Frame System in Banach Spaces

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A sequence \( \{x_k\}_{k=1}^{\infty} \) in \( \mathcal{H} \) is a frame for \( \mathcal{H} \) if \( \exists A, B > 0 \) such that
\[
A \|x\|^2 \leq \sum_{k=1}^{\infty} |<x, x_k>|^2 \leq B \|x\|^2, \quad \forall x \in \mathcal{H}.
\]
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- \( A \) and \( B \) → frame bounds.
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\( A \) and \( B \) \( \rightarrow \) frame bounds.

For frame \( \{x_n\} \) in \( \mathcal{H} \), \( T : l^2(\mathbb{N}) \rightarrow \mathcal{H} \),

\[
T(\{c_k\}) = \sum_{k=1}^{\infty} c_k x_k.
\]

is pre-frame operator.
The adjoint operator of $T$, $T^* : \mathcal{H} \rightarrow l^2$

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• Frame operator $S = TT^* : \mathcal{H} \rightarrow \mathcal{H}$ defined as

$$S(x) = \sum_{k=1}^\infty < x, x_k > x_k \quad x \in \mathcal{H}. $$
Let \( \{x_n\} \) be a frame for \( \mathcal{H} \) with frame operator \( S \). Then

\[
x = \sum_{k=1}^{\infty} \langle x, S^{-1}x_k \rangle x_k, \quad x \in \mathcal{H}.
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\]

Let \( \{x_n\} \) in \( \mathcal{E} \) and \( \{f_n\} \) in \( \mathcal{E}^* \). Then \((\{f_n\}, \{x_n\})\) is an atomic decomposition of \( \mathcal{E} \) with respect to \( \mathcal{E}_d \), if

(a) \( \{f_n(x)\} \in \mathcal{E}_d, \forall x \in \mathcal{E} \).

(b) \( A\|x\|_E \leq \|\{f_n(x)\}\|_{E_d} \leq B\|x\|_E, \quad x \in \mathcal{E} \)

(c) \( x = \sum_{k=1}^{\infty} f_k(x)x_k, \forall x \in \mathcal{E} \).

\( A, B \rightarrow \) atomic bounds.
Let \( \{f_n\} \) in \( \mathcal{E}^* \) and \( S : \mathcal{E}_d \to \mathcal{E} \). Then \((\{f_n\}, S)\) is Banach frame for \( \mathcal{E} \) w.r.to \( \mathcal{E}_d \) if

(a) \( \{f_n(x)\} \in \mathcal{E}_d \), \( \forall x \in \mathcal{E} \).

(b) \( \|x\|_{\mathcal{E}} \leq \|\{f_n(x)\}\|_{\mathcal{E}_d} \leq B \|x\|_{\mathcal{E}} \), \( x \in \mathcal{E} \).

(c) \( S(f_n(x)) = x \), \( \forall x \in \mathcal{E} \).

A,B \rightarrow \text{frame bounds.}
Let \( \{f_n\} \) in \( \mathcal{E}^* \) and \( S: \mathcal{E}_d \rightarrow \mathcal{E} \). Then \( (\{f_n\}, S) \) is Banach frame for \( \mathcal{E} \) w.r.to \( \mathcal{E}_d \) if

(a) \( \{f_n(x)\} \in \mathcal{E}_d, \forall x \in \mathcal{E} \).
Banach Frame

Let \( \{f_n\} \) in \( \mathcal{E}^* \) and \( S : \mathcal{E}_d \rightarrow \mathcal{E} \). Then \((\{f_n\}, S)\) is Banach frame for \( \mathcal{E} \) w.r.t. \( \mathcal{E}_d \) if

(a) \( \{f_n(x)\} \in \mathcal{E}_d, \forall x \in \mathcal{E} \).

(b) \( A\|x\|_\mathcal{E} \leq \|\{f_n(x)\}\|_{\mathcal{E}_d} \leq B\|x\|_\mathcal{E}, \quad x \in \mathcal{E} \).
Let $\{f_n\}$ in $\mathcal{E}^*$ and $S : \mathcal{E}_d \rightarrow \mathcal{E}$. Then $(\{f_n\}, S)$ is Banach frame for $\mathcal{E}$ w.r.to $\mathcal{E}_d$ if

(a) $\{f_n(x)\} \in \mathcal{E}_d, \forall x \in \mathcal{E}$.

