3.2 Components

If a space X is not connected, then the knowledge about its maximal connected subspaces becomes indispensable in any description of the complete structure of the topology of X. We now consider these "pieces" of the space X.

Definition 3.2.1 Let X be a space, and $x \in X$. The *component* $C(x)$ of x in X is the union of all connected subsets of X which contain x.

By Theorem 3.1.9, the component $C(x)$ is connected. And, it is evident from its very definition that $C(x)$ is not properly contained in any connected subset of X. Thus $C(x)$ is a maximal connected subset of X; we shall see in a moment that the components of different points of X are either equal or disjoint. So we refer to them as the *components* of X.

A connected space X has only one component, viz., X itself, and at the opposite extreme, the components of a discrete space are the one-point subsets.

Proposition 3.2.2 Let X be a space. Then:

(a) Each component of X is closed.

(b) Each connected subset of X is contained in a component of X.

(c) The set of all components of X forms a partition of X.

Proof. (a): If $C(x)$ is a component of $x \in X$, then $C(x)$ is connected. So $\overline{C(x)}$ is also connected. By the maximality of $C(x)$, we have $C(x) = \overline{C(x)}$, for $C(x) \subset \overline{C(x)}$. Hence $C(x)$ is closed in X.

(b): If A is a nonempty connected subset of X, then $A \subset C(a)$ for any $a \in A$; this follows from the definition of a component.

(c): By definition, each point $x \in X$ belongs to a component of X, viz., $C(x)$. If $C(x)$ and $C(x')$ intersect, then $C(x) \cup C(x')$ is connected. The maximality of $C(x)$ and $C(x')$ implies that $C(x) = C(x) \cup C(x') = C(x')$. Thus any two components of X are either equal or disjoint. $\diamondsuit$

Proposition 3.2.2 provides a decomposition of the space X into connected pieces which are also closed. However, the components of this decomposition do not have to be open. If the number of components is finite, then obviously each component is open as well.

Example 3.2.1 Let $\mathbb{Q}$ be the space of rational numbers with the usual topology. If $X \subset \mathbb{Q}$ has two distinct points x and y, then, for an irrational number c between x and y, $X = [X \cap (-\infty, c)] \cup [X \cap (c, +\infty)]$ is a separation of X, and therefore X is disconnected. It follows that the components of $\mathbb{Q}$ are its one-point subsets. But the points in $\mathbb{Q}$ are not open, for every nonempty open subset of $\mathbb{Q}$ is infinite.

We next determine the components of a product space in terms of those of its factor spaces.

Proposition 3.2.3 Let X_α, $\alpha \in A$, be a family of spaces. Then the component of a point $x = (x_\alpha)$ in $\prod X_\alpha$ is $\prod C(x_\alpha)$, where $C(x_\alpha)$ is the component of x_α in X_α.

Proof. Let E be the component of x in $\prod X_\alpha$. By Theorem 3.1.13, $\prod C(x_\alpha)$ is connected, so we have $\prod C(x_\alpha) \subseteq E$. To prove the reverse inclusion, assume that $y \in E$. If $p_\beta : \prod X_\alpha \to X_\beta$ is the projection map, then $p_\beta(E)$ is a connected subset of X_β containing both x_β and y_β. Consequently, $y_\beta \in C(x_\beta)$, and we have $y \in \prod C(x_\alpha)$. ◊

It is of some interest to observe that the components of the product of discrete spaces X_α, $\alpha \in A$, are one-point sets, although $\prod X_\alpha$ is not discrete when A is infinite.

If $f : X \to Y$ is continuous, and $C(x)$ is the component of x in X, then $f(C(x)) \subseteq C(f(x))$, the component of $f(x)$ in Y. If f is a homeomorphism, then we also have $f^{-1}(C(f(x))) \subseteq C(x)$, and therefore $f(C(x)) = C(f(x))$. This shows that the number of components of a space and structure of each component are topological invariants. So the knowledge about components can be used for distinguishing between spaces.

Definition 3.2.4 A space X is called *totally disconnected* if its components are singleton sets.

