Lie Groups and their Applications

DU Talk: A Gentle Introduction to Lie Groups

At Delhi University, Department of Mathematics, February 16, 2024

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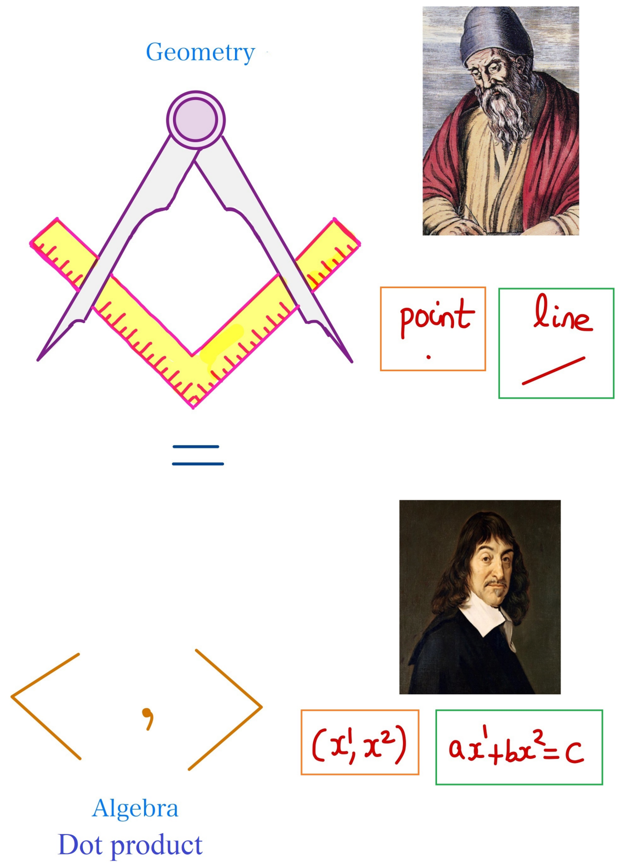
Story so far: The Euclidean Space

- $\mathbb{R}^n$ - a single coordinate system $(x_1, x_2, \ldots, x_n)$ for the entire space - geometry is taken over by algebra of coordinates



Coordinates - so what?

- All of geometry is taken over by algebra of coordinates

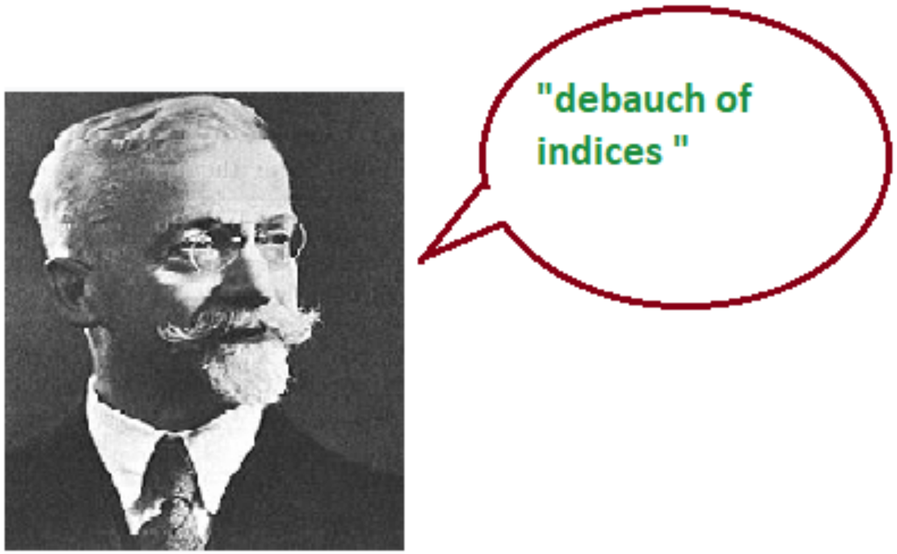


Figure: Élie Cartan

- Points $\rightarrow$ real numbers
- straight lines $\rightarrow$ linear equations
- conics $\rightarrow$ quadratic equations...

Not Intellectually Satisfying!

