Your Roll Number: ....

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations, December 2016
Part I Semester I
MATH14-103: Measure and Integration

Time: 3 hours | Maximum Marks: 70

Instructions: All sections are compulsory. Attempt any **three** questions from Section **I**, **two** questions from Section **II** and **two** questions from Section **III**. Symbols have their usual meaning. $[a, b]$ denotes a finite interval.

Section I (Answer any **Three** questions: 3×10 marks $= 30$ marks)

1. (a) What is meant by a measurable set? Show that every interval is measurable. [5 marks]

   (b) Prove that the class of Lebesgue measurable sets is a σ-algebra. [5 marks]

2. (a) Show that if $\mathbb{Q} = \bigcup\limits_{n=1}^{\infty} \{q_n\}$, $G = \bigcup\limits_{n=1}^{\infty} \left(q_n - \frac{1}{n^2}, q_n + \frac{1}{n^2}\right)$ and F is any closed set, then $m(G \triangle F) > 0$. [4 marks]

   (b) Show that the Cantor set is measurable. [2 marks]

   (c) Prove that not every measurable set is a Borel set. [4 marks]

3. (a) Show that $f \leq \text{ess sup} f$, a.e. [3 marks]

   (b) Show that if f is a measurable function, then $\{x : f(x) = \alpha\}$ is measurable for each extended real number α. [2 marks]

   (c) Prove that if f is a non-negative measurable function, then there exists a sequence $\{\phi_n\}$ of measurable simple functions such that, for each x, $\phi_n(x) \uparrow f(x)$ and $\lim \int \phi_n dx = \int f dx$. [5 marks]

4. (a) Let $\{f_n\}$ be a sequence of integrable functions such that [5 marks]

$$\sum_{n=1}^{\infty} \int |f_n| dx < \infty.$$

   Show that if the series $\sum_{n=1}^{\infty} f_n(x)$ converges a. e., then its sum $f(x)$ is integrable. Is $\int f dx = \sum_{n=1}^{\infty} \int f_n dx$? Justify your answer.

   (b) Show that if f is bounded and measurable function on $[a, b]$, then for each $\epsilon > 0$ there exists a continuous function g such that g vanishes outside a finite interval and $\int_a^b |f - g| dx < \epsilon$. [5 marks]

Section II (Answer any **Two** questions: 2×12 marks $= 24$ marks)

5. (a) What is meant by a function of bounded variation? Show that $f \in BV[a,b]$ if and only if f is the difference of two finite-valued monotone increasing functions on $[a,b]$ [5 marks]

   (b) Let f be a function defined on $[0,1]$ by $f(x) = 0$ if x is rational and $f(x) = 1$, otherwise. Find four derivatives at any x. [3 marks]

   (c) Let $f \in BV[a,b]$. Is f continuous on $[a,b]$? Is f monotone on $[a,b]$? Justify your answer. [4 marks]

6. (a) Show that the product of two absolutely continuous functions is absolutely continuous. [3 marks]

   (b) Show that if f is a finite-valued monotone increasing function defined on $[a,b]$, then f' is measurable and $\int_a^b f'dx \leq f(b) - f(a)$. What happens if f is monotone decreasing on $[a,b]$? Justify your answer. [7 marks]

   (c) Show that every σ-algebra is a σ-ring. Is the converse true? Justify your answer. [2 marks]

7. (a) Prove that a function F is an indefinite integral if and only if it is absolutely continuous. [7 marks]

   (b) Show that if f' exists and is bounded on $[a,b]$, then $f \in BV[a,b]$. [2 marks]

   (c) Let μ be a measure on a ring $\mathcal{R}$ and let $A, B, C \in \mathcal{R}$ be such that $A, B \subset C$. Show that if $\mu(A) = \mu(C) < \infty$, then $\mu(B) = \mu(A \bigcap B)$ [3 marks]

Section III (Answer any **Two** questions: 2×8 marks $= 16$ marks)

8. (a) What is meant by a complete measure? Show that the Lebesgue measure on $\mathcal{M}$ is σ-finite but not finite. [4 marks]

   (b) State and prove the Minkowski's inequality. [4 marks]

9. (a) Show that if $f_n \to f$ a. u., then $f_n \to f$ a. e. [2 marks]

   (b) Show that if f is a non-negative measurable function defined on a measurable space $(X, \mathcal{S}, \mu)$, then $\phi(E) = \int_E f d\mu$ is a measure on $\mathcal{S}$. Also show that if $\int f d\mu < \infty$,then for each $\epsilon > 0$, there exists a $\delta > 0$ such that if $A \in \mathcal{S}$ and $\mu(A) < \delta$, then $\phi(A) < \epsilon$. [6 marks]

10. (a) Show that if $\mu(X) < \infty$ and $0 < p < q \leq \infty$, then $L^q(\mu) \subset L^p(\mu)$. [3 marks]

    (b) Prove that $L^p(\mu)$ is complete, where $1 \leq p < \infty$. [5 marks]

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