On the finiteness of the set of the first Hilbert coefficients with respect to m-primary ideals

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Outline

1. Introduction
2. Finiteness of $\Lambda(M)$: A sufficient condition
3. Finiteness of $\Lambda(M)$: Necessary conditions

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- $(A, \mathfrak{m})$ be Noetherian local ring of dimension $d > 0$.
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Hilbert polynomial of M

There exist integers $e_i(I, M)_{0 \leqslant i \leqslant d}$ such that

$$H_{I,M}(n) = e_0(I, M)\binom{n+d}{d} - e_1(I, M)\binom{n+d-1}{d-1} + \ldots + (-1)^d e_d(I, M)$$

for all $n \gg 0$.

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- Define $\Pi(M) := \{e_1(I,M) \mid I \text{ is a parameter ideal for } M\}$.

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- For $n \gg 0$,
$H^K_{I,M}(n) = g^K_0(I)\binom{n+d}{d} - g^K_1(I)\binom{n+d-1}{d-1} + \ldots + (-1)^d g^K_d(I)$.

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- Define $\Lambda_K(A) := \{g^K_1(I) \mid I \text{ is an } \mathfrak{m} - \text{primary ideal and } I \subseteq K\}$.

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Theorem (L. Ghezzi et al., 2014)

Let A be a Noetherian local ring and M be a finitely generated unmixed A-module with dimension $d \geq 2$. Then

1. *M is Cohen-Macaulay if and only if $0 \in \Pi(M)$.*
2. *M is Buchsbaum if and only if $|\Pi(M)| = 1$.*
3. *M is generalized Cohen-Macaulay if and only if $\Pi(M)$ is a finite set.*

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Theorem (K. Asuki and T. Naoki, preprint, 2013)

Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring with dim $A = 1$ and $A/\mathfrak{m}$ is infiinte. Then $\Lambda(A) = \{\lambda_A(B/A) | A \subseteq B \subseteq \bar{A}$ is an intermediate ring which is a module finite extension of $A\}$.

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Corollary

Let A be a Cohen-Macaulay local ring of dimension one. Then, $\Lambda(A)$ is a finite set if and only if A is analytically unramified.

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Let $(A, \mathfrak{m})$ *be a Noetherian local ring with dim* $A = 1$*. Then*

$$\begin{aligned} sup\ \Lambda(A) &= \lambda(\bar{B}/B) - \lambda(H^0_{\mathfrak{m}}(A)) \ \ and \\ inf\ \Lambda(A) &= -\lambda(H^0_{\mathfrak{m}}(A)) \end{aligned}$$

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Theorem (K. Asuki and T. Naoki, preprint, 2013)

Let $(A, \mathfrak{m})$ *be a Noetherian local ring of dimension* $d \geq 1$. *Then TFAE.*

1. $\Lambda(A)$ *is a finite set;*
2. $d = 1$ *and* $A/H^0_{\mathfrak{m}}(A)$ *is analytically unramified.*

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Finiteness of $\Lambda(M)$: A sufficient condition

- Define $\Lambda_A(M) := \{e_1(I,M) \mid I \textit{ is an } \mathfrak{m}-\textit{primary ideal of } A\}$.
- For simplicity, we write $\Lambda(M)$ if A is unambiguous.

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Proof. We have

- $e_1(I^k, M) = \frac{d-1}{2}.e_0(I, M).k^d + \frac{2e_1(I,M)-(d-1)e_0(I,M)}{2}.k^{d-1}$.

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- $\Lambda(M)$ is finite implies the set $\{e_1(I^k, M) \mid k \geq 1 \textit{ is an integer}\}$ is finite.
- This is possible only if $d = 1$.

Finiteness of $\Lambda(M)$: A sufficient condition

We know that

$$\sum_{n\geq 0} \lambda(M/I^{n+1}M)z^n = \frac{h_M^I(z)}{(1-z)^{d+1}}.$$

Theorem (Tony J. Puthenpurakal, 2003)

Let (A, m) be a Cohen-Macaulay local ring of dimension d, I be an m-primary ideal and M be a finitely generated module of dimension d. Let $depth(M) \geq d-1$. Then

1. $\sum_{n\geq 0} \lambda(Tor_1^A(M, A/I^{n+1})z^n = \frac{h_N^I(z) - \mu(M)h_A^I(z) + h_M^I(z)}{(1-z)^{d+1}}$ *where* $N = Syz^1(M)$.

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2. $\lambda(Tor_1^A(M, A/I^{n+1}))$ *is given by a polynomial* $t_M^I(z)$ *for all* $n \gg 0$ *and*

$$t_M^I(z) = (\mu(M)e_1(I,A) - e_1(I,M) - e_1(I, Syz^1(M))\frac{z^{d-1}}{(d-1)!} + \textit{ lower terms}$$

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3. $\mu(M)e_1(I, A) \geq e_1(I, M) + e_1(I, Syz^1(M))$.

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- $\mu(M)e_1(I,A) \geq e_1(I,M) + e_1(I, Syz^1(M))$.

Corollary

Let (A,m) be a Noetherian local ring of dimension one, I be an m-primary ideal and M be a finitely generated module of dimension one. Then

$$e_1(I,A)\mu(M') + \lambda((H_m^0(A))\mu(M') \geq e_1(I,M) + \lambda(H_m^0(M)) + e_1(J, Syz^1(M'))$$

where $A' = A/H_m^0(A)$, $M' = M/H_m^0(M)$ and $J = IA'$.

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- Since $e_1(J, Syz^1(M')) \geq 0$, we get an upper bound on $e_1(I,M)$.