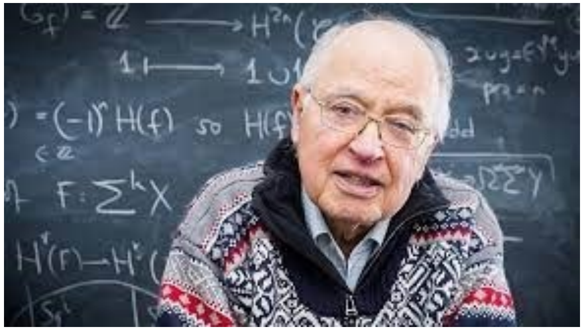


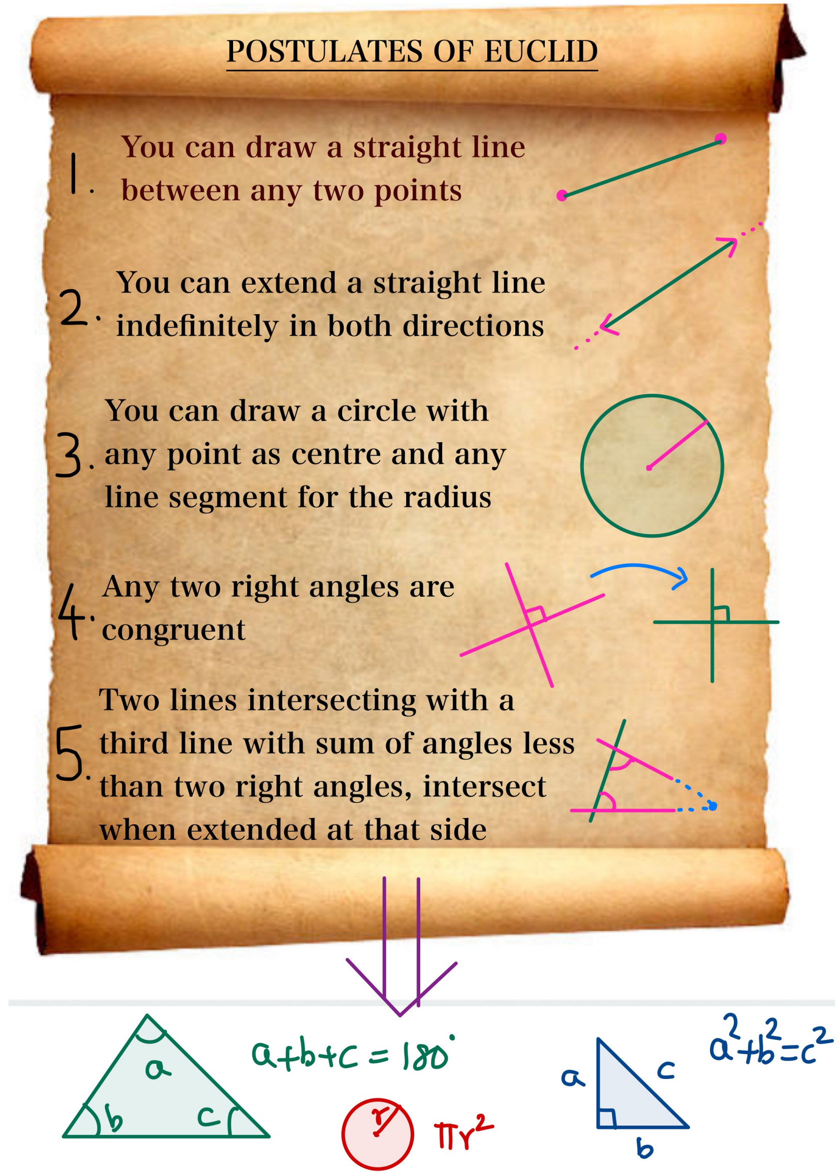
Figure: Sir Michael Atiyah

Mathematics in the 20th Century

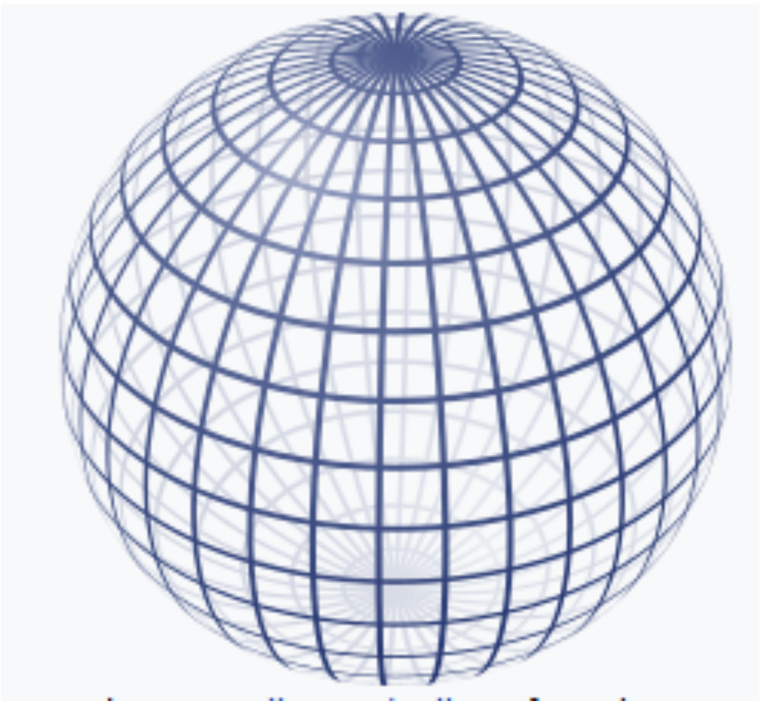
Algebra is the offer made by the devil to the mathematician. The devil says: I will give you this powerful machine, it will answer any question you like. All you need to do is give me your soul: give up geometry and you will have this marvellous machine. (Nowadays you can think of it as a computer!)

Looking beyond Euclidean space

- Is the Euclidean plane, the only surface worthy of a geometry?

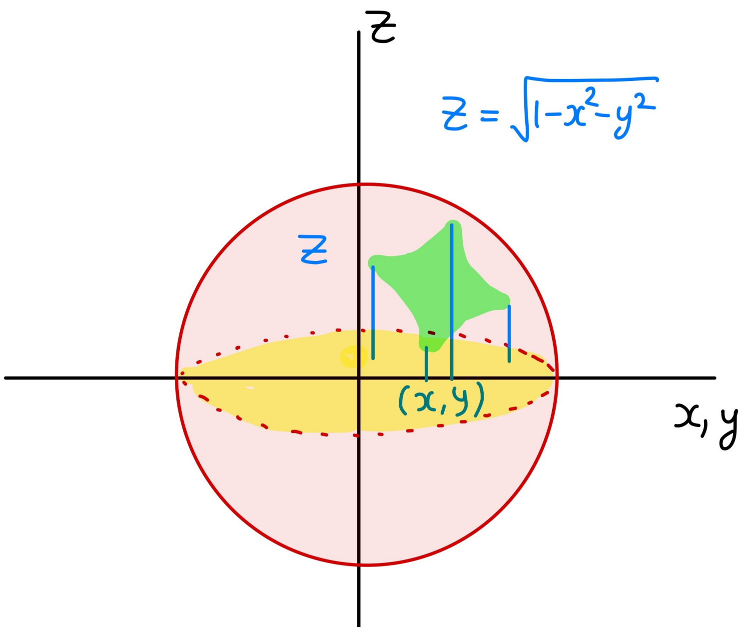


How about a sphere?



- How about the surface of the sphere $\mathbb{S}^2 := \{(x, y, z) \in \mathbb{R}^3 \text{ such that } x^2 + y^2 + z^2 = 1\}$? Assume that a particle is moving on the surface of a sphere...

Locally, Descartes works



Local Coordinates possible on $\mathbb{S}^2$

- In any small neighborhood of $\mathbb{S}^2$, two coordinates can be chosen independent with the third coordinate as a function of those two - e.g. $z = \sqrt{1 - x^2 - y^2}$

Question: Can a constraint like $x^2 + y^2 - z^2 - 1 = 0$ always be used to locally eliminate one variable and write it as a smooth function of the other variables like $z = \pm\sqrt{1 - x^2 - y^2}$?

$$x^2 + y^2 + z^2 - 1 = 0 \text{ (implicit) } \rightarrow \; z = \sqrt{1 - x^2 - y^2} \text{ (explicit)}$$

Mathematical Detour: Implicit Function Theorem

Implicit Function Theorem

- Let $f_i|_{1\leq i\leq p} : \mathbb{R}^m \to \mathbb{R}$ be p smooth functions in $\mathbb{R}^m$
- Consider the set G defined by p constraints - $f_i(z) = 0$
- Consider the matrix of partial derivatives $J = \frac{\partial f_i}{\partial x_j}(z_0)|_{1\leq i\leq p, 1\leq j\leq m}$
- If the matrix J is of rank p, then locally around x_0, p variables $(y_1, y_2, \dots, y_p)$ in $\mathbb{R}^m$ can be eliminated and expressed in terms of $n = m - p$ other variables $(x_1, x_2, \dots, x_n)$ using the constraints. i.e.

$$f(x_1, \dots, x_n, y_1, \dots, y_p) = 0 \Leftrightarrow y_i = y_i(x_1, x_2, \dots, x_n) \ : 1 \leq i \leq p$$

Example: $\mathbb{S}^2 \subseteq \mathbb{R}^3$: $m = 3, p = 1, n = m - p = 2$ and $f_1 = x_1^2 + x_2^2 + x_3^2 - 1 = 0$

But globally, Descartes fails

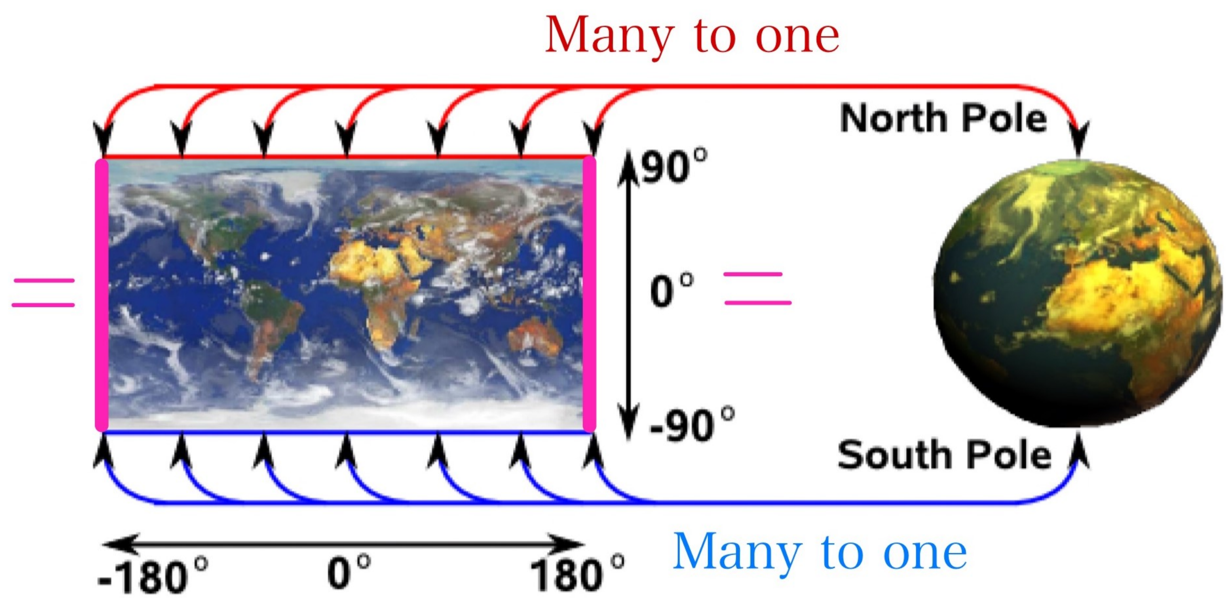


Figure: No matter whatever one tries, one cannot smoothly map a sphere ($\mathbb{S}^2$) into a bunch of two real numbers ($\mathbb{R}^2$) - this figure shows the failure of the standard Mercator Map

Satellites and Robots

Question

Is the set of all orientations of a rigid body amenable to a description by independent coordinates?