(b) $A \|x\|_\mathcal{E} \leq \|\{f_n(x)\}\|_{\mathcal{E}_d} \leq B \|x\|_\mathcal{E}, \quad x \in \mathcal{E}$.

(c) $S$ is bounded linear operator s.t.

$$S(f_n(x)) = x \quad x \in \mathcal{E}.$$

$A, B \rightarrow$ frame bounds.
(i) Tight frame if $A = B$
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(ii) Normalized tight frame if $A = B = 1$
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(ii) Normalized tight frame if $A = B = 1$

(iii) Exact frame if there exists no reconstruction operator $S_0$ such that $(\{f_n\}_{n \neq i}, S_0) (i \in \mathbb{N})$ is a Banach frame for $\mathcal{E}$. 
Examples

- For $\mathcal{E} = c_0$ and $\{f_n\}$ in $\mathcal{E}^*$ defined as
  
  $$f_n(x) = \xi_n, \ (n \in \mathbb{N}) \quad x = \{\xi_n\} \in \mathcal{E}.$$ 

  Then there exist $\mathcal{E}_d = \{\{f_n(x)\} : x \in \mathcal{E}\}$ with norm
  
  $$\|\{f_n(x)\}\|_{\mathcal{E}_d} = \|x\|_\mathcal{E}$$

  and $S : \mathcal{E}_d \rightarrow \mathcal{E}$ defined by

  $$S(\{f_n(x)\}) = x, \quad x \in \mathcal{E}.$$

  Thus $(\{f_n\}, S)$ is a Banach frame for $\mathcal{E}$ w. r. to $\mathcal{E}_d$.

- $\ell^\infty/c_0$ does not have any Banach frame.
Few Results on Banach Frames

1. If a Banach space $\mathcal{E}$ has an atomic decomposition, then $\mathcal{E}$ has a Banach frame. The converse need not be true.
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(2) A Banach space with an atomic decomposition is separable. However, the converse need not be true.
Few Results on Banach Frames

(1) If a Banach space $E$ has an atomic decomposition, then $E$ has a Banach frame. The converse need not be true.

(2) A Banach space with an atomic decomposition is separable. However, the converse need not be true.

(3) Every separable Banach space has a normalized tight and exact Banach frame.
Few Results on Banach Frames

(1) If a Banach space $E$ has an atomic decomposition, then $E$ has a Banach frame. The converse need not be true.

(2) A Banach space with an atomic decomposition is separable. However, the converse need not be true.

(3) Every separable Banach space has a normalized tight and exact Banach frame.

(4) Let $E$ be a Banach space having a Banach frame. Then, $E$ has a normalized tight and exact Banach frame.
Frames in Subspaces of Banach Spaces

- Every Banach space $E$ has a closed subspace having a Banach frame.
Frames in Subspaces of Banach Spaces

- Every Banach space $\mathcal{E}$ has a closed subspace having a Banach frame.
- Let $(\{f_n\}, \mathcal{S})(\{f_n\} \subset \mathcal{E}^*, \mathcal{S}: \mathcal{E}_d \to \mathcal{E})$ be a Banach frame for $\mathcal{E}$ and $\mathcal{G}$ be a closed subspace of $\mathcal{E}$. Then, $\exists \mathcal{G}_d$ and $T: \mathcal{G}_d \to \mathcal{G}$ such that $(\{f_n|_\mathcal{G}\}, T)$ is a Banach frame for $\mathcal{G}$ w.r.to $\mathcal{G}_d$. 
Frames in Subspaces of Banach Spaces