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Let (A,m), I and M be as above. Then

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3. *$A/H_m^0(A)$ is analytically unramified implies $\Lambda(M)$ is finite.*

Outline

Case 1: When M is an $\mathfrak{m}$-primary ideal

- Let I and K be $\mathfrak{m}$-primary ideals and $dim(K) = d$.
- Define $\tau_1^{I,K}(n) := \lambda_A(Tor_1(A/I^{n+1}, A/K))$.

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Theorem

Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension d. Let I and K be $\mathfrak{m}$-primary ideals and $dim(K) = d$. Then degree of the polynomial $\tau_1^{I,K}$ is $d - 1$. Further,

$$\tau_1^{I,K}(n) = \sum_{i=0}^{d-1} (-1)^i A_i^I(K) \binom{n + d - (i+1)}{d - (i+1)}$$

where $A_i^I(K) = e_{i+1}(I) - e_{i+1}(I, K)$ *for* $i = 0 \leq i \leq d - 2$

and $A_{d-1}^I(K) = e_d(I) - e_d(I, K) + (-1)^{d-1}\lambda(A/K)$.

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Let I and K be as above. Then $e_0(I) = e_0(I,K)$. Suppose $d \geq 2$, then $e_1(I) > e_1(I,K)$.

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In case of $d = 1$, $\tau_1^{I,K}(n) = e_1(I) - e_1(I,K) + \lambda(A/K)$ for $n \gg 0$,

Consequently, $e_1(I,K) < e_1(I) + \lambda(A/K)$.

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Consequently, $e_1(I,K) < e_1(I) + \lambda(A/K)$.

Corollary

Let A and K be as above. Then

1. $sup\ \Lambda(K) \leq\ sup\ \Lambda(A) + \lambda(A/K)$.
2. $A/H^0_{\mathfrak{m}}(A)$ *is analytically unramified implies* $\Lambda(K)$ *is finite.*

Case 1: When M is an $\mathfrak{m}$-primary ideal

Example ($sup\ \Lambda(K) \neq\ sup\ \Lambda(A) + \lambda(A/K) - 1$)

Let $A = k[[t^2, t^3]]$, then $\Lambda(A) = \{0, 1\}$. Let $K = \mathfrak{m}$.

$$\begin{aligned} \lambda(\mathfrak{m}/I^n\mathfrak{m}) &= \lambda(A/I^n) + \lambda(I^n/\mathfrak{m}I^n) - \lambda(A/\mathfrak{m}). \\ So\ e_1(I, \mathfrak{m}) &= e_1(I) + 1 - \mu(I^n). \end{aligned}$$

If possible, $e_1(I, \mathfrak{m}) = 1$ for some I, then $\mu(I^n) = e_1(I) \leq 1$. This implies $\mu(I^n) = 1$ for all n. Therefore I is a parameter ideal which is a contradiction.

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Theorem

Let (A, m) be Cohen-Macaulay local ring of dimension one and K be an m-primary ideal. Then, for all m-primary ideals I and for all $n \gg 0$,

$$\lambda(Tor_1(A/I^n, A/K)) \leq e(K).$$

and equality holds if I is such that there exists a non-zero-divisor $x_I \in K$ such that $(x_I) \cap I^r K^s = x_I I^r K^{s-1}$ for all $r \geq 0$ and $s > 0$.

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Let (A, m) be Cohen-Macaulay local ring of dimension one and K be an m-primary ideal. Then,

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Let A and K be as above. Then, for each $\mathfrak{m}$-primary ideal I,

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Let $(A, \mathfrak{m})$ *be a Noetherian local ring of dimension one and I and K be* $\mathfrak{m}$*-primary ideals such that* $I \subseteq K$*. Then* $\sup \Lambda_K(A) \leq \sup \Lambda(A) - 1$.

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Theorem

Let $(A,\mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 1$ and K be an $\mathfrak{m}$-primary ideal. Then $\Lambda_K(A)$ is finite if and only if $d = 1$ and A is analytically unramified.

Case 2: When M is an A-algebra

Theorem

Let (A, m) be one dimensional reduced Noetherian local ring. Let $B \supseteq A$ be a one dimensional Noetherian local ring such that B is finitely generated A-module and $\lambda(B/A)$ is finite. Then, for each m-primary ideal I,

1. $e_1(I) \leq e_1(I, B) + \lambda(B/A)$.
2. *$\Lambda_A(B)$ is finite if and only if $A/H^0_m(A)$ is analytically unramified.*

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Proposition

Let (A, m) and (B, n) be one dimensional Noetherian local rings such that $A \to B \to 0$ is exact. Then TFAE.

1. *$\Lambda_A(B)$ is finite.*
2. *$\Lambda_B(B)$ is finite .*
3. *$B/H_n^0(B)$ is analytically unramified.*

Case 2: When M is an A-algebra

Corollary

Let $M = (x)$ be an ideal of dimension one. Then $\Lambda(M)$ is finite if and only if $\bar{A}/H^0_m(\bar{A})$ is analytically unramified where $\bar{A} = A/ann(x)$.

Case 2: When M is an A-algebra

Corollary

Let $M = (x)$ be an ideal of dimension one. Then $\Lambda(M)$ is finite if and only if $\bar{A}/H^0_m(\bar{A})$ is analytically unramified where $\bar{A} = A/ann(x)$.

Proposition

Let $f : (A, m) \to (B, \mathfrak{n})$ be a local homomorphism such that B is a finite A-module. Then $|\Lambda_A(B)| \leq |\Lambda(B)|$.

Thank You