No! - Gimbal Lock!

Smooth Manifold: General Definition

- **Sphere**: "2"-dimensional "surface" in "$\mathbb{R}^3$" , one smooth constraint $(x^2 + y^2 + z^2 - 1) = 0$
- **Smooth Manifold**: *abstraction* - a "n" dimensional "entity" in "$\mathbb{R}^m$" - $p = m - n$ smooth constraints

Smooth Manifold G

A smooth manifold G is a subset of $\mathbb{R}^m$ defined by p smooth constraints

$$G = \{x \in \mathbb{R}^m \ : \ f_i(x) = 0\} \ \ f_i : \mathbb{R}^m \to \mathbb{R}(i = 1, \ldots, p)$$

where the matrix $\{\frac{\partial f_i}{\partial x_j}\}$ is full rank $n = m - p$ at all points in G[a]

[a] The full rank comes in so that we can apply implicit function theorem and write some p coordinates as a function of the remaining $m - p$ coordinates locally so that we have available, a ready-made local coordinate system around every point

- Dimension of a smooth manifold n:= number of independent local coordinates - $n = m - p$ (Implicit Function Theorem)

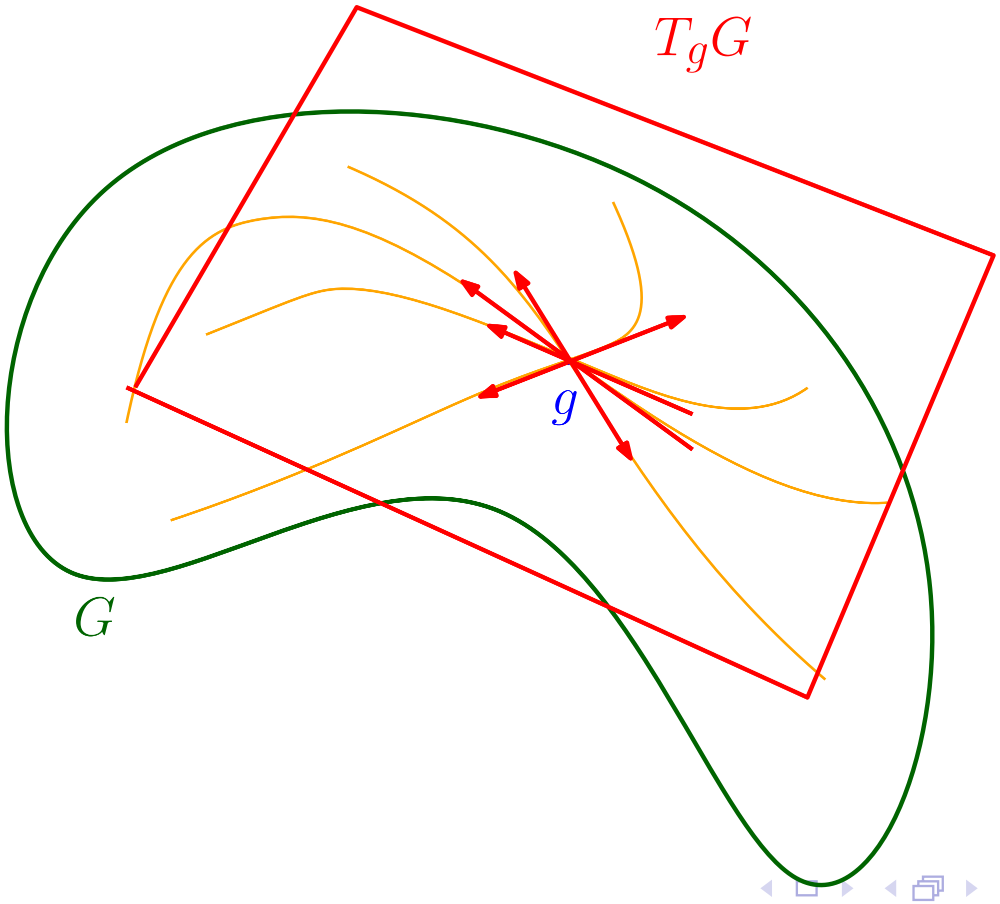
What happens in a Manifold?

On a smooth manifold, we have

- non-availability of a single globally covering coordinate system
- **forced** to think coordinate invariantly and geometrically for global analyses
- Mechanical systems are a system of interconnected rigid bodies - so configuration spaces typically non Euclidean

Velocity and the Tangent Space

- Imagine a particle constrained to move only on the surface of a sphere (or on a manifold) - what are its possible velocities at a point g?
- The tangent space of a manifold G at a point g, denoted by $T_g G$ is defined as the set of all velocities that a particle can have when passing through g and staying in G



The Tangent Space as a local Euclidean approximation

- $T_g G$: Euclidean space that best approximates G around g
- G: approximated arbitrarily by gluing together small sections of tangent spaces at various points

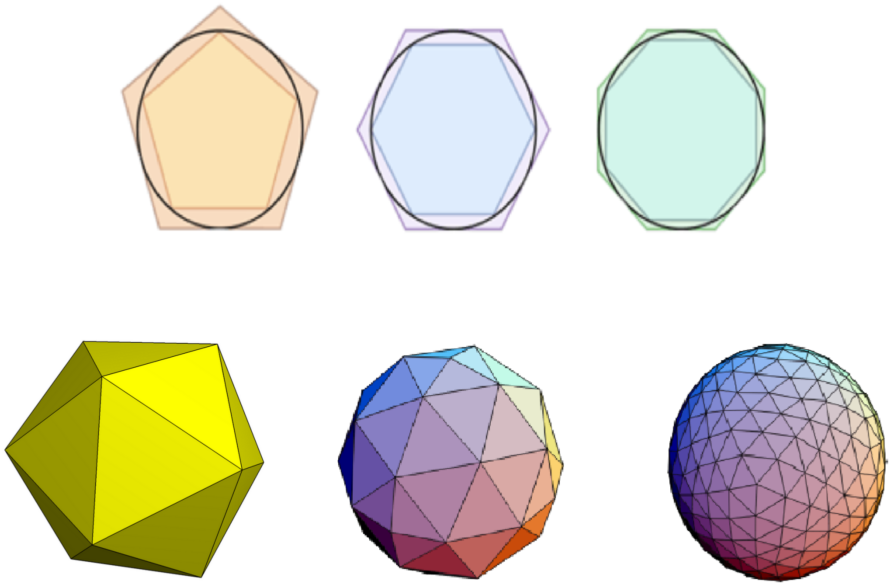


Figure: Polygonal approximations of a circle and polyhedral approximations of a sphere by using tangent lines and tangent planes respectively

Computing $T_g G$

- Let $G \subset \mathbb{R}^m$ be a n dimensional manifold defined by p smooth constraints $f_i = 0$ for $i = 1, 2, \ldots, p$ where $n = m - p$
- By implicit function theorem, it has $n-$independent local coordinates and hence we have that $T_g G$ is $n-$dimensional.
- $$0 = \frac{d}{dt}(0) = \left.\frac{df_i}{dt}\right|_{v \in T_g G} = \underbrace{\sum_{j=1}^{m} \frac{\partial f_i}{\partial x_j} v_j}_{\text{by chain rule}} = \langle grad(f_i), v \rangle \tag{1}$$
- So, $T_g G \subseteq \cap_{i=1}^{p} grad(f_i)^{\perp}$ whose dimension is also $m - p = n$
- So, $T_g G = \cap_{i=1}^{p} grad(f_i)^{\perp}$ - a vector space of dimension n

