Your Roll Number: ....

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations, November-December 2016
Part II Semester III
MATH14-303(B): Measure Theory

Time: 3 hours Maximum Marks: 70

Instructions: • Write your roll number on the space provided at the top of this page immediately on receipt of this question paper • Answer any **three** questions from Section I, **two** from Section II and **two** from Section III. • The symbols used have their usual meanings. Unless otherwise stated, $(X, \mathcal{A}, \mu)$ will be a given measure space, and for Section III, X will denote a locally compact Hausdorff space.

SECTION I

(1) (a) Give an example to show that the Hahn decomposition need not be unique. Also show that the Hahn decomposition is unique except for null sets. [5 marks]

(b) State Jordan decomposition theorem for a signed measure and prove only the uniqueness part. [1+4 marks]

(2) (a) Let ν be a signed measure on $(X, \mathcal{A})$ and $E \in \mathcal{A}$ such that $0 < \nu E < \infty$. Show that there is a positive set A contained in E with $\nu A > 0$. [6 marks]

(b) Let μ be σ-finite, and ν, ν_1, ν_2 be σ-finite measures on $(X, \mathcal{A})$ absolutely continuous with respect to μ. Show that for any nonnegative measurable function f, the Radon-Nikodym derivatives satisfy the following [2 + 2 marks]

$$\int f d\nu = \int f \left[\frac{d\nu}{d\mu}\right] d\mu, \quad \left[\frac{d(\nu_1 + \nu_2)}{d\mu}\right] = \left[\frac{d\nu_1}{d\mu}\right] + \left[\frac{d\nu_2}{d\mu}\right]$$

(3) (a) Define absolute continuity and mutual singularity of measures, give one example each, and show that if ν is a measure on $(X, \mathcal{A})$ such that ν and μ are mutually singular and $\nu \ll \mu$ then $\nu = 0$. [5 marks]

(b) Let $f \in L^p(\mu)$, $1 \leq p < \infty$. Show that given $\varepsilon > 0$, there is a simple function φ vanishing outside a set of finite measure such that $\|f - \varphi\|_p < \varepsilon$. [5 marks]

(4) (a) Let T be a bounded linear functional on $L^p(\mu)$, $1 < p < \infty$. Show that there is a g in $L^q(\mu)$ with $1/p + 1/q = 1$ such that $T(f) = \int fg d\mu, \ \forall f \in L^p(\mu)$. [6 marks]

(b) Suppose μ is σ-finite and there exist nonnegative measurable functions f and g such that $\int_E f \, d\mu = \int_E g \, d\mu$ for every measurable set E. Show that $f = g$ a.e. (μ). [4 marks]

SECTION II

(5) (a) State Caratheodory extension theorem and prove the uniqueness part. [2 + 4 marks]

(b) Let μ^* be an outer measure and $\mathcal{A}$ be the collection of all μ^*-measurable sets. Show that $\mathcal{A}$ is a σ-algebra. [4 marks]

(6) (a) Let $\mathcal{C}$ be a semi algebra of sets and μ a nonnegative set function on $\mathcal{C}$ with $\mu(\emptyset) = 0$ satisfying the following conditions [6 marks]

(i) If $C \in \mathcal{C}$ is the union of a finite disjoint collection (C_j) of sets in $\mathcal{C}$, then $\mu C = \sum_j \mu C_j$.

(ii) If $C \in \mathcal{C}$ is the union of a countable disjoint collection (C_j) of sets in $\mathcal{C}$, then $\mu C \leq \sum_j \mu C_j$.

Show that μ has a unique extension to a measure on the algebra $\mathcal{A}$ generated by $\mathcal{C}$.

(b) Define the Lebesgue outer measure m^* on $\mathbb{R}^n$ for $n > 1$, and show that it is translation invariant. [4 marks]

(7) (a) Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be complete measure spaces and $\mathcal{R}$ be the class of measurable rectangles in $X \times Y$. Let E be a set in $\mathcal{R}_{\sigma\delta}$ with [6 marks]

$\mu \times \nu(E) < \infty$. Assuming that $E_x \in \mathcal{B}\ \forall x \in X$, show that $g(x) = \nu(E_x)$ defines a measurable function of $x \in X$ and that $\int g\ d\mu = \mu \times \nu(E)$.

(b) Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be complete measure spaces, g and h be integrable functions on X and Y, and define $f(x, y) = g(x)h(y)$ for $(x, y) \in X \times Y$. Show that f is integrable on $X \times Y$ and [4 marks]

$$\int f\, d(\mu \times \nu) = \int g\, d\mu \int h d\nu.$$

SECTION III

(8) (a) Define the class $\mathcal{B}a(X)$ of Baire sets of X and show that $\mathcal{B}a(X)$ is the σ-algebra generated by the compact G_δ sets of X. [1+5 marks]

(b) Let μ be a Baire measure on X and $\mathcal{R}$ be the collection of all Baire sets E of X such that [4 marks]

(i) $\forall\, \varepsilon > 0$, there is a σ-compact open set $U \subset X$ with $E \subset U$ and $\mu(U \setminus E) < \varepsilon$.

(ii) $\mu E = \sup\ \{\mu K : K \subset E, K \text{ is a compact } G_\delta\}$.

Then show that $\mathcal{R}$ contains all compact G_δ sets of X.

(9) (a) Let F be a positive linear functional on $C_c(X)$ and μ' be the regular Borel measure on X induced by F. Then show that [6 marks]

$$F(f) = \int f d\mu', \ \ f \in C_c(X).$$

(b) Let $K, U \subset X$ be such that K is compact, U is open and $K \subset U$. Show that there is σ-compact open set V and a compact G_δ set H such that $K \subset V \subset H \subset U$. [4 marks]

(10) (a) Let F be a positive linear functional on $C_c(X)$. Define the outer measure μ^* induced by F and show that μ^* is an outer measure on X. [1+5 marks]

(b) Show that the Lebesgue measure m on $\mathbb{R}$ is regular. [4 marks]

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