Your Roll Number: ....

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examination, November-December 2016
Part II Semester III

MATH14-304(D): MATHEMATICAL PROGRAMMING

Time: 3 hours Maximum Marks: 70

Instructions: • Write your roll number on the space provided at the top of this page immediately on receipt of this question paper. • This question paper has three sections. • Section A is compulsory. • Answer any **Three** questions from Section B. • Answer any **Two** questions from Section C.• Use of calculator is allowed.

Section A

(1) (a) If $f : C \to \mathbb{R}$ where C is an open convex subset of $\mathbb{R}^n$ is a convex Gâteaux differentiable function then prove that for all $x, y \in C$ [2 Marks]

$$\langle \nabla f(y) - f(x), y - x \rangle \geq 0.$$

(b) Solve the following problem geometrically [2 Marks]

$$\text{Minimize } x^2 - y$$
$$\text{s.t. } x^2 - y^2 = 0.$$

(c) State Motzkin's transposition theorem. [2 Marks]

(d) Define Slater's constraint qualification for the problem [2 Marks]

$$\text{Minimize } f(x)$$
$$\text{s.t. } g_i(x) \leq 0, i = 1, 2, ..., r,$$
$$h_j(x) = 0, j = 1, 2, ..., m,$$

where f, g_i and h_j are real valued functions defined on $\mathbb{R}^n$.

(e) State a second order necessary conditions for the problem considered in part (d), assuming that the functions f, g_i and h_j have continuous second order partial derivatives on $\mathbb{R}^n$. [2 Marks]

Section B (Answer any three questions.)

(2) (a) If $f : \mathbb{R}^n \to \mathbb{R}$ is a continuous function such that it has a nonempty compact sublevel set then prove that it achieves a global minimum on $\mathbb{R}^n$. [8 Marks]

(b) Show that the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined as [6 Marks]

$$f(x,y) = \begin{cases} \dfrac{x^3 y}{x^4 + y^2}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0), \end{cases}$$

is Gâteaux differentiable but not Fréchet differentiable at $(0, 0)$.

(3) (a) Prove that a function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if epi(f) is a convex set in $\mathbb{R}^{n+1}$. [7 Marks]

(b) Find the critical points of the function $f(x, y, z) = xyze^{-x-y-z}$ defined on $\mathbb{R}^3$ and determine their nature. [7 Marks]

(4) (a) If C and D are two disjoint nonempty convex sets in $\mathbb{R}^n$ then prove that there exists a hyperplane that separates C and D. [7 Marks]

(b) Consider the problem [7 Marks]

$$\begin{aligned} &\text{Minimize } x \\ \text{s.t. } &(x-3)^2+(y-2)^2 \geq 13 \\ &(x-4)^2+y^2 \leq 16. \end{aligned}$$

Find all the points satisfying the KKT conditions.

(5) (a) Let $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ be convex sets and let $L : A \times B \to \mathbb{R}$ be a convex-concave function. Prove that the set of saddle points of L is a set of the form $A_0 \times B_0$ where A_0 and B_0 are convex sets. [7 Marks]

(b) If A is $m \times n$ matrix, Q is $n \times n$ symmetric positive definite matrix, $c \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$, determine the dual of the problem [7 Marks]

$$\begin{aligned} &\text{Minimize } \langle Qx, x\rangle + \langle c, x\rangle \\ &\text{s.t. } Ax = b, x \geq 0. \end{aligned}$$

Section C (Answer any two questions.)

(6) Explain Wolfe's method to solve a quadratic problem with linear constraints and use it solve the problem [9 Marks]

$$\begin{aligned} &\text{Maximize } z = 2x_1 + x_2 - x_1^2 \\ &\text{s.t. } x_1 + x_2 \leq 2 \\ &x_1 \geq 0, x_2 \geq 0. \end{aligned}$$

(7) (a) Use convex simplex method to solve the following problem [7 Marks]

$$\begin{aligned} &\text{Minimize } z = x_1^2 + x_2^2 - 6x_1 - 4x_2 \\ &\text{s.t. } x_1 + x_2 \leq 2 \\ &-x_1 + 2x_2 \leq 3 \\ &x_1 \geq 0, x_2 \geq 0. \end{aligned}$$

(b) Solve the problem in part (a) geometrically. [2 Marks]

(8) Let $f, g_i, i = 1, 2, ...r, h_j, j = 1, 2, ...m$ be continuous function defined on $\mathbb{R}^n$ and X be a nonempty compact set in $\mathbb{R}^n$. Let α be a continuous penalty function on $\mathbb{R}^n$ such that for each μ, there is $x_\mu \in X$ such that [9 Marks]

$$\theta(\mu) = \inf\{f(x) + \mu\alpha(x) : x \in X\} = f(x_\mu) + \mu\alpha(x_\mu).$$

Prove that

$$\inf\{f(x) : g(x) \leq 0, h(x) = 0, x \in X\} = \sup_{\mu\geq 0}\theta(\mu) = \lim_{\mu\to\infty}\theta(\mu).$$

Also, prove that if x is the limit of any convergent subsequence of $\{x_\mu\}$ then

$$\sup_{\mu\geq 0}\theta(\mu) = f(x).$$

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