Distance and angle in a manifold

- Euclidean space: Angle = putting an inner product $\langle , \rangle$.
- Manifold G: - approximation by gluing $T_g G$s - hence assign an inner product $\langle , \rangle$ in each $T_g G$ - i.e. by assigning speeds to curves and angles between them
- Once speed of a curve $\gamma : [a, b] \to G$ is defined, its length is

$$Length[\gamma] := \int_a^b ||\dot{\gamma}(t)||dt$$

- Once, lengths are gotten, distance between two points is simply the length of that curve which is of the shortest possible length between the two points

Riemannian Manifold

Smooth manifold with a smoothly varying inner product $\langle , \rangle_g$ at each $T_g G$

Groups - the other strand

Now, let us jump to another concept called 'GROUP THEORY' which arose as an abstraction of symmetry

Beginnings: Euclid

The Third Postulate of Euclidean Geometry

All right angles are congruent

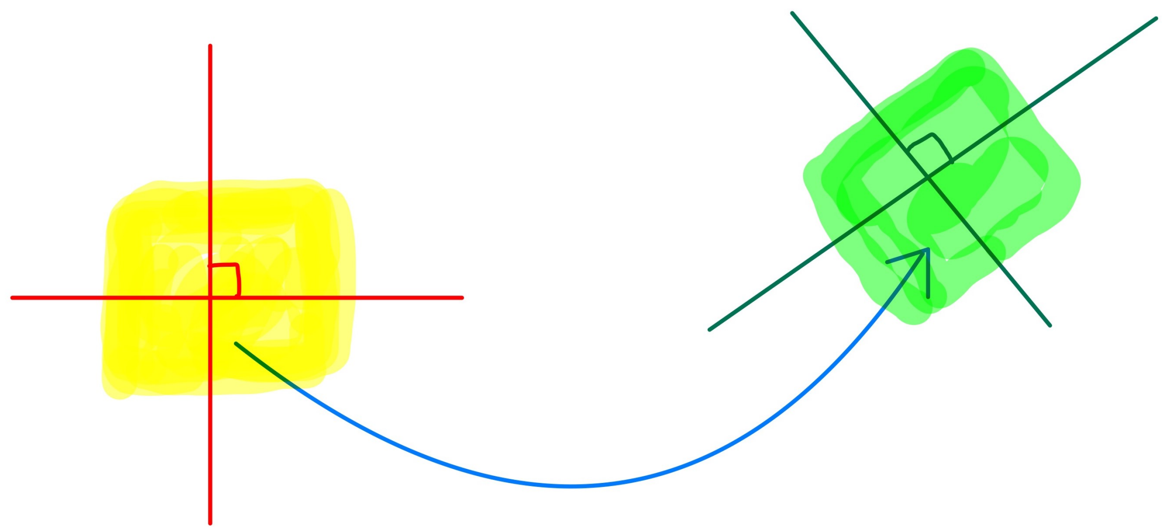


Figure: The third postulate - what does it really mean?

- What Euclid has said in modern terms is that 'given any two right angles at any two points, there is a transformation taking one to the other, preserving all the geometrical properties of the plane'.

Symmetry: Beginnings of Group Theory

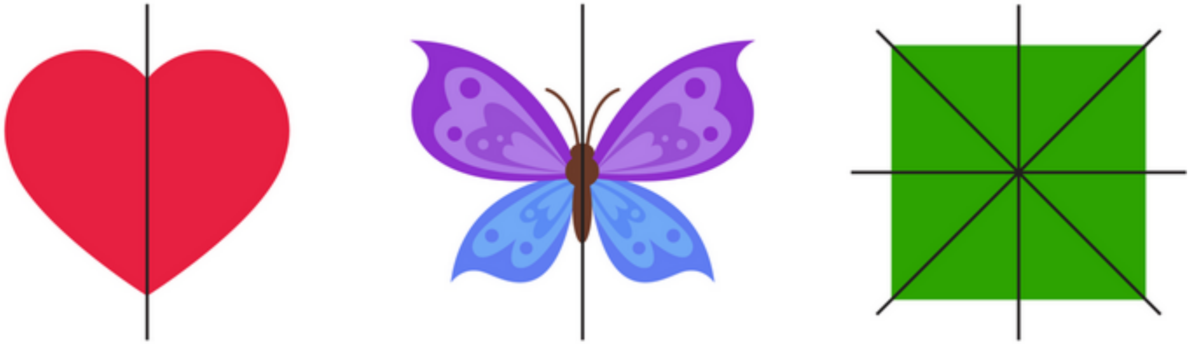


Figure: What exactly is symmetry?
'If no one asks me, I know. But if I wish to explain it to one that asketh, I know not' - St Augustine

- Etymology:
  Greek *syn* (same/together) + *metre* (measure)

En arkhêi ên ho lógos, kaì ho lógos ên pròs tòn theón, kaì theòs ên ho lógos
(In the beginning was the Word, and the Word was with God, and the Word was God)

The Gospel of John, 1:1

What was there in the beginning? In the beginning, was symmetry!

Werner Heisenberg

Defining Symmetry

Symmetry of an object

A symmetry of an object, is a **”transformation”** that leaves it **”unchanged”**

- Note: Transformation - you do something to an object
- Note: Unchanged - some property of it remains unchanged after the transformation
- So, the definition of symmetry depends on what transformations are allowed and what properties one is concerned about

Starting Simple: Equilateral Triangle

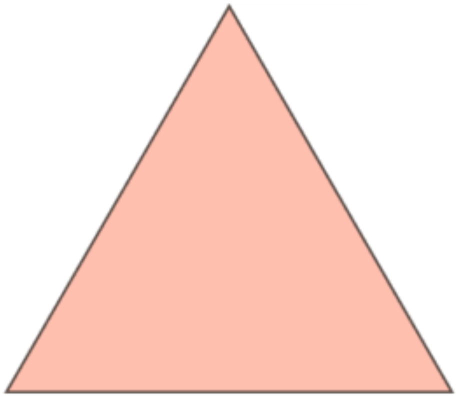


Figure: What are the symmetries of an equilateral triangle?

Property = geometric property

Symmetry Transformations of an Eq Triangle

- It turns out that the following are the symmetry operations of an equilateral triangle:

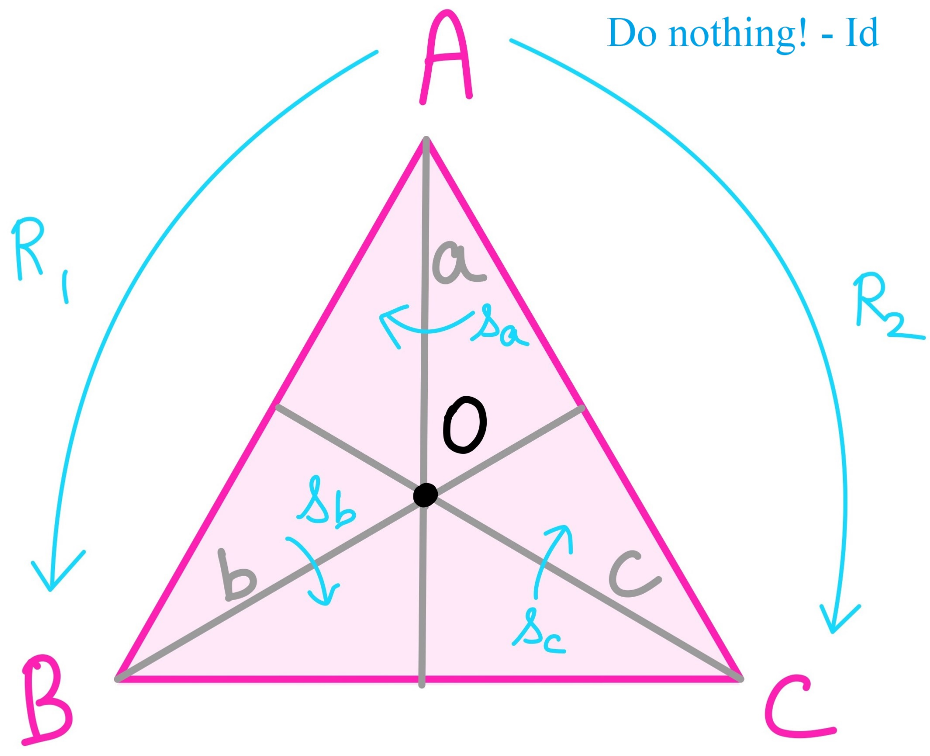


Figure: Symmetry transformations of an equilateral triangle

- I, R_1, R_2 : Rotations through $0^0, 120^0, 240^0$ respectively
- s_a, s_b, s_c : Reflections through medians a, b, c respectively