Obviously, a discrete space is totally disconnected. More interesting examples are the Cantor set, the set $\mathbb{Q}$ of rationals, and the set $\mathbb{R} - \mathbb{Q}$ of irrationals with the relative topologies induced from $\mathbb{R}$. Note that a one-point space is both connected and totally disconnected. If X is totally disconnected and Y is a subspace of X, then Y is also totally disconnected, for a component of Y must be contained in some component of X. Also, we see from Proposition 3.2.3 that the product of a family of totally disconnected spaces is totally disconnected.

Returning to the general discussion, let $X = A \cup B$ be a separation of X. If x and y belong to a component of X, then both points are either in A or in B (ref. Exercise 3.1.2). However, there are spaces X with distinct points x and y, which do not lie together in a connected set, and yet they cannot be separated by any separation of X.

3.3 Path-connected Spaces

Path-connectedness corresponds to our intuitive notion of considering a space one piece if one can move in the space from any point to any other point. This kind of property has led to many sophisticated algebraic techniques for the study of geometry of the spaces.

Let X be a space. A *path* in X is a continuous function $f : I \to X$, where I is the unit interval with the usual subspace topology. The point $f(0)$ is referred to as the *origin* (or *initial point*) and the point $f(1)$ as the *end* (or *terminal point*) of f. We usually say that f is a path in

X from $f(0)$ to $f(1)$ (or joining them). It is important to note that a path is a function, not the image of the function.

Definition 3.3.1 A space X is called *path-connected* if for each pair of points $x_0, x_1 \in X$, there is a path in X joining x_0 to x_1.

Example 3.3.1 An indiscrete space and the Sierpinski space are obviously path-connected.

Example 3.3.2 Given $x, y \in \mathbb{R}^n$, the set of points $(1-t)x + ty \in X$, $0 \le t \le 1$, is called a (closed) *line segment* between x and y. A set $X \subseteq \mathbb{R}^n$ is called *convex* if it contains the line segment joining each pair of its points. Thus, if X is a convex subspace of $\mathbb{R}^n$, then, given $x, y \in X$, we have $(1-t)x + ty \in X$ for every $t \in I$. So there is a path in X joining x to y, namely, $t \mapsto (1-t)x + ty$. In particular, the euclidean space $\mathbb{R}^n$, the n-disc $\mathbb{D}^n$, and the open balls in $\mathbb{R}^n$ are all path-connected.

Lemma 3.3.2 A space X is path-connected if and only if there is a point $x_0 \in X$ such that each $x \in X$ can be joined to x_0 by a path in X.

Proof. The necessity is obvious. To establish the sufficiency, let $x, y \in X$. Suppose that f and g are paths in X joining x and y to x_0, respectively. Consider the function $h : I \to X$ defined by

$$h(t) = \begin{cases} f(2t) & 0 \le t \le 1/2, \\ g(2-2t) & 1/2 \le t \le 1. \end{cases}$$

For $t = 1/2$, we have $f(2t) = x_0 = g(2-2t)$. By the Gluing lemma, h is continuous, and thus defines a path in X joining x to y. So X is path-connected. ◇

Example 3.3.3 Consider the subspace A of $\mathbb{R}^2$ consisting of the points (x, y), where $(x = 1/n,\ n \in \mathbb{N}$, and $0 \leq y \leq 1)$ or $(0 \leq x \leq 1$ and $y = 0)$ (see Figure 3.3). For each $(x, y) \in A$, the function $f : I \to X$ defined by

$$f(t) = \begin{cases} (2xt, 0) & 0 \leq t \leq 1/2 \\ (x, (2t-1)y) & 1/2 \leq t \leq 1 \end{cases}$$

is a path from $(0, 0)$ to (x, y) in X. From the preceding lemma, it is immediate that A is path-connected. Its closure $\overline{A} = A \cup (\{0\} \times I)$ is also path-connected, since A and $\{0\} \times I$ have a common point. The subspace $\overline{A} \subset \mathbb{R}^2$ is called the *Comb space.*

