Your Roll Number: ................

Department of Mathematics, University of Delhi
M.A./M.Sc. Mathematics Examinations, November-December 2015
Part II Semester III
**MATH14-301(A): Algebraic Topology**

Time: 3 hours | Maximum Marks: 70

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**Instructions:** • Write your roll number on the space provided at the top of this page immediately on receipt of this question paper • Answer **any five** questions • Each question carries 14 marks.

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(1) (a) Let $f$, $g$ and $h$ be paths in $X$ such that $f(1) = g(0)$ and $g(1) = h(0)$. Show that $(f * g) * h \simeq f * (g * h)$ rel $\{0, 1\}$. [5 Marks]

(b) Show that any two continuous maps from the plane $\mathbb{R}^2$ into the 2-sphere $\mathbb{S}^2$ are homotopic. [3 Marks]

(c) Let $A$ be a subspace of a space $X$. Give an example that $A$ is a deformation retract of $X$ but not a strong deformation retract. [6 Marks]

(2) (a) Let $X$ be any space. Show that $\mathbb{R}^2 \times X$ is homotopically equivalent to $X$. [5 Marks]

(b) Let $X$ be starlike with respect to origin and compact subspace of the euclidean space $\mathbb{R}^n$. If $f : X \to \mathbb{S}^1$ is a continuous map such that $f(0) = 1$ then show that, for each integer $m$, there exists $\tilde{f} : X \to \mathbb{R}$ such that $p\tilde{f} = f$ and $\tilde{f}(0) = m$, where $p$ is the exponential map. [9 Marks]

(3) (a) If $\phi : X \to Y$ is a homotopy equivalence then show that for any $x_0 \in X$, $\phi_\sharp : \pi_1(X, x_0) \to \pi_1(Y, \phi(x_0))$ is an isomorphism. [5 Marks]

(b) Determine the fundamental groups of (i) $\mathbb{R}^3$ with $x$-axis removed, and (ii) the punctured disc $\mathbb{D}^2 - \{(0, 0)\}$. [5 Marks]

(c) Show that the exponential map $p : \mathbb{R} \to \mathbb{S}^1$ is open. [4 Marks]

(4) (a) Show that a space $X$ is contractible if and only if the identity map on $X$ is homotopic to a constant map of $X$ into itself. [4 Marks]

(b) Define the degree of a loop in $\mathbb{S}^1$ based at 1. Let $f$ and $g$ be two loops in $\mathbb{S}^1$ based at 1. Show that the degree of $f$ is equal to the degree of $g$ if and only if $f \simeq g$ rel $\{(0, 1)\}$. [6 Marks]

(c) Show that every continuous map $f : \mathbb{D}^2 \to \mathbb{D}^2$ has at least one fixed point. [4 Marks]

(5) (a) Show that the quotient map $\mathbb{S}^n \to \mathbb{R}P^n$, which identifies pair of antipodal points, is a covering map. [7 Marks]

(b) Let $p : \widetilde{X} \to X$ be a covering map, and $f : I \to X$ be a path [7 Marks] with origin $x_0$. If $\tilde{x}_0 \in p^{-1}(x_0)$ then show that there exists unique path $f : I \to \widetilde{X}$ with $p\tilde{f} = f$ and $\tilde{f}(0) = \tilde{x}_0$.

(6) (a) Let $p : \widetilde{X} \to X$ be a covering map. If $X$ is locally path connected and $\widetilde{C}$ a path component of $\widetilde{X}$ then show that $p(\widetilde{C})$ is a path component of $X$ and $p|_{\widetilde{C}} : \widetilde{C} \to p(\widetilde{C})$ is a covering map. [6 Marks]

(b) State the Covering Homotopy Theorem. State and prove the Monodromy Theorem. [5 Marks]

(c) Prove that any two continuous maps from a simply connected and locally path connected space $X$ into $\mathbb{S}^1$ are homotopic. [3 Marks]

(7) (a) Let $p : \widetilde{X} \to X$ be an $n$-sheeted covering map. If $\widetilde{X}$ is simply connected and $n$ is a prime then prove that $\pi_1(X, x_0) \cong \mathbb{Z}_n$. [4 Marks]

(b) Let $p : \widetilde{X} \to X$ be a regular covering map and $\tilde{x} \in \widetilde{X}$. Show that the group of deck transformations $\Delta(p)$ is isomorphic to $\pi_1(X, p(\tilde{x}))/p_{\#}\pi_1(\widetilde{X}, \tilde{x})$. Deduce that $\pi_1(\mathbb{S}^1, 1) \cong \mathbb{Z}$. (Assume that all spaces are path connected and locally path connected) [7+3 Marks]