Symmetries Combine

- Note that 'doing nothing' or 'identity transformation' always is a symmetry transformation as doing nothing changes nothing and hence does not alter any property
- If S and T are two symmetry transformations that leave something unchanged, then so does $S \circ T$
- If S is a symmetry transformation that leaves something unchanged, then the same holds true for performing its inverse transformation S^{-1} as well
- Symmetries have an algebra - they combine with one another

Symmetries Combine: Example

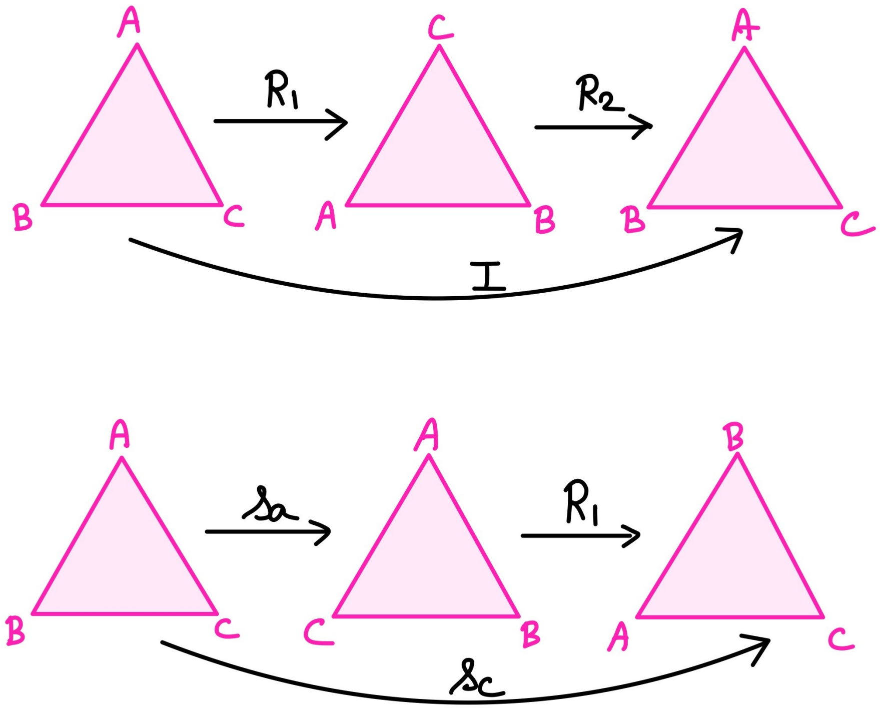


Figure: Combining two symmetry operation gives another symmetry operation

- $Id = R_2 \circ R_1$
- $s_c = R_1 \circ s_a$

A Multiplication Table for Symmetries

- One can build a table for symmetries by seeing the result of any two combinations

$*$	I	S_a	S_b	S_c	R_1	R_2
I	I	S_a	S_b	S_c	R_1	R_2
S_a	S_a	I	R_1	R_2	S_b	S_c
S_b	S_b	R_2	I	R_1	S_c	S_a
S_c	S_c	R_1	R_2	I	S_a	S_b
R_1	R_1	S_c	S_a	S_b	R_2	I
R_2	R_2	S_b	S_c	S_a	I	R_1

Figure: Multiplication Table for the symmetry transformations of an equilateral triangle

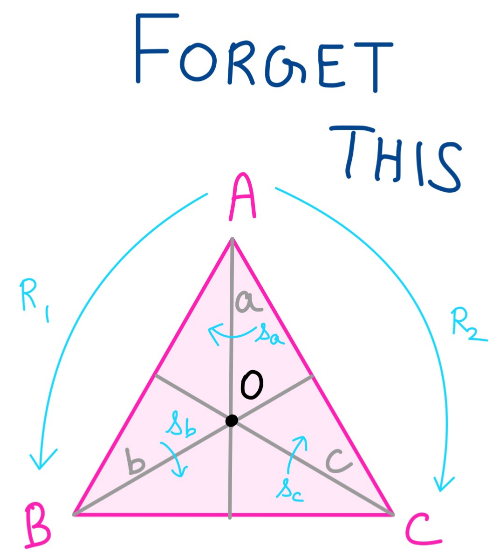
- Once this table is known, one can calculate the result of any sequence of symmetry transformations and their inverses

From Symmetry to a Group

- The concept of a group arises when one just takes the multiplication table and forgets the actual physical details of the transformation

<u>Abstract theory of groups</u>

$*$	I	S_a	S_b	S_c	R_1	R_2
I	I	S_a	S_b	S_c	R_1	R_2
S_a	S_a	I	R_1	R_2	S_b	S_c
S_b	S_b	R_2	I	R_1	S_c	S_a
S_c	S_c	R_1	R_2	I	S_a	S_b
R_1	R_1	S_c	S_a	S_b	R_2	I
R_2	R_2	S_b	S_c	S_a	I	R_1



- D_3 - dihedral group of order 3

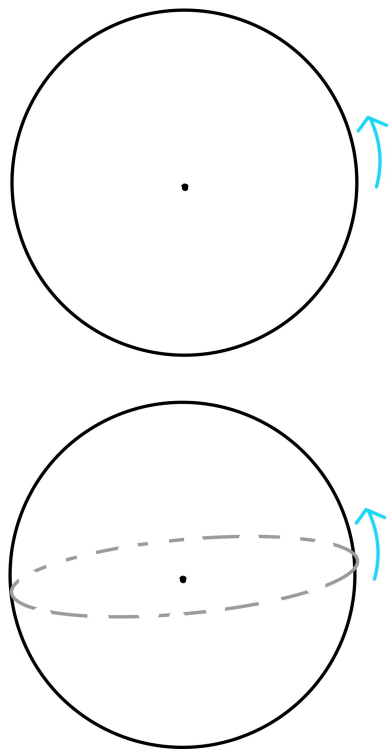
Groups versus Symmetries: A Comparison

- Combining two symmetry transformations ($\circ$) gives another symmetry transformation
- The identity (doing nothing) is a symmetry transformation always
- The inverse of a symmetry transformation is another symmetry operation

- A group G is a set with a binary operation ($*$) that takes two elements of the set to give another element in that set itself with the following two properties:
- There is an identity element $e \in G$ which has the property that $g * e = e * g = g$ for all $g \in G$
- For every $g \in G$, there is an element g^{-1} that satisfies $g^{-1} * g = g * g^{-1} = e$

From Discrete to Continuous Groups

- The dihedral group D_3 is discrete in the sense that its elements can be enumerated by natural numbers (in particular - finitely many natural numbers).
- But things are really interesting when we deal with groups that got to be specified by continuous parameters or **local coordinates**! So, we see the theory of groups merging with the theory of smooth manifolds!!



any continuous rotation is a symmetry

$SO(3)$: A bridge to generic "Lie" Groups

- Continuous groups (technically called "Lie groups" are dificult to handle - multiplication table cannot be specified for each element separately!
- So, efficient ways of studying the group properties needed
- Hence, we start by studying a particular example of a continuous group $SO(3)$ which happens to be the symmetry group of a sphere in $\mathbb{R}^3$

Group + Manifold = Lie Group

- A Lie group is a manifold that is also a group where in the group operation and inversion are smooth

PLATO

God ever geometrizes

KLEIN

Geometrical properties are characterised by their invariance under a group of transformations @Erlangen Programme

IAN STEWART

If Plato and Klein are correct, then God must be a group theorist . Is she? (Fearful Symmetry)

$SO(3)$: All possible rotations

- We now analyze the rotational symmetries of a sphere in $\mathbb{R}^3$ - a sphere is invariant under any rotation

The Special Orthogonal Group

$SO(3) := \{R \in \mathbb{R}^{3\times 3}$ such that for all $x, y \in \mathbb{R}^3$, $\langle Rx, Ry \rangle = \langle x, y \rangle$ and $\det(R) = 1\}$
(this definition immediately yields that $SO(3)$ is a group under matrix multiplication - no surprise)

- w.r.t standard inner product,

$$SO(3) = \{R \in \mathbb{R}^{3\times 3} \text{ such that }, \ R^T R = I, \ det(R) = 1\}$$

- it is a continuous group - rotation about any continuous angle and in any continuous direction possible

$SO(3)$: Also a Smooth Manifold!