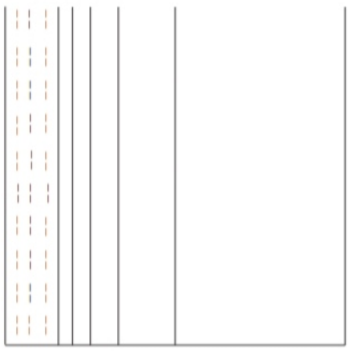

FIGURE 3.3: Comb space

Example 3.3.4 The n-sphere $\mathbb{S}^n$, $n > 0$, is path-connected. For, if $H_1 = \{x \in \mathbb{S}^n | x_{n+1} \geq 0\}$ is the upper hemisphere and $y \in H_1$ is the north pole, then

$$t \to ((1-t)x + ty) \,/\, \|(1-t)x + ty\|$$

is a path in $\mathbb{S}^n$ joining x to y. So H_1 is path-connected. Similarly, the lower hemisphere $H_2 = \{x \in \mathbb{S}^n | x_{n+1} \leq 0\}$ is also path-connected. Obviously, $H_1 \cap H_2 \neq \varnothing$, and hence $\mathbb{S}^n = H_1 \cup H_2$ is path-connected.

Theorem 3.3.4 Every path-connected space is connected.

Proof. Suppose that X is path-connected, and choose a point $x_0 \in X$. For each $x \in X$, there is a path f_x in X from x_0 to x. Since I is connected, so is $\mathrm{im}(f_x)$. Clearly, X is the union of the connected subsets $\mathrm{im}(f_x)$, $x \in X$. By Theorem 3.1.9, X is connected. $\diamond$

In general, the converse of the above theorem is false, although it is true for all subspaces of $\mathbb{R}$. Thus path-connectedness is a stronger form of connectedness.

Example 3.3.5 We have already seen that the subspace $S = \{(x, \sin 1/x) \,|0 < x \leq 1\} \subset \mathbb{R}^2$ is connected (refer to Ex. 3.1.7), and therefore the closed topologist's sine curve $\overline{S}$ is connected. But, it is not path-connected. To see this, consider the points $z_0 = (0,0)$ and $z_1 = (1/\pi, 0)$ in $\overline{S}$ and suppose that $f : I \to \overline{S}$ is a path with origin z_0 and end z_1. Let $p_i : \mathbb{R}^2 \to \mathbb{R}^1$, $i = 1, 2$, be the projection maps. Then the compositions $p_1 f$ and $p_2 f$ must be continuous. Let t_0 be the supremum of the points $t \in I$ such that $p_1 f(t) = 0$. We assert that $p_2 f$ is discontinuous at t_0, a contradiction. By the continuity of $p_1 f$, it is easily seen that $p_1 f(t_0) = 0$; accordingly, $t_0 < 1$. Since $f(t) \in S$ for $t > t_0$, the image of each interval $[t_0, t_1)$ under $p_1 f$ contains numbers of the form $2/n\pi$ for all large integers n. Thus $p_2 f$ takes on both values $+1$ and -1 in $[t_0, t_1)$. Hence the open interval $(y_0 - 1/3, y_0 + 1/3)$, where $y_0 = p_2 f\,(t_0)$, cannot contain the image of $[t_0, t_1)$ under $p_2 f$, and this proves our assertion.

The invariance properties of path-connectedness are similar to those of connectedness. The simple proof of the folowing theorem is left to the reader.

Theorem 3.3.5 If $f : X \to Y$ is continuous and X is path-connected, then $f(X)$ is path-connected.

The preceding theorem shows that path-connectedness is a topological invariant. Like connectedness, this property is also productive, but not hereditary.

Theorem 3.3.6 The product of a family of path-connected spaces is path-connected.

Proof. Let X_α, $\alpha \in A$, be a family of path-connected spaces, and suppose that $x, y \in \prod X_\alpha$. Then, for each $\alpha \in A$, there is a path $f_\alpha : I \to X_\alpha$ joining x_α to y_α. Define $\phi : I \to \prod X_\alpha$ by $\phi(t) = (f_\alpha(t))$, $t \in I$. If $p_\beta : \prod X_\alpha \to X_\beta$ is the projection map, then $p_\beta \phi = f_\beta$ is continuous. So ϕ is continuous, by Theorem 2.2.10. Also, we have $\phi(0) = x$ and $\phi(1) = y$, obviously. Thus x and y are joined by a path in $\prod X_\alpha$, and the theorem follows. ♢

Note that there is no analogue of Theorem 3.1.12 for path-connectedness, as shown by the subset S in Example 3.3.5.

