Your Roll Number: ....

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examinations, November-December 2016
Part I Semester I
MATH14-102 : COMPLEX ANALYSIS

Time: 3 hours | Maximum Marks: 70

Instructions: • Write your roll number on the space provided at the top of this page immediately on receipt of this question paper • Answer ***six*** (6) questions in all, choosing ***four*** (4) from **Section-A** and ***two*** (2) from **Section-B** •

Section A

(Answer any four (4) questions. Each question carries **11 marks**.)

(1) (a) Suppose $f : G \longrightarrow \mathbb{C}$ is analytic and G is connected such that $f(z)$ is real for all z in G. Show that f is constant. What happen if G is not connected? [3 Marks]

(b) Define *cross ratio* of four numbers in the extended complex plane. Show that a Mobius transformation preserves the cross ratio. [5 Marks]

(c) If $T(z) = \dfrac{az+b}{cz+d}$, find z_2, z_3, z_4 in terms of a, b, c, d such that $T(z) = (z, z_2, z_3, z_4)$. [3 Marks]

(2) (a) Let $f : G \longrightarrow \mathbb{C}$ be analytic and suppose $\overline{B}(a; r) \subset G$ $(r > 0)$. If $\gamma(t) = a + re^{it}, 0 \leq t \leq 2\pi$, then for $|z - a| < r$ show that [7 Marks]

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\omega)}{\omega - z} d\omega$$

(b) If $p(z)$ is a non constant polynomial, then show that there is a complex number a with $p(a) = 0$. [4 Marks]

(3) (a) Let γ be a closed rectifiable curve in $\mathbb{C}$. Show that $n(\gamma; a)$ is a constant for a belonging to a component of $G = \mathbb{C} \setminus \{\gamma\}$ and that $n(\gamma; a) = 0$ for a in the unbounded component of G. [7 Marks]

(b) Let γ be a closed rectifiable curve in G which is homotopic to zero. Show that $n(\gamma; \omega) = 0$ for every ω in $\mathbb{C} \setminus G$. Does the converse hold? Justify. [4 Marks]

(4) (a) Let G be simply connected and $f : G \longrightarrow \mathbb{C}$ be a non vanishing analytic function. Show that there is analytic branch of $\log f(z)$ on G. [4 Marks]

(b) Suppose f is analytic in $B(a; R)$ such that $f(z) - f(a)$ has a zero of order m at $z = a$. Show that there is an $\epsilon > 0$ and $\delta > 0$ such that for $|\zeta - f(a)| < \delta$ the equation $f(z) = \zeta$ has m simple roots in $B(a; \epsilon)$ and hence deduce the Open Mapping Theorem. [7 Marks]

(5) (a) Let f be analytic in $B(a;R)$. Show that a is a zero of f of multiplicity m if and only if $f^{(k)}(a) = 0$ for $k = 0, 1, ., m-1$ and $f^{(m)}(a) \neq 0$. [4 Marks]

(b) State and prove *Goursat Theorem.* [7 Marks]

(6) (a) State the three types of *isolated singularities* giving one example of each. Suppose $z = a$ is an isolated singularity of f. How does Laurent series of f in $\operatorname{ann}(a; 0, R)$ classify the singularities? [5 Marks]

(b) State and prove *Casorati-Weierstrass Theorem.* [6 Marks]

Section B

(Answer any (2) questions. Each question carries **13 marks**.)

(7) (a) What do you mean by *residue* of a function f at a point a ? Determine residue of the function $f(z) = 2e^z/z^4$ at $z = 0$. [3 Marks]

(b) Evaluate the integral [10 Marks]

$$\int_0^\infty \frac{\sin x}{x} dx$$

(8) (a) State *Argument Principle.* [2 Marks]

(b) Suppose f and g are meromorphic in a neighbourhood of $\overline{B}(a;R)$ with no zeros or poles on the circle $\gamma = \{z : |z-a| = R\}$. If Z_f, Z_g (P_f, P_g) are the number of zeros (poles) of f and g inside γ counted according to multiplicities and if $|f(z) - g(z)| < |f(z)| + |g(z)|$ on γ, then show that [5 Marks]

$$Z_f - P_f = Z_g - P_g$$

(c) Let G be a bounded open set in the complex plane and suppose f is continuous on $\overline{G}$ and analytic on G. Show that maximum value of $|f(z)|$ for z in $\overline{G}$ occurs when z is on the boundary of G. What happen when G is not bounded? [6 Marks]

(9) (a) Let G be a region and suppose that $f : G \longrightarrow \mathbb{C}$ is analytic and $a \in G$ be such that $|f(a)| \leq |f(z)|$ for all z in G. Show that either $f(a) = 0$ or f is a constant. [4 Marks]

(b) State *Schwarz's Lemma.* [2 Marks]

(c) Let $\mathbb{D}$ be the open unit disk $\{z : |z| < 1\}$. Let $f : \mathbb{D} \longrightarrow \overline{\mathbb{D}}$ be analytic. For $a \in \mathbb{D}$ deduce the maximum possible value for $|f'(a)|$. Is it possible to have an analytic function f on $\mathbb{D}$ into $\mathbb{D}$ such that $f(\frac{1+i}{2}) = \frac{3}{5}$ and $f'(\frac{1+i}{2}) = \frac{2}{3}$? [7 Marks]

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