Your Roll Number: ....

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations, November-December 2016
Part II Semester III
MATH14-302(C): Theory of Bounded Operators

Time: 3 hours | Maximum Marks: 70

Instructions: • Write your roll number on the space provided at the top of this page immediately on receipt of this question paper • Question 1 is compulsory • Answer **five** questions in all • All symbols have their usual meaning.

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(1) Give an example of each of the following with brief justification.

(a) A bounded operator whose residual spectrum is empty. [2 Marks]

(b) A compact operator T such that the null space of T is not finite dimensional. [2 Marks]

(c) Two projections whose product is not a projection. [2 Marks]

(d) An approximate eigen value for the left shift operator on l^2. [2 Marks]

(e) A spectral family for the operator T on a Banach space X given by $Tx = \alpha x,\ \forall\ x \in X$, where $\alpha \in \mathbb{R}$ is fixed. [3 Marks]

(f) Give an example of a partial isometry which is neither an isometry nor a projection. [3 Marks]

(2) Let X be a Banach space and $T \in \mathcal{B}(X)$.

(a) Prove that $\sigma(p(T)) = \{p(\lambda) : \lambda \in \sigma(T)\}$ for any polynomial p. Hence or otherwise determine the spectrum of an orthogonal projection P defined on a complex Hilbert space H. [6+2 Marks]

(b) Let $X = C[0,1]$ and T be defined as $Tf = qf$, where $q(t) = t^2$. Find $\sigma(T)$. [6 Marks]

(3) Let T be a bounded linear operator on a Banach space X.

(a) Show that if T is a finite rank operator then it must be compact. If X is a Hilbert space and T is given by $Tx = \langle x, y\rangle z$ where $y, z \in X$ are fixed, is T compact? [4 Marks]

(b) Let $1 \leq p < \infty$ and $X = l^p$. Let T be given by $T(x_1, x_2, \ldots) = (x_1/3, x_2/3^2, x_3/3^3, \ldots)$. Is T compact? [5 Marks]

(c) Show that if T is compact and $\{x_n\} \subset X$ is a weakly convergent sequence then $\{Tx_n\}$ is (norm) convergent. [5 Marks]

(4) Let T be a compact operator defined on a Banach space X.

(a) Show that the set of eigen values of T is countable. Hence deduce that the backward shift on l^2 cannot be compact. [6+3 Marks]

(b) Show that $x \in X$ is a solution of the equation $Tx - \lambda x = y$ if and only if $y \in X$ is such that $f(y) = 0$, for all $f \in X'$ satisfying $T^*f - \lambda f = 0$. [5 Marks]

(5) Let $T : H \to H$, be a bounded self adjoint operator on a complex Hilbert space H.

(a) Show that $\sigma(T)$ is real. Prove further that $\sigma(T) \subset [0, \infty)$ if and only if T is positive. [3+5 Marks]

(b) Show that if T is positive and S is a positive operator commuting with T then ST is also positive. [6 Marks]

(6) Let H be a Hilbert space.

(a) Let P_2, P_1 be two projections on H. Give a necessary and sufficient condition for the sum $P_2 + P_1$ to be a projection. When $P_2 + P_1$ is a projection, what is its range? [6 Marks]

(b) Let $\{P_n\}$ be a monotonically decreasing sequence of projections on H. Show that $\{P_n\}$ converges strongly to a projection P and find the range and null space of P. Illustrate the above with an example. [6+2 Marks]

(7) (a) Let W be a partial isometry on a Hilbert space H. Show that W^* is also a partial isometry and find the initial and final space of W^*. Further prove that W^*W is a projection onto $(KerW)^{\perp}$. [4+3 Marks]

(b) Show that every operator T on an infinite dimensional Hilbert space can be written as $A = W|A|$, where W is a partial isometry. Is this decomposition unique? Justify. [4+3 Marks]

(8) (a) Show that every compact operator defined on a Hilbert space admits a singular value decomposition. [6 Marks]

(b) What is meant by a *trace class operator*. Define the trace trA of a trace class operator A on a Hilbert space H and show that it is independent of the choice of orthonormal basis of H. Find the trace of the operator $T : l^2 \to l^2$ where $T(x_1, x_2, \ldots) = (x_1, x_2, 0, 0, \ldots)$. [1+5+2 Marks]