- Every Banach space \( \mathcal{E} \) has a closed subspace having a Banach frame.
- Let \((\{f_n\}, S)(\{f_n\} \subset \mathcal{E}^*, S : \mathcal{E}_d \to \mathcal{E})\) be a Banach frame for \( \mathcal{E} \) and \( \mathcal{G} \) be a closed subspace of \( \mathcal{E} \). Then, \( \exists \mathcal{G}_d \) and \( T : \mathcal{G}_d \to \mathcal{G} \) such that \((\{f_n|_\mathcal{G}\}, T)\) is a Banach frame for \( \mathcal{G} \) w.r.t \( \mathcal{G}_d \).
- Let \((\{f_n\}, S)(\{f_n\} \subset \mathcal{E}^*, S : \mathcal{E}_d \to \mathcal{E})\) be a Banach frame for \( \mathcal{E} \). Then, for every \( \{n_k\} \), \( \exists \) a closed subspace \( \mathcal{G} \) of \( \mathcal{E} \) and \( T : \mathcal{G}_d \to \mathcal{G} \) such that \((\{f_{n_k}|_\mathcal{G}\}, T)\) is Banach frame for \( \mathcal{G} \) w.r.t \( \mathcal{G}_d \).
Frames in Subspaces of Banach Spaces

Let \( \{f_n\} S(\{f_n\} \subset \mathcal{E}^*, S: \mathcal{E} \to \mathcal{E}) \) be a exact Banach frame for \( \mathcal{E} \). Then, for every \( \{n_k\} \), \( \exists \) a closed subspace \( \mathcal{G} \) of \( \mathcal{E} \) and \( T: \mathcal{G}_d \to \mathcal{G} \) such that \( (\{f_{n_k}|_{\mathcal{G}}\} T) \) is exact Banach frame for \( \mathcal{G} \) w.r.to \( \mathcal{G}_d \).
Frames in Subspaces of Banach Spaces

- Let \( (\{f_n\}_S)(\{f_n\}_E E^*, S: E_d \rightarrow E) \) be a exact Banach frame for \( E \). Then, for every \( \{n_k\}_E \), \( \exists \) a closed subspace \( G \) of \( E \) and \( T: G_d \rightarrow G \) such that \( (\{f_{n_k}|_G\}_T) \) is exact Banach frame for \( G \) w.r.to \( G_d \).

- Let \( G \subset E \) such that \( G \) and \( E/G \) have Banach frames. Then \( E \) also have a Banach Frame.
For $\mathcal{E} = \ell^\infty$ and $\mathcal{G} = c_0$. $\mathcal{E}$ and $\mathcal{G}$ have Banach frame, however $\mathcal{E}/\mathcal{G}$ has no Banach frame.
Let \((\{f_n\}, S)(\{f_n\} \subset \mathcal{E}^*, S : \mathcal{E}_d \to \mathcal{E})\) is a Banach frame for \(\mathcal{E}\) w.r.to \(\mathcal{E}_d\). Let \(\{\phi_n\}\) in \(\mathcal{E}^{**}\) be such that

\[
\phi_i(f_j) = \delta_{i,j}, \quad i, j \in \mathcal{N}.
\]
Let \((\{f_n\}, S)(\{f_n\} \subset \mathcal{E}^*, S : \mathcal{E}_d \to \mathcal{E})\) is a Banach frame for \(\mathcal{E}\) w.r.to \(\mathcal{E}_d\). Let \(\{\phi_n\}\) in \(\mathcal{E}^{**}\) be such that

\[
\phi_i(f_j) = \delta_{i,j}, \quad i, j \in \mathbb{N}.
\]

If \(\exists \mathcal{E}_d^*\) and \(\mathcal{T} : \mathcal{E}_d^{**} \to \mathcal{E}^{**}\) such that \((\{\phi_n\}, \mathcal{T})\) is a Banach frame for \(\mathcal{E}^{**}\) w.r.to \(\mathcal{E}_d^{**}\), then

\[
((\{f_n\}, S), (\{\phi_n\}, \mathcal{T}))
\]

is a Banach frame system for \(\mathcal{E}\).