- $SO(3)$ is a subset of the Euclidean space $\mathbb{R}^{3\times 3}$, which is a 9-dimensional Euclidean space
- It is defined by the constraint $R^T R = I$ which amounts to six independent quadratic constraints [2]
- The gradient condition can be verified
- Hence, turns out $SO(3)$ is not just a group - but a smooth manifold as well!
- Its dimension n =no. of entries - no. of constraints = 9-6=3

[2]($R^T R$ is symmetric and hence only 6 of its 9 entries are independent)

Conjugation: A Motivation

- A rotation of 12^0 about $x-$axis is 'essentially' same as a rotation of 12^0 about $y-$axis - it is just a relabelling of the axes. How do we formalize this idea?

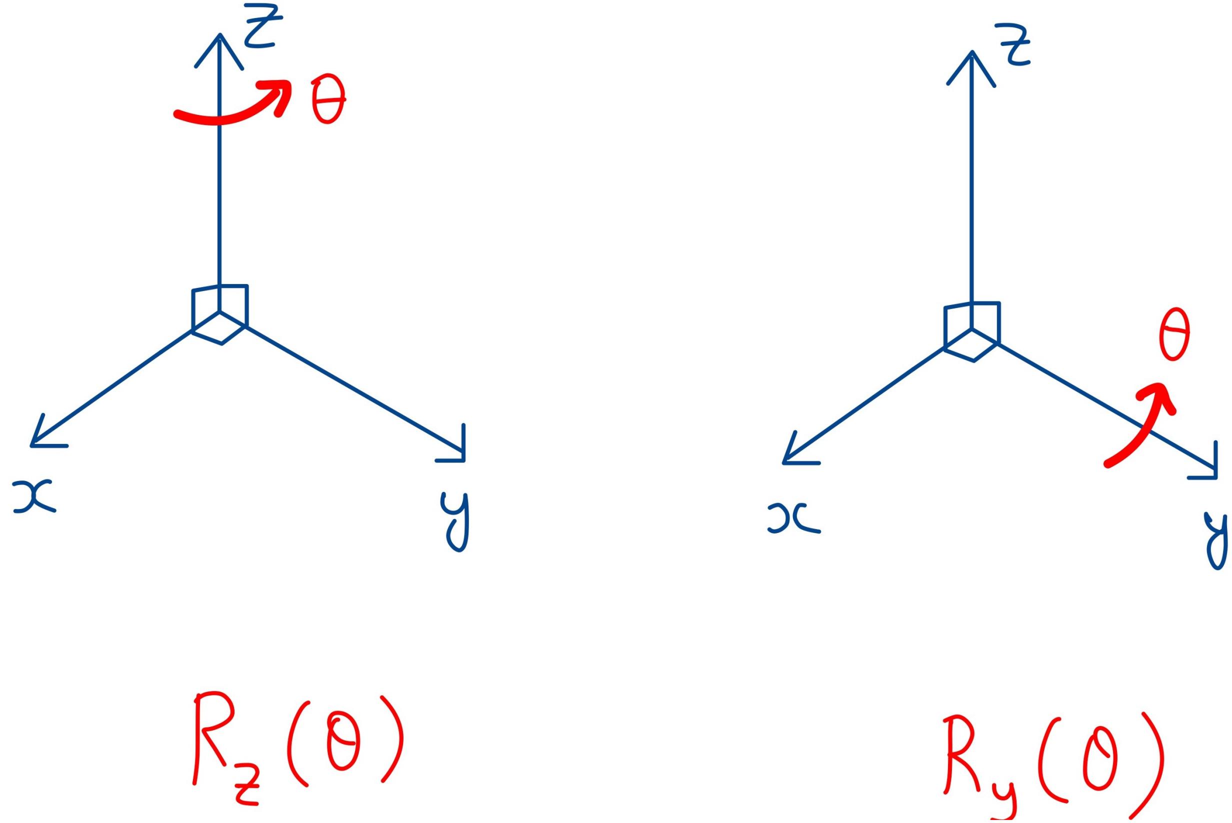


Figure: A rotation of angle θ about $x-$ axis is 'equivalent' to rotation of angle θ about y-axis - same transformation in a rotated coordinate system

Conjugation

ACTIVE

$$\underbrace{R^{\substack{z\\ \uparrow\\ y}} \; R_y(\theta) \; R^{\substack{y\\ \uparrow\\ z}}}$$

ACTIVE

$$\underbrace{R^{-1} \; R_y(\theta) \; R}$$

ACTIVE

$$= R_z(\theta) = R^{-1} R_y(\theta) R$$

Figure: Mathematical equivalence between $R_z(\theta)$ and $R_y(\theta)$

Conjugation: Abstract Definition

Conjugacy Relation

Two elements g_1, g_2 in a group G are said to be conjugate if there exists another transformation in the group $g' \in G$ such that

$$g_1 = g' * g_2 * g'^{-1}$$

- Conjugacy is an equivalence relation
- The equivalence class of an element g under the conjugacy relation is the set of all group elements equivalent to g
- Physically, g_1, g_2 are said to be conjugate in the group, if g_1 looks like g_2 in another coordinate system which is achieved passively through an element g' in the group itself

