Your Roll Number: ....

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI
M.A./M.Sc. Mathematics Examinations, Nov./Dec. 2016
Part II Semester III

MATH14-301(C): COMMUTATIVE ALGEBRA

Time: 3 hours | Maximum Marks: 70

Instructions: • Write your roll number on the space provided at the top of this page immediately on receipt of this question paper. • Throughout the question paper the word "ring" shall mean commutative ring with nonzero identity.

Section A

(Answer any one question)

(1) (a) Define the Jacobson radical $J(R)$ of R. Show that

$$J(R) = \{a \in R \mid 1 - ab \text{ is unit in } R, \text{ for all } b \in R\}.$$

Hence or otherwise find the Jacobson radical of $\mathbb{Z}/12\mathbb{Z}$. [5+2 Marks]

(b) Let M, N be R-modules and let M', N' be submodules of M, N respectively. Let $x_i \in M, y_i \in N$ be such that $\sum_{i=1}^{n} x_i \otimes y_i = 0$ in $M \otimes_R N$. Is $\sum_{i=1}^{n} x_i \otimes y_i = 0$ in $M' \otimes_R N'$? Justify your answer. If an element is zero in $M' \otimes_R N'$, will the same is zero in $M \otimes_R N$? Find the finitely generated submodules M_0, N_0 of M, N, respectively, such that $\sum_{i=1}^{n} x_i \otimes y_i = 0$ in $M_0 \otimes_R N_0$. [3+4 Marks]

(2) (a) State and prove Chinese remainder. Use it show

$$\frac{\mathbb{Z}}{6\mathbb{Z}} \cong \frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{3\mathbb{Z}}.$$

[5+2 Marks]

(b) Let $f : A \to B$ be a ring homomorphism and let M be an A-module. Show that $M \otimes_A B$ is a B-module. If M is flat over A, is $M \otimes_A B$ flat over B? Justify. [4+3 Marks]

Section B

(Answer any three questions)

(3) (a) Let J be an ideal in $S^{-1}R$, where S is a multiplicative closed subset of R. Show that $J = S^{-1}\mathfrak{a}$ for some ideal $\mathfrak{a}$ in R. Find all prime ideals $\mathfrak{p}$ of $\mathbb{Z}$ such that $S^{-1}\mathfrak{p}$ is a prime ideal in $S^{-1}\mathbb{Z}$, where $S = \mathbb{Z} \setminus 5\mathbb{Z}$. [3+3 Marks]

(b) For any ideal $\mathfrak{a}$ in R, show that $S^{-1}\mathfrak{a} = \cup_{s\in S}(\mathfrak{a} : s)$. [4 Marks]

(c) Let R be a domain with quotient field K. Show that the quotient field of $R_{\mathfrak{p}}$ is isomorphic to $K_{\mathfrak{p}}$. [4 Marks]

(4) Prove or disprove the following:

(a) A power of prime ideal need not be primary. [5 Marks]

(b) Every primary ideal is a power of prime ideal. [4 Marks]

(c) Decomposable ideals have unique primary decomposition. [5 Marks]

(5) Let $A \subset B$ be rings.

(a) Let B be integral over A. If $\mathfrak{p}$ is a prime ideal of A, then show that there exists a prime ideal $\mathfrak{q}$ of B such that $\mathfrak{q}^c = \mathfrak{p}$. For any prime ideal $\mathfrak{p}_1 \supset \mathfrak{p}$ of A, show that there exists a prime ideal $\mathfrak{q}_1 \supset \mathfrak{q}$ such that $\mathfrak{q}_1^c = \mathfrak{p}_1$. [4+5 Marks]

(b) Let C be integral closure of A in B. For any ideal $\mathfrak{a}$ in A, show that the integral closure of $\mathfrak{a}$ in B is the radical of $\mathfrak{a}^e$ in C. [5 Marks]

(6) (a) Let $A \subset B$ be integral domains, B finitely generated over A. Let v be a nonzero element of B. Show that there exists a nonzero $u \in A$ with the following property:
any homomorphism f from A into an algebraically closed field Ω such that $f(u) \neq 0$ can be extended to a homomorphism g of B into Ω such that $g(v) \neq 0$. [6 Marks]

(b) Let k be a field and B a finitely generated k-algebra. Show that if B is a field then it is finite algebraic extension of k. [4 Marks]

(c) Let B be a valuation ring of a field K. Show that B is local and integrally closed in K. [4 Marks]

Section C

(Answer any one questions)

(7) (a) Show that the nilradical is nilpotent in Noetherian and Artinian ring. [4+5 Marks]

(b) Let A be the valuation ring of discrete valuation v on a field K. Show that the ideals in A are of the form $\{a \in A \mid v(a) \geq n\}$, for some $n \in \mathbb{N}$. [5 Marks]

(8) (a) Let R be an integrally closed, Noetherian, local domain of dimension one with $\mathfrak{m}$ its maximal ideal. Show that every ideal in R is a power of $\mathfrak{m}$. [4 Marks]

(b) Let R be a discrete valuation ring. Show that R is a Noetherian local ring of dimension one in which every nonzero ideal is a power of the maximal ideal. [6 Marks]

(c) Prove or disprove that an Artin ring is a zero dimensional ring. [4 Marks]

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