By Corollary 3.3.3, for each point x in a given space X, there exists a maximal path-connected subset of X containing x, viz., the union of all path-connected subsets of X which contain x.

Definition 3.3.7 Let X be a space, and $x \in X$. The *path component* of x in X is the maximal path-connected subset of X containing x.

Clearly, the path component of a point x in a space X is the set of all points $y \in X$ which can be joined to x by a path in X. If $P(x)$ and $P(y)$ are path components of two points x and y in a space X, and $z \in P(x) \cap P(y)$, then $P(x) \cup P(y)$ is path-connected. By the maximality of $P(x)$, we have $P(x) = P(x) \cup P(y)$. So $P(x) \subseteq P(y)$. Similarly, $P(y) \subseteq P(x)$ and the eqality holds. It follows that the path components of the space X partition the set X, and X is path-connected if and only if it has no more than one path component. Since a path component is connected, it is contained in a component. Accordingly, each component of X is a disjoint union of its path components.

3.4 Local Connectivity

The path components of a space X are not necessarily closed subsets of X, nor are they necessarily open, as shown by the path components S and $\{0\} \times [-1, 1]$ of the closed sine curve (refer to Ex. 3.1.7). Also, the components of a space need not be open (cf. Ex. 3.2.1). We will study here the localized version of connectedness and path-connectedness under which components and path components, respectively, are open and hence closed.

LOCAL CONNECTEDNESS

Definition 3.4.1 A space X is called *locally connected at* $x \in X$ if, for each open neighbourhood U of x, there is a connected open set V such that $x \in V \subset U$. The space X is *locally connected* if it is locally connected at each of its points.

It is clear that the family of all connected open sets in a locally connected space forms a basis. Conversely, if a space X has a basis consisting of connected sets, then it is obviously locally connected.

Example 3.4.1 A discrete space is locally connected, and so is an indiscrete space.

Example 3.4.2 The euclidean space $\mathbb{R}^n$ is locally connected, since its base of open balls consists of connected sets. Similarly, $\mathbb{S}^n$ is also locally connected.

Example 3.4.3 The subspace $X = \{0\} \cup \{1/n | n = 1, 2, \ldots\}$ of the real line is not locally connected. Because the singleton set $\{1/n\}$ are clopen in X, and any neighbourhood of 0 contains them for large values of n. Thus there is no connected neighbourhood of the point 0 in X.

A locally connected space need not be connected, as shown by the subspace $[0, 1/2) \cup (1/2, 1]$ of I. On the other hand, the following examples show that a connected space need not be locally connected.

Example 3.4.4 For each integer $n > 0$, let L_n be the line segment in $\mathbb{R}^2$ joining the origin 0 to the point $(1, 1/n)$, and L_0 be the segment $\{(s, 0) | 0 \leq s \leq 1\}$ (Figure 3.4). Then each L_n, $n \geq 0$, is connected and contains the point 0. Hence $X = \bigcup L_n$ is a connected subset of $\mathbb{R}^2$. We

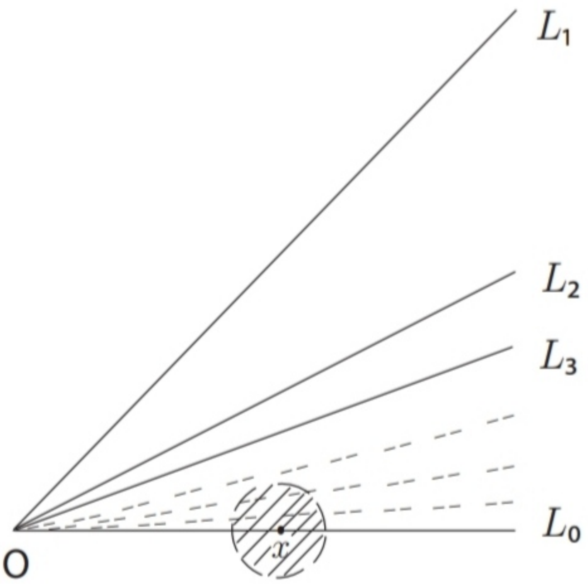

FIGURE 3.4: A connected space which is not locally connected.

show that X is not locally connected at any point x of L_0 other than 0. Consider a small ball B about x such that $0 \notin B$. For $1/n$ less than the radius of B, $L_n \cap B \neq \varnothing$, and this intersection is a component of $U = X \cap B$, for it is connected, closed and open in U. Now, if V is open and $x \in V \subset U$, then clearly $V \cap L_n \neq \varnothing$ for all large n. Therefore V cannot be connected, and we see that X is not locally connected at x.