Geometry begins: Evaluating Tangent Space of $SO(3)$

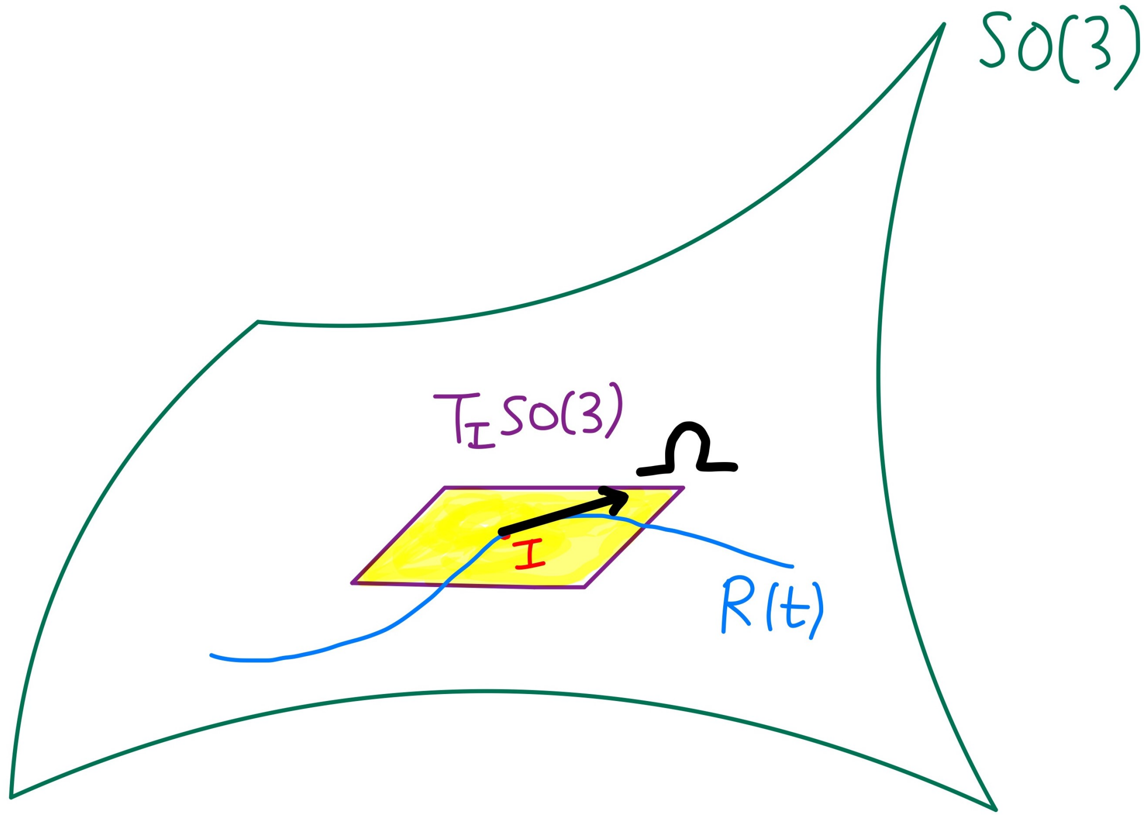


Figure: Determining $T_I SO(3)$

$T_I SO(3) := \mathfrak{so}(3)$

- Let us now consider a curve $R(t)$ in $SO(3)$ through I - $R(0) = I$. Let Ω be its velocity at time $t = 0$ - $\dot{R}(0) = \Omega$
- $$0 = \frac{d}{dt}\Big|_{t=0} (R^T R) = R(0)^T \Omega + \Omega^T R(0) = \Omega + \Omega^T$$
- So we get that any curve in $SO(3)$ passing through velocity can have velocity Ω that should satisfy
$$\Omega = -\Omega^T \ \text{(skew-symmetric)}$$
- So, $T_I SO(3) := \mathfrak{so}(3)$:=set of all 3×3 skew-symmetric matrices

Tangent Space at other points

- Consider a curve with $R(0) = R$ and velocity $\dot{R}(0) = R\Omega$. Then,

$$0 = \frac{d}{dt}\bigg|_{t=0} (R^T R) = R(0)^T [R\Omega] + [\Omega^T R^T] R(0)$$

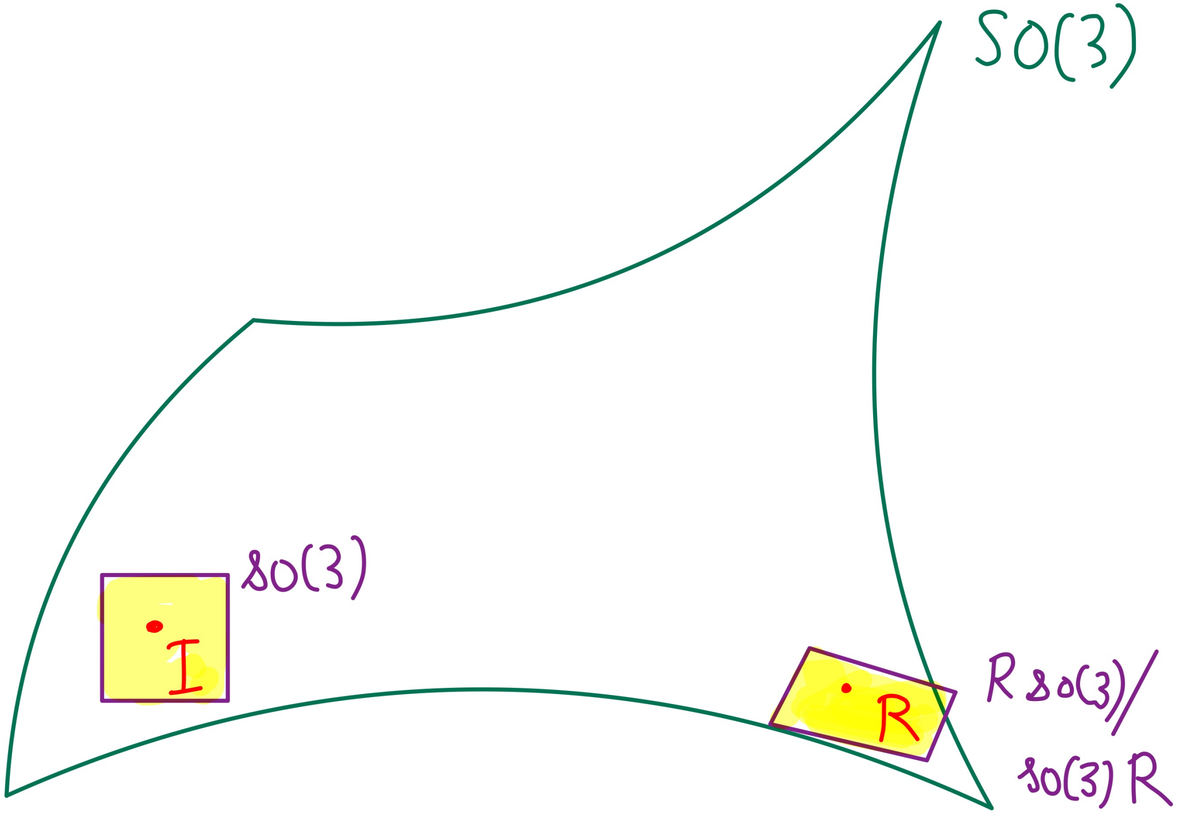
- Repeat with $\dot{R} = \Omega R$ with

$$0 = \frac{d}{dt}\bigg|_{t=0} (R R^T) = R(0)[R^T \Omega^T] + [\Omega R] R(0)^T$$

$T_R SO(3)$

$$T_R SO(3) = \{R\Omega \mid \Omega \in \mathfrak{so}(3)\} = \{\Omega R \mid \Omega \in \mathfrak{so}(3)\}$$

Visualizing $SO(3)$, $\mathfrak{so}(3)$ and $T_R SO(3)$



$\mathfrak{so}(3)$ - Angular velocity

- Consider a rigid body with an attached body coordinate (B) that is rotated by a rotation $R(t)$

