Your Roll Number: ....

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations, November-December 2016
Part I Semester I
MATH14-104: Differential Equations

Time: 3 hours | Maximum Marks: 70

Instructions: • Write your roll number on the space provided at the top of this page immediately on receipt of this question paper • Question No.1 is compulsory • Answer **any four** questions from the remaining five questions.

(1) State whether each of the following statements are true or false and justify your assertions:

(a) The zeros of the solution of the equation $y''-y=0$ are not alternate on $(-\infty,\infty)$. [2 Marks]

(b) The solutions of the initial value problem $y'=f(x,y),\ y(x_0)=y_0$ may not be unique although $f(x,y)$ is continuous. [2 Marks]

(c) The matrix function $\psi(t)=\begin{pmatrix} e^t & e^{2t} \\ e^t & 2e^{2t}\end{pmatrix}$ is a fundamental matrix for the system $Y'=AY$ with $A=\begin{pmatrix} 0 & 1 \\ -2 & 3\end{pmatrix}$. [2 Marks]

(d) Necessary condition for the existence of the solution of the Neumann problem $\nabla^2 u=0$ in D and $\frac{\partial u}{\partial \eta}=f(s)$ on B is $\int_B f(s)ds=0$. [2 Marks]

(e) The function $V=\frac{1}{4}(2x^2+2y^2+x^4)$ is a Lyapunov function for the system $\dot{x}=y,\ \dot{y}=-x-y-x^3$. [2 Marks]

(2) (a) (i) State and prove the continuity theorem for an IVP of first order. [3 Marks]

(ii) Consider the IVP $y'=f(x,y), y(0)=0$ on $R:|x|\le\frac{1}{2}, |y|\le\frac{1}{2}$. If $f(x,y)=1-2xy$, show that $|f(x,y)|\le 2$, and that all the successive approximations to the solution exist on $|x|\le\frac{1}{2}$. Also show that f satisfies Lipschitz condition on R. [4 Marks]

(b) (i) State and prove the Sturm separation theorem. Show that every solution of $y''+x^2y=0$ has infinitely many zeros in $[1,\infty)$. [3+2 Marks]

(ii) Show that every non trivial solution of the equation $y''+(\sinh x)y=0$ has at most one zero in $(-\infty,0)$ and infinitely many zeros in $(0,\infty)$. [3 Marks]

(3) (a) (i) Define a standard fundamental matrix. Let ψ be a solution matrix of the system $Y'=AY$ on J, then prove that for some $t_0\in J$, [1+4 Marks]

$$\det\psi(t)=\det\psi(t_0)\exp\left[\int_{t_0}^{t}\sum_{i=1}^{n}a_{ii}(\tau)d\tau\right],\quad \forall\, t\in J.$$

(ii) Find the general solution of the following system by the method of eigen values $y_1'=y_3,\ y_2'=y_1-3y_3,\ y_3'=y_2+3y_3$. [3 Marks]

(b) (i) Define the Green's function $G(x,\xi)$ for a BVP: $L[y]=-f(x)$, $a\le x\le b$, where [1+3 Marks]

$$L=\frac{d}{dx}\left[p(x)\frac{d}{dx}\right]+q(x),\quad a_1y(a)+a_2y'(a)=0,\quad b_1y(b)+b_2y'(b)=0.$$

Show that the Green's function $G(x,\xi)$ for the above BVP is symmetric.

(ii) Solve the non-homogeneous BVP using the Green function $y''=-x^4$, $y(0)=0$, $y(1)+y'(1)=4$. [3 Marks]

(4) (a) (i) Find the Green's function of a BVP $y'' = \sin 3x$, $y(0) = 0$, $y(1)+2y'(1) = 0$. [3 Marks]

(ii) Define the periodic Sturm-Liouville system. Determine the eigenvalues and eigen function of the Sturm-Liouville system $y'' + \lambda y = 0,\ 0 \leq x \leq 1, y(0) = 0, y(1) + hy'(1) = 0, h > 0, a$ is constant. [5 Marks]

(b) (i) Let $\dot{X} = AX$ is linear autonomous system with (n x n) real non-singular constant coefficient matrix A. Let the critical point be at the origin in $\mathbb{R}^n$, then prove that the critical point is (i) strictly stable if the real part of the eigen values of A are negative, (ii) stable if A has atleast one pair of pure imaginary eigen values of multiplicity one, (iii) unstable otherwise. [$4\frac{1}{2}$ Marks]

(ii) Determine stability of the critical point of the system $\dot{x} = -x + 2y$, $\dot{y} = x - y$. [$2\frac{1}{2}$ Marks]

(5) (a) (i) If there exists a Lyapunov function $V(X)$ in Ω, then the origin is stable. [$2\frac{1}{2}$ Marks]

(ii) Show that $u(x, y)$ attains its maximum on the boundary B of D, if $u(x, y)$ is harmonic in a bounded domain D and continuous in $\bar{D} = D \cup B$. [$2\frac{1}{2}$ Marks]

(iii) Using the method of Fourier transform, find the solution of the Dirichlet problem for the upper half plane [3 Marks]

$$\begin{aligned} u_{xx} + u_{yy} &= 0, \ -\infty < x < \infty, \ y > 0 \\ u(x, 0) &= f(x), \ -\infty < x < \infty \end{aligned}$$

with the condition that u is bounded as $y \to \infty$, u and u_x vanish as $|x| \to \infty$.

(b) State and prove the Helmholtz first theorem. Using the theory of Green's function, find the solution of the space form of wave equation $(\nabla^2 + K^2)\psi = 0$ for $z \geq 0$, when $\psi = f(x, y)$ on $z = 0$. [3+4 Marks]

(6) (a) Use the theory of Green's function to find the solution of the three dimensional diffusion equation in a volume V bounded by the simple surface S under the initial and boundary condition. Show that the [5+3 Marks]

$$\theta(x, t) = \frac{1}{\sqrt{2\pi\kappa t}} e^{-\frac{(x-\xi)^2}{4\kappa t}}$$

is a solution of diffusion equation

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\kappa}\frac{\partial \theta}{\partial t}, \ \kappa > 0.$$

(b) Show that the solution of the interior Dirichlet problem $\nabla^2\psi = 0$ within a finite volume V enclosed by a surface S when value of ψ is prescribed on S, is given by [4+3 Marks]

$$\psi(\bar{r}) = -\frac{1}{4\pi}\int_S \psi(\bar{r'}) \frac{\partial G(\bar{r}, \bar{r'})}{\partial \eta} ds'$$

Also, find the Green's function for Dirichlet problem in a semi-infinite space.

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