Example 3.4.5 The closed topologist's sine curve (ref. Ex. 3.1.5) is not locally connected at any point $p = (0, y)$, $-1 \leq y \leq 1$. Consider the open nbd $U = \overline{S} \cap B$ of p, where B is an open ball of radius less than 1/2 and centered at p (see Figure 3.5). Clearly, the boundary of B divides the wiggly part of $\overline{S}$ into infinitely many arcs, and $\overline{U} = \overline{S} \cap \overline{B}$

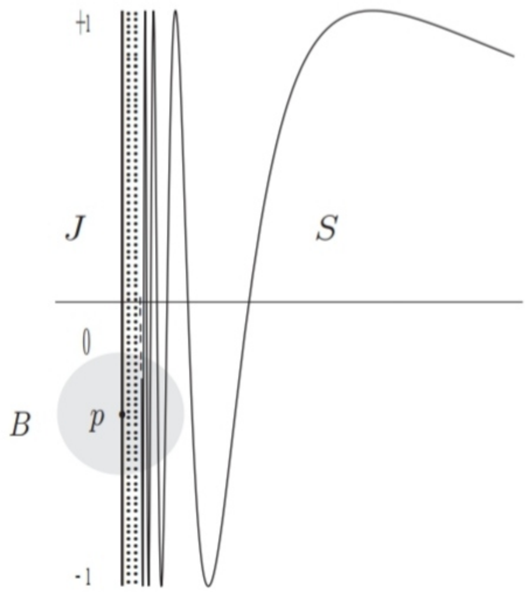

FIGURE 3.5: The closed topologist's sine curve is not locally connected.

consists of a segment C of $\{(0, y) | -1 \leq y \leq 1\}$, and some of the arcs of S. Each arc of S lying in $\overline{B}$ is closed and connected, being a continuous image of a closed interval. Since C is separated from every arc contained in $\overline{U}$, it follows that C is the component of p in $\overline{U}$. Obviously, every neighbourhood of p intersects S; so there is no connected open set V in $\overline{S}$ containing p and contained in U. Thus $\overline{S}$ is not locally connected at p.

Similarly, one sees that the sine curve $S \cup \{0\}$ is also not locally connected at 0, while the subspace S is.

Now, we turn to see a characterisation of locally connected spaces in terms of the components of open subsets of the space. By the components of a subset $U \subset X$, we mean the components of the subspace U of X. Thus, the component of a point x in U is a subset of U.

Theorem 3.4.2 A space X is locally connected if and only if the components of each open subset of X are open.

Proof. Suppose that X is locally connected. Let U be an open subset of X, and C be a component of U. If $x \in C$, then there is a connected open set $V \subseteq X$ such that $x \in V \subseteq U$, by our hypothesis. Since C is a component of x in U and V is a connected subset of U containing x, we have $V \subseteq C$. Thus C is a neighbourhood of each of its points, and therefore open.

To prove the converse, let $U \subset X$ be open, and $x \in U$. By our hypothesis, the component V of x in U is open. So X is locally connected at x. This is true for every $x \in X$, and X is locally connected. ♢

As a particular case of the above theorem, we see that each component of a locally connected space is open.

Next, we come to the usual questions involving continuous functions, products and subspaces of locally connected spaces. Although, local connectedness is not hereditary, every open subset of a locally connected space is locally connected. It is noteworthy that every connected subspace of the real line $\mathbb{R}$ is locally connected. It is also clear from the definition that a continuous open image of a locally connected space is locally connected. Thus local connectedness is a topological invariant. Moreover, we have

Theorem 3.4.3 Let $f : X \to Y$ be a continuous closed surjection. If X is locally connected, then so is Y.

