Your Roll Number: ....

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examinations, November-December 2016
Part-II Semester-III

MATH302(i): FOURIER ANALYSIS

Time: 3 hours | Maximum Marks: 70

Instructions: • Write your roll number on the space provided at the top of this page immediately on receipt of this question paper • Attempt any five questions. • All questions carry equal marks.

---

(1) (a) For $f \in L^1(\mathbb{T})$ and $\delta > 0$, determine $\lim\limits_{n\to\infty} \int_\delta^\pi f(t)D_n(t)\, dt$, $D_n(t)$ being the Dirichlet kernel. [3 Marks]

(b) Let $f(t) = \begin{cases} e^{-t} & \text{if } t \in [0,\infty), \\ 0 & \text{otherwise.} \end{cases}$ Compute $\widehat{f}(y)$, $y \in \mathbb{R}$ and show that $\widehat{f} \notin L^1(\mathbb{R})$. [4 Marks]

(c) Show that $L^1(G)$ has left approximate identity, G being a locally compact Hausdorff topological group having left Haar measure. [4 Marks]

(d) Let $G = \mathbb{R} \sim \{0\}$ be the multiplicative group. Find the left and right Haar measure of $A = [1,3] \sim \left\{2, \frac{5}{2}\right\}$. [3 Marks]

(2) (a) If $f \in L^1(\mathbb{R})$, show that $\widehat{f}$ is a continuous function that vanishes at infinity. [5 Marks]

(b) Show that for $f \in L^1(\mathbb{T})$, the sequence of partial sum of the Fourier series of f at $t_0 \in \mathbb{T}$ converges to s if and only if [5 Marks]

$$\lim_{N\to\infty} \int_0^\delta [f(t_0+t)+f(t_0-t)-2s]D_N(t)\, dt=0, \quad (\delta > 0),$$

$D_N(t)$ being the Dirichlet kernel.

(c) Define Poisson kernel and show that it is summability kernel. [4 Marks]

(3) (a) Define Fejer kernel on $\mathbb{T}$ and show that it is an approximate identity. [5 Marks]

(b) Let $f \in L^1(\mathbb{T})$ be such that $\widehat{f}(n) = 0$ for all $n \in \mathbb{Z}$. Show that $f = 0$ a.e. [4 Marks]

(c) Show that there exists a function in $L^1(\mathbb{T})$ whose Fourier series diverges. [5 Marks]

(4) (a) Let $f(t) = \begin{cases} 1 & \text{if } -\pi \le t \le \pi, \\ 0 & \text{otherwise.} \end{cases}$ Verify Riemann Lebesgue lemma for functions in $L^1(\mathbb{R})$. [5 Marks]

(b) For $f \in L^1(\mathbb{R})$, define $F(x) = \displaystyle\int_{-\infty}^{x} f(t)\, dt$. If $F \in L^1(\mathbb{R})$, show that [4 Marks]

$$\widehat{F}(\xi) = \frac{1}{i\xi}\, \widehat{f}(\xi),\ \xi \neq 0,\ \xi \in \mathbb{R}.$$

(c) Let $\{K_\lambda\}$ be a family of continuous functions on $\mathbb{R}$ which is a summability kernel. Show that $\{K_\lambda\}$ is a bounded approximate identity. [5 Marks]

(5) (a) Let $f \in L^1(\mathbb{R})$ with $\widehat{f} \in L^1(\mathbb{R})$. Show that [4 Marks]

$$\int_{\mathbb{R}} \widehat{f}(\xi) e^{i\xi x} d\xi = f(x) \ \text{ a.e.}$$

(b) Define Fejer kernel on $\mathbb{R}$. Show that it is a summability kernel. [5 Marks]

(c) Let f be a continuous function with compact support on $\mathbb{R}$. Show that [5 Marks]

$$\|f\|_2 = \|\widehat{f}\|_2.$$

(6) (a) Let $f \in L^1(G)$ and $g \in L^\infty(G)$. Show that the convolution $f * g$ exists and is bounded and uniformly continuous function. [5 Marks]

(b) Let G be a locally compact abelian group. Show that the map $\widehat{G} \to (L^1(G))\widehat{\ }$ defined by $r \to h_r$, where $h_r(f) = \displaystyle\int_G f(x)\overline{r(x)}\, dx$ is a well-defined map of the dual group of G onto maximal ideal space of $L^1(G)$. [5 Marks]

(c) Define Δ-topology and P-topology on the character group of an abelian locally compact T_2 group. Show that these topologies are equal. [4 Marks]

(7) (a) Find the left Haar integral of a discrete infinite group. [4 Marks]

(b) Let G be a locally compact Hausdorff topological group with left Haar measure λ. For $f \in L^p(G)$, $1 \le p < \infty$, show $x \to {}_xf$ is uniformly continuous. [5 Marks]

(c) Let G be a topological group. Let A be a closed subset of G and B a compact subset of G. Show that AB is closed in G. Is the result true if A and B are closed subsets of G? Justify your answer. [5 Marks]

✠