Your Roll Number: ....

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examinations, November-December 2016
Part I Semester I
MATH14-101: FIELD THEORY

Time: 3 hours | Maximum Marks: 70

Instructions: • Write your roll number on the space provided at the top of this page immediately on receipt of this question paper. • This question paper has three sections with three, four and two questions. • Answer **Two** questions from Sections A, **Three** questions from Section B and **One** question from Section C.

1. (a) Prove that every field has a unique least subfiled which is isomorphic to either $\mathbb{Q}$ or $\mathbb{F}_p$ according as characteristic of the field is zero or p. [2,3 Marks]

   (b) Prove that every finite field extension is algebraic. Does the converse hold? Justify. [1,3 Marks]

   (c) Let E/k be a field extension and let α in E be algebraic over k. Prove that if the degree of the minimal polynomial of α over k is odd, then $k(\alpha^2) = k(\alpha)$. [3 Marks]

2. (a) Let α and β be two zeros of the same irreducible polynomial over a field k (in some extension of k). Prove that there exist a unique k-isomorphism between $k(\alpha)$ and $k(\beta)$ mapping α to β. Does the result holds if α and β are zeros of two different irreducible polynomials over k? Justify. [5,3 Marks]

   (b) Prove that every field extension of degree two is normal. Does the result hold for degree three exstensions? Justify. [2,2 Marks]

3. (a) Prove that every field has an algebraic closure. [4 Marks]

   (b) Prove that every algebraically closed field is perfect. [2 Marks]

   (c) Prove that a field k is perfect if and only if every irreducible polynomial in $k[x]$ is separable. [6 Marks]

Section B

4. (a) Prove that any non empty set of distinct automorphisms of a field E is linearly independent over E. Also, deduce that if E/k is a field extension of degree n, then there are atmost n k-automorphisms of E. [6,3 Marks]

   (b) Show that a finite field extension F/k is Galois if and only if the only elements of F fixed under all automorphisms of F over k are the elements of k. [3 Marks]

5. (a) Let E/k be a Galois extension with Galois group G. Let F be a field between k and E and H denote the corresponding subgroup of G. Prove that F/k is normal if and only if H is a normal subgroup of G. Also, show that $\mathrm{Gal}(F/k) \cong G/H$. [4,3 Marks]

(b) Prove that the extension $\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}$ is Galois and find its Galois group. [2,3 Marks]

6. (a) Write $x^{12} - 1$ as a product of irreducible polynomials over $\mathbb{Q}$. [4 Marks]

   (b) Prove that every finite extension of a finte field is Galois with cyclic Galois group. [6 Marks]

   (c) Prove that if a field contains a primitive n-th root of unity, where n is an odd prime, then k also contains a primitive $2n$-th root of unity. [2 Marks]

7. (a) Prove that every finite extension of a field of characteristic zero is simple. [8 Marks]

   (b) Show that if $r|n$, then there exist an irreducible polynomial of degree r over $\mathbb{F}_p$ dividing $x^{p^n} - x$. [2 Marks]

   (c) Prove that any two finite fields of same cardinality are isomorphic. [2 Marks]

Section C

8. (a) Let $f(x)$ be a separable over a field $k \subseteq K$ with E its minimal splitting field over K. If L is the minimal splitting field of $f(x)$ over k contained in E, then prove that $\mathrm{Gal}(E/K)$ is isomorphic to a subgroup of $\mathrm{Gal}(L/k)$ corresponding to the subfield $K \cap L$. Also, deduce that if char $k = 0$ and ζ is a primitive n-th root of unity, then $k(\zeta)/k$ is an abelian extension. [5,2 Marks]

   (b) Let the group G of $f(x)$, over a field k acts transitively on the roots. Prove that the degree of $f(x)$ divides the order of G. [3 Marks]

9. (a) If the group of $f(x)$ over a field k of characteristic zero is solvable, then prove that $f(x) = 0$ is solvable by radicals. [6 Marks]

   (b) Using the fact that S_n is not solvable for $n \geq 5$, find a polynomial equation over a field which is not solvable by radicals. [2 Marks]

   (c) Does the following statement: *"The discriminant Δ of a polynomial $f(x)$ is a square in k if and only if the group of $f(x)$ over k consist of even permutations only"* holds when characteristic of k is two? Justify. [2 Marks]