous first and second (partial) derivatives with respect to all its arguments then, among all functions $y(x)$ which are continuously differentiable for $a \le x \le b$ and satisfy the boundary conditions $y(a) = A, y(b) = B$, find the function for which the functional $J[y] = \int_a^b F(x,y,y')dx$ has a weak extremum. [5 Marks]

(b) Find the eigen values and eigen functions of the homogeneous integral equation $\phi(x) = \cos x + \lambda \int_0^{\pi} \sin(x-t)\phi(t)dt$. [5 Marks]

(c) Show that the function $\phi(x) = xe^x$ is a solution of the volterra integral equation $\phi(x) = \sin x + 2\int_0^{2\pi} \cos(x-t)\phi(t)dt$. [4 Marks]

(6) (a) Discuss the Stability and Population Dynamics along with its Bifurcation. [7 Marks]

(b) Using the recursion relation, find the resolvent kernel and solve the integral equation $\phi(x) = x + \lambda \int_0^1 (4xt - x^2)\phi(t)dt$, where $\lambda = 1$. [4 Marks]

(c) Solve $\phi(x) = 1 + \int_0^x (x-t)\phi(t)dt$ by using the method of successive approximation. [3 Marks]

(7) (a) Show that the function $\phi(x) = \sin\frac{\pi x}{2}$ is a solution of Fredholm-type integral equation $\phi(x) = \frac{x}{2} + \frac{\pi^2}{2}\int_0^1 K(x,t)\phi(t)dt$ where [7 Marks]

$$K(x,t) = \begin{cases} \frac{x(2-t)}{2}, & 0 \le x \le t \\ \frac{t(2-x)}{2}, & t \le x \le 1 \end{cases}.$$

(b) Solve the integral equation $\phi(x) = \sinh x - \int_0^x \cosh(x-t)\phi(t)dt$ by using method of Laplace transformation. [7 Marks]

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