\((\{\phi_n\}, \mathcal{T})\) is called an admissible Banach frame to

\((\{f_n\}, S)\).
Examples of Banach frame System

Let $\mathcal{E} = l^1$ and $\{f_n\} \subset \mathcal{E}^*$. Define $\{g_n\} \subset \mathcal{E}^{**}$ by

$$g_n(x) = \xi_n, \quad x = \{\xi_n\} \in \ell^\infty, \ n \in \mathbb{N}.$$ 

Let $\mathcal{E}_d = \{\{f_n(x)\} : x \in \mathcal{E}\}$ and $(\mathcal{E}^*)_d = \{\{g_n(f) : f \in \mathcal{E}^*\}$. Then $\mathcal{E}_d$ and $(\mathcal{E}^*)_d$ are associated Banach space with norms given by $\|\{f_n(x)\}\|_{\mathcal{E}_d} = \|x\|_{\mathcal{E}}, \ x \in \mathcal{E}$ and $\|\{g_n(f)\}\|_{(\mathcal{E}^*)_d} = \|f\|_{\mathcal{E}^*}, \ f \in \mathcal{E}^*$ respectively. Define $S : \mathcal{E}_d \to \mathcal{E}$ by $S(\{f_n(x)\}) = x, \ x \in \mathcal{E}$ and $T : (\mathcal{E}^*)_d \to \mathcal{E}^*$ by $T(\{g_n(f)\}) = f, \ f \in \mathcal{E}^*$. Then $((\{f_n\}, S), (\{g_n\}, T))$ is a Banach frame system for $\mathcal{E}$. 
Examples of Banach frame System

\( \ell^\infty \) has no Banach frame system however it possesses a Banach frame.
Towards Uniqueness

As \([f_n] \neq \mathcal{E}^*\), let \(e \in \mathcal{E}^* \setminus [f_n]\), \(\exists g_0 \in \mathcal{E}^{**}\) such that

\[g_0(e) \neq 0\]

and

\[g_0([f_n]) = 0.\]
Towards Uniqueness

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Define \(\{\phi_n\} \subset \mathcal{E}^{**}\) by

\[ \phi_1 = g_1 - g_0, \phi_n = g_n, n = 2, 3, 4, .. \]
Towards Uniqueness

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Define \(\{\phi_n\} \subset \mathcal{E}^{**}\) by

\[ \phi_1 = g_1 - g_0, \phi_n = g_n, n = 2, 3, 4, .. \]

Then \(\{\phi_n\}, T_0\) is another admissible Banach frame to

\(\{f_n\}, S\), where

\[ T_0 : (\mathcal{E}^*)_d \to \mathcal{E}^* \]

by \(T_0(\{g_n(f)\}) = f, f \in \mathcal{E}^*\)
Theorem

Let \(((\{f_n\}, S), (\{g_n\}, T))\) be a Banach frame system for \(\mathcal{E}\). Then, \((\{g_n\}, T)\) is the unique admissible Banach frame to \((\{f_n\}, S)\) if and only if \([f_n] = \mathcal{E}^*\)
Theorems

**Theorem**

Let $((\{f_n\}, S), (\{g_n\}, T))$ be a Banach frame system for $E$. Then, $(\{g_n\}, T)$ is the unique admissible Banach frame to $(\{f_n\}, S)$ if and only if $[f_n] = E^*$.

**Theorem**

If $(\{f_n\}, S)$ is an exact Banach frame for $E$ such that

$$\bigcap_{n=1}^{\infty} \left[ f_i \right]_{i=n+1}^{\infty} = \{0\}.$$

Then, $E$ has a Banach frame system.
Theorem

If \(((\{f_n\}, S), (\{\phi_n\}, T))\) is a Banach frame system for \(E\) such that each \(\phi_n\) is weak\(^*\) continuous, then

\[
\bigcap_{n=1}^{\infty} [f_i]_{i=n+1}^\infty = \{0\}.
\]
Theorem

If \(((\{f_n\}, S), (\{\phi_n\}, T))\) is a Banach frame system for \(E\) such that each \(\phi_n\) is weak* continuous, then

\[
\bigcap_{n=1}^{\infty} \overline{\{f_i\}_{i=n+1}} = \{0\}.
\]

Theorem

If \(E^*\) has a Banach frame \((\{\phi_n\}, T)\), where \(\phi_n\) is weak* continuous for each \(n \in \mathbb{N}\), then \(E\) has a Banach frame system.


THANK YOU