Your Roll Number: ....

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations, November-December 2016
Part II Semester III
MATH14-303(A): Advanced Complex Analysis

Time: 3 hours | Maximum Marks: 70

Instructions: • Write your roll number on the space provided at the top of this page immediately on receipt of this question paper • **Question 1 is compulsory** and it carries 10 marks; each part has 2 marks • **Answer any five questions from questions 2–7** • Each question from question 2–7 carries 12 marks.

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(1) (a) Let $f\colon [a,b] \to \mathbb{R}$ be differentiable. Show that f is convex if f' is increasing. [2 marks]

(b) If an entire function f satisfies $|f(z)| \leq 3 + 4\sqrt{|z|}$ for all $z \in \mathbb{C}$, then show that f is constant. [2 marks]

(c) Show that $\frac{1}{2}|z| \leq |\log(1+z)| \leq \frac{3}{2}|z|$ for $|z| < \frac{1}{2}$. [2 marks]

(d) If u is harmonic, show that $f = u_x - iu_y$ is analytic. [2 marks]

(e) Prove that an analytic function $f\colon B(a,r) \to \mathbb{C}$ satisfying [2 marks]

$$|f'(z) - f'(a)| < |f'(a)| \text{ for all } z \in B(a,r)$$

is one-to-one.

(2) (a) Prove that a function is convex if and only if the portion of the plane lying above the graph of the function is a convex set. Using this, determine an upper bound for a convex function on $[a,b]$. [8 marks]

(b) Represent $G = \{z\colon 0 < \operatorname{Re} z < 1\}$, as union of compact sets K_n satisfying (i) $K_n \subseteq \operatorname{int} K_{n+1}$, (ii) every component of $\mathbb{C}_\infty \setminus K_n$ contains a component of $\mathbb{C}_\infty \setminus G$. [4 marks]

(3) (a) Let $a \geq 1/2$ and $G = \{z\colon |\arg z| < \pi/(2a)\}$. Suppose $f\colon G \to \mathbb{C}$ is analytic and $|f(z)| \leq Pe^{|z|^b}$ for all z with $|z|$ sufficiently large for some $P > 0$, $b < a$. Prove that $|f(z)| \leq M$ for all $z \in G$. [4 marks]

(b) If $f_n \in \mathcal{H}(G)$, $f_n \to f \in \mathcal{C}(G, \mathbb{C})$, show that $f \in \mathcal{H}(G)$ and $f_n^{(k)} \to f^{(k)}$ for each integer $k \geq 1$. Further show that if each f_n is one-to-one, then f is one-to-one or f is constant. [8 marks]

(4) (a) Let $\mathbb{D} := \{z\colon |z| < 1\}$, G be a simply connected region and $G \neq \mathbb{C}$. Assume that the family $\mathcal{F} = \{f \in \mathcal{H}(G)\colon f \text{ is one-to-one}, f(a) = 0, f'(a) > 0, f(G) \subseteq \mathbb{D}\}$ is non-empty and $\overline{\mathcal{F}} = \mathcal{F} \cup \{0\}$. Prove that there exists a function $f \in \mathcal{F}$ with $f(G) = \mathbb{D}$. [8 marks]

(b) Let K be a compact subset of $\mathbb{C}$ and E be a subset of $\mathbb{C} \setminus K$ that meets every component of $\mathbb{C}_\infty \setminus K$. Show that $(z-a)^{-1} \in B(E)$ for $a \in \mathbb{C} \setminus K$. [4 marks]

(5) (a) Find the infinite product representation of $\cos \pi z$ and hence find the infinite product representation of $\cosh z$. [6 marks]

(b) Let G be an open connected subset of $\mathbb{C}$. Let $n(\gamma; a) = 0$ for every closed rectifiable curve γ in G and for every $a \in \mathbb{C} \setminus G$. Show that $\mathbb{C}_\infty \setminus G$ is connected. [4 marks]

(c) State Mittag-Leffler's Theorem. [2 marks]

(6) (a) Let G be a region and u be a continuous real valued function on G that has MVP. If, for each $a \in \partial_\infty G$, $\limsup_{z \to a} u(z) \leq 0$, show that either $u(z) < 0$ for all $z \in G$ or $u(z) = 0$ for all $z \in G$. [6 marks]

(b) If the function f is analytic on a region containing the closure of $\mathbb{D}$, $f(0) = 0$, and $f'(0) = 1$, show that $f(\mathbb{D})$ contains a disc of radius L, where L is the Landau's constant. [6 marks]

(7) (a) Let G be a region in $\mathbb{C}$. If $u_n \in \text{Har}(G)$, $u_1 \leq u_2 \leq \cdots$, show that either $u_n \to \infty$ uniformly on compact subsets of G or $\{u_n\}$ converges in $\text{Har}(G)$ to a harmonic function. [6 marks]

(b) Let G be simply connected, the function $f\colon G \to \mathbb{C} \setminus \{0, 1\}$ be analytic. Show that there exists an analytic function g on G such that [6 marks]

$$f(z) = -e^{i\pi \cosh(2g(z))} \quad (z \in G).$$

Does $g(G)$ contains a disc of radius 1?

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