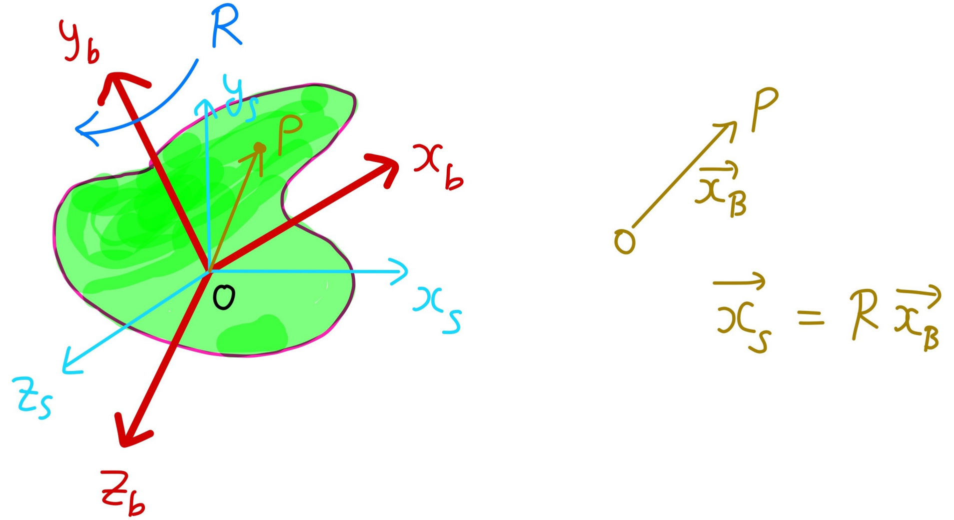


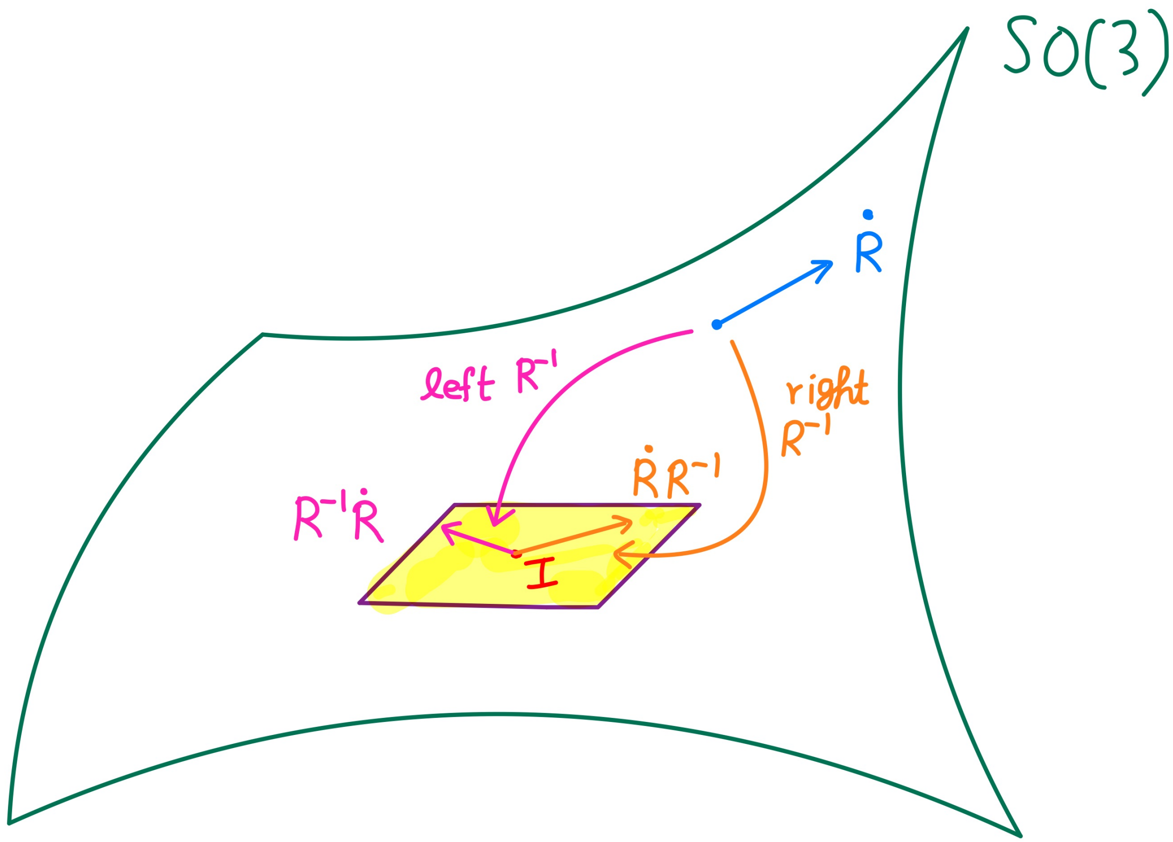
Figure: A rotating rigid body with an attached coordinate system

$$\dot{x}_S = \overbrace{(\dot{R}R^{-1})}^{\in\,\mathfrak{so}(3)} x_S$$
$$\dot{x}_B := R^{-1}\dot{x}_S = \underbrace{(R^{-1}\dot{R})}_{\in\,\mathfrak{so}(3)} .x_B$$

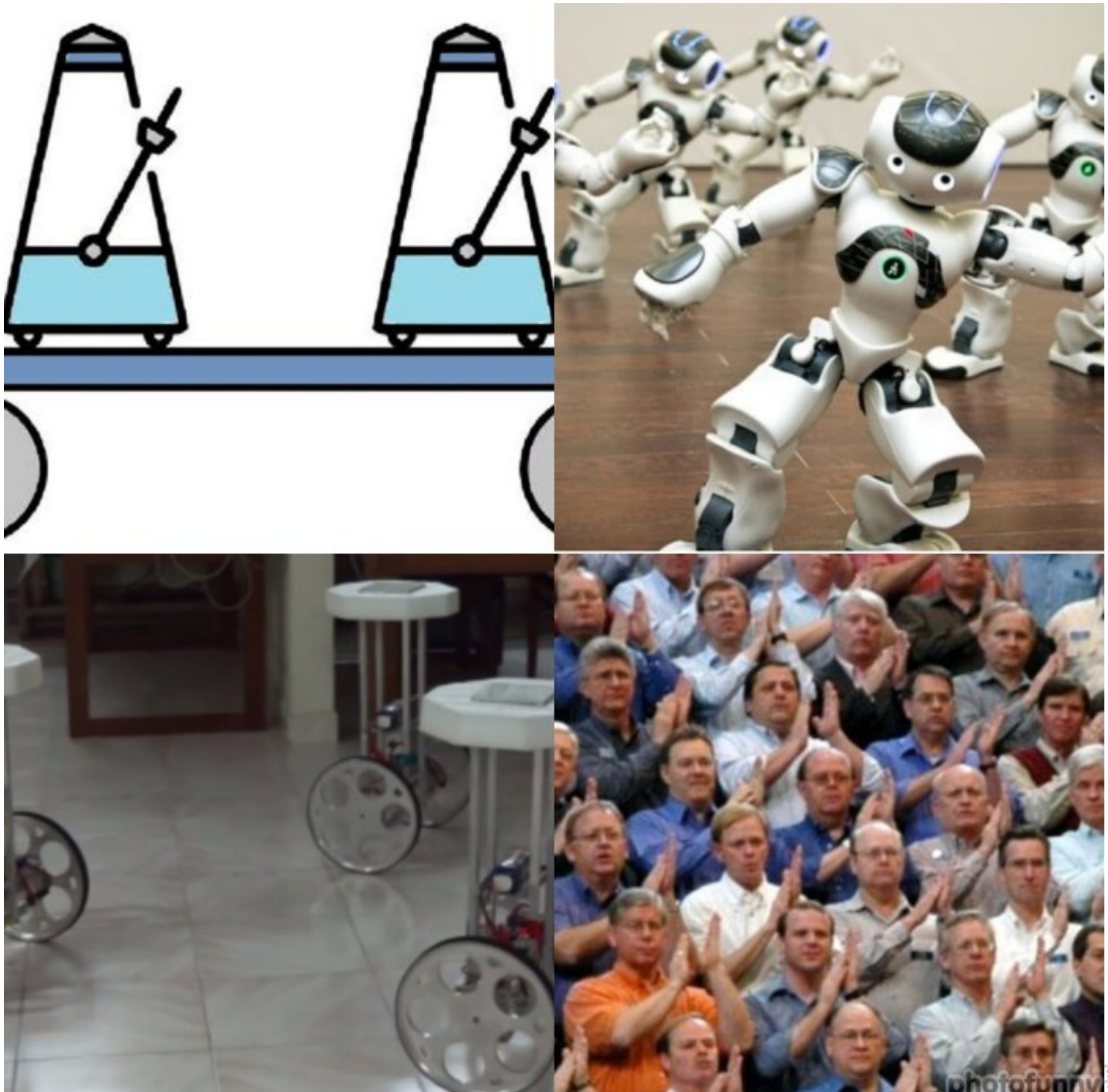
Angular Velocities

Nomenclature

- $R^{-1}\dot{R}$ - angular velocity in the body frame
- $\dot{R}R^{-1}$ - angular velocity in the space frame



Consensus, Coordination and Synchronization Problems on Lie Groups



Broad Outline of Consensus

- Finitely many agents : $i = 1, 2, \ldots, n$
- Each agent evolves in a configuration space G (a Lie group in our setting)
- Some of the pairs of agents are coupled and interact with each other

Consensus

All the agents should asymptotically converge to the same point and come to rest

$$\lim_{t \to \infty} g_i(t) = g_0 \in G$$