Your Roll Number: ....

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations, November-December, 2016
Part II Semester III
MATH14-302(B): MATRIX ANALYSIS

Time: 3 hours | Maximum Marks: 70

Instructions: Write your roll number on the space provided at the top of this page immediately on receipt of this question paper. • Attempt <u>**six**</u> questions in all. • **Question no. 1 is compulsory**. • Attempt all parts of one question at one place. • The symbols have their usual meaning unless otherwise stated.

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(1) True or False, justify your claim: [$2 \times 5 = 10$ Marks]

(a) ℓ_1-norm on M_n is an induced matrix norm.

(b) For $A \in M_n$, AA^* is unitarily similar to A^*A.

(c) The group $\mathbb{R}^n$ under vector addition is a matrix Lie group.

(d) The function $A \to \rho(A)$ is a matrix norm on M_n.

(e) A 2-by-2 doubly stochastic matrix has equal diagonal entries.

(2) (a) Define a complex symplectic group $Sp(n, \mathbb{C})$. Prove that $A \in Sp(n, \mathbb{C})$ if and only if $A^{tr}JA = A$, where $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. Is $Sp(n, \mathbb{C}) \cap U(2n)$ compact? Justify your answer. [6 Marks]

(b) Let $|||\cdot|||_\alpha$ and $|||\cdot|||_\beta$ be induced matrix norms on M_n. Prove that [6 Marks]

$$\max_{A \neq 0} \frac{|||A|||_\alpha}{|||A|||_\beta} = \max_{A \neq 0} \frac{|||A|||_\beta}{|||A|||_\alpha}$$

(3) (a) For $A \in M_n$, prove that the eigenvalues of A lie in the union of its Gershgorin discs. Hence or otherwise prove that [3+2 Marks]

$$\rho(A) \leq \min_{p_1,\ldots,p_n > 0} \max_{1 \leq i \leq n} \frac{1}{p_i} \sum_{j=1}^{n} p_j |a_{ij}|.$$

(b) Prove that the maximum column sum matrix norm on M_n is a minimal matrix norm. [7 Marks]

(4) (a) Let $A = [a_{ij}], B = [b_{ij}] \in M_n$ be non-negative matrices such that $A \leq B$. Prove that $\rho(A) \leq \rho(B)$. Hence or otherwise prove that there exists an eigenvalue λ of B such that $a_{ii} \leq |\lambda|$ for all $1 \leq i \leq n$. [3+3 Marks]

(b) State and prove Singular Value Decomposition Theorem for rectangular matrices. [1+5 Marks]

(5) (a) Define a doubly stochastic matrix. If $I \neq A = [a_{ij}] \in M_n$ is a [1+4 Marks]

doubly stochastic matrix, then prove that there is a permutation σ of $\{1, 2, \ldots, n\}$ different from identity permutation such that $a_{1\sigma(1)} \cdots a_{n\sigma(n)} > 0$.

(b) Let $A = [a_{ij}]$ be a positive semidifinite matrix such that $a_{kk} = 0$ for some $k \in \{i, \ldots, n\}$. Prove that $a_{ik} = a_{ki} = 0$ for each $i = 1, 2, \ldots, n$. [3 Marks]

(c) Define the group $O(2)$. Show that every element A of $O(2)$ is precisely of the two forms: [4 Marks]

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \text{or} \quad A = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

(6) (a) For $0 < A \in M_n$, prove that $\rho(A)$ is its eigenvalue of strictly largest modulus. [6 Marks]

(b) For $X = \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix}$, $a \in \mathbb{R}$, evaluate e^X. [3 Marks]

(c) Prove that $\|\cdot\|_1^D = \|\cdot\|_\infty$, where $\|\cdot\|^D$ denotes the dual norm. [3 Marks]

(7) (a) Let $A \succeq B \succeq 0$. Which of the following is true? Justify your answer in each case. [8 Marks]

(i) $A^2 \succeq B^2$

(ii) $A^{\frac{1}{2}} \succeq B^{\frac{1}{2}}$

(iii) $\det A \geq \det B$

(b) If $A, B \in M_n$ are normal, prove that $\rho(AB) \leq \rho(A)\rho(B)$. Give examples of matrices which do not satify this inequality. [4 Marks]