Your Roll Number: ....

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations, November-December 2015
Part II Semester III
MATH14-301(B): Representation Theory of Finite Groups

Time: 3 hours | Maximum Marks: 70

Instructions: • Write your roll number on the space provided at the top of this page immediately on receipt of this question paper • **Q.1** is compulsory, attempt any **six** from the rest. All questions carry equal marks • G will denote a finite group and F a scalar field throughout the question paper.

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1 **a**) If x is a non identity element of the group G, then prove that $\chi(g) \neq \chi(1)$ for some irreducible character χ of G. [2 marks]

b) Show that the 2−dimentional subspace

$$W = \langle 1 + \omega^2 a + \omega a^2, b + \omega^2 ab + \omega a^2 b \rangle$$

is an irreducible $\mathbb{C}G$-submodule of the regular $\mathbb{C}G$-module when $G = D_6$ and $\omega = e^{2\pi i/3}$. [3 marks]

c) If G is a non-abelian group of order 6, then find dimensions of all the irreducible $\mathbb{C}G$-modules. [2 marks]

d) Prove that a non-zero, non trivial character χ satisfying $\chi(g) \geq 0$ for all $g \in G$, is reducible. Also show that $\langle \psi_{reg}, \psi \rangle = \psi(1)$ for any character ψ of G. [2+1 marks]

2 **a**) Show that the 2-dimensional FG−module F^2 when $G = D_8$ is irreducible. [3 marks]

b) Determine the regular representation of $C_2 \times C_2$ over F. [2 marks]

c) Give an example of a faithful representation of G when $G = D_8$ [2 marks]

d) Define an FG-module. Show that for an FG−module V, the mapping $g \longmapsto [g]_\beta$ for $g \in G$ is a representation of G over F, where β is a basis of V. [1+2 marks]

3 **a**) If $G = S_3$ and $V = \langle v_1, v_2, v_3 \rangle$ is the permutation module then find an FG-submodule W of V such that $V = U \oplus W$ where U is the 1−dimensional space $\langle v_1 + v_2 + v_3 \rangle$ and F is $\mathbb{R}$ or $\mathbb{C}$. [3 marks]

b) If $G = \langle (1,2), (3,4) \rangle$ is a subgroup of S_4 then prove that the permutation module for G over F is not isomorphic to the regular FG−module. [7 marks]

4 **a**) Determine all the irreducible representations of the group $G = C_2 \times C_2$ and the values $\{vg \mid v \in$ irreducible $\mathbb{C}G$-modules and $g \in G\}$ [3 marks]

b) Find all the non-isomorphic irreducible representations of the dihedral group D_6 and describe every irreducible representation of this group over $\mathbb{C}$. Also express the regular $\mathbb{C}G$-module $\mathbb{C}G$ as direct sum of irreducible $\mathbb{C}G$-submodules. [4+2+1 marks]

5 **a**) State Schur's lemma. Interpret and prove Schur's lemma and its converse in terms of representations. [1+2+3 marks]

b) Prove that the dimension of the space of $\mathbb{C}G$-homomorphisms from the regular $\mathbb{C}G$-module to any other $\mathbb{C}G$-module U is equal to $\dim U$. [4 marks]

6 **a**) If χ is a character of the representation $\rho : G \longrightarrow GL(n, \mathbb{C})$ then prove that for $g \in G$, $|\chi(g)| = \chi(1)$ if and only if $g\rho = \lambda I_n$ for some $\lambda \in \mathbb{C}$. [2 marks]

b) Define regular character of the group G. Also determine the regular character of S_3. [1+2 marks]

c) Prove that the set of irreducible characters of group G forms a basis of the vector space of all class functions on G. [6 marks]

7 **a**) Define a lift of a character. If N is a normal subgroup of a group G, then show that the irreducible characters of G/N corresponds bijectively to irreducible characters of G which have N in their kernel. [1+4 marks]

b) Lifting the irreducible characters of the quotient group S_n/S_n' find all the linear character of the group $S_n (n \geq 2)$ where S_n' is the derived subgroup of S_n. [3 marks]

c) Find the permutation character and all linear characters of S_5. Are they irreducible? [2 marks]

8 **a**) Consider a $\mathbb{C}G$-module V with character χ. Prove that $S(V \otimes V)$ and $A(V \otimes V)$ are $\mathbb{C}G$-submodules of $V \otimes V$ of dimensions $n(n+1)/2$ and $n(n-1)/2$ respectively where $S(V \otimes V)$ is the symmetric part and $A(V \otimes V)$ is the antisymmetric part of of $V \otimes V$ respectively and n is the dimension of V. Also determine the characters of both $S(V \otimes V)$ and $A(V \otimes V)$ [4+2 marks]

b) Determine the complete character table of the alternating group A_4. [4 marks]