Proof. Suppose that X is locally connected. By Theorem 3.4.2, we need to show that the components of each open set $U \subset Y$ are open. Let C be a component of U. We assert that $f^{-1}(C)$ is open. If $x \in f^{-1}(C)$, then there exists a connected open set V in X such that $x \in V \subseteq f^{-1}(U)$, since X is locally connected and $f^{-1}(U)$ is open in X. It follows that $f(x) \in f(V) \subseteq C$, for $f(V)$ is connected. So $x \in V \subseteq f^{-1}(C)$, and $f^{-1}(C)$ is a neighbourhood of x. This proves our assertion. Now, since f is closed, $Y - C = f\left(X - f^{-1}(C)\right)$ is closed. So C is open and the theorem follows. ♢

However, local connectedness is not preserved by a continuous map, as shown by the following.

Example 3.4.6 Let $Y = \mathbb{N} \cup \{0\}$ with the discrete topology, and X be the space in Ex. 3.4.3. It is known that Y is locally connected, and X is not. Obviously, the function $f : Y \to X$, defined by $f(n) = 1/n$, $n \in \mathbb{N}$, and $f(0) = 0$, is a continuous bijection.

The property of local connectedness is not transmitted to arbitrary products, as shown by an infinite product of discrete spaces. In this regard, we have the following.

Theorem 3.4.4 Let X_α, $\alpha \in A$, be a collection of spaces. Then the product $\prod X_\alpha$ is locally connected if and only if each X_α is locally connected and all but finitely many spaces X_α are also connected.

Proof. Suppose that $\prod X_\alpha$ is locally connected. Then each component C of $\prod X_\alpha$ is open. Let $p_\beta : \prod X_\alpha \to X_\beta$ denote the projection map for every index $\beta \in A$. Find a basic open set $B = \bigcap_{i=1}^n p_{\alpha_i}^{-1}(U_{\alpha_i})$, say, contained in C. Then, for $\alpha \neq \alpha_1, \ldots, \alpha_n$, $X_\alpha = p_\alpha(B) = p_\alpha(C)$ is connected. To see the local connectedness of X_β, let $x \in X_\beta$ be arbitrary, and U_β be an open neighbourhood of x in X_β. Then $p_\beta^{-1}(U_\beta)$ is open in $\prod X_\alpha$ and contains a point ξ with $x = \xi_\beta$. By our assumption, there is a connected open set V in $\prod X_\alpha$ such that $\xi \in V \subseteq p_\beta^{-1}(U_\beta)$. We have $x \in p_\beta(V) \subseteq U_\beta$. The set $p_\beta(V)$ is connected and open, for p_β is continuous and open. Thus, X_β is locally connected at x.

To prove the converse, let $\xi = (x_\alpha) \in \prod X_\alpha$, and let $B = \bigcap_1^n p_{\alpha_i}^{-1}(U_{\alpha_i})$ be a basic neighbourhood of ξ. By our hypothesis, there are at most finitely many indices $\alpha \neq \alpha_1, \ldots, \alpha_n$ such that X_α is not connected. Assume that these indices are $\alpha_{n+1}, \ldots, \alpha_{n+m}$. By local connectedness of the X_α, there exists a connected open neighbourhood V_{α_i} of x_{α_i}, $1 \le i \le n+m$, such that $V_{\alpha_i} \subset U_{\alpha_i}$ for $1 \le i \le n$. By Theorem 3.1.13, $C = \bigcap_1^{n+m} p_{\alpha_i}^{-1}(V_{\alpha_i})$ is a connected open set, and $\xi \in C \subseteq B$. So $\prod X_\alpha$ is locally connected at ξ, and this completes the proof. ◇

LOCAL PATH-CONNECTEDNESS

Definition 3.4.5 A space X is said to be *locally path-connected at a point* $x \in X$ if each open neighbourhood of x contains a path-connected neighbourhood of x. The space X is called *locally path-connected* if it is locally path-connected at each of its points.

A discrete space is obviously locally path-connected. The euclidean space $\mathbb{R}^n$ is locally path-connected, since the open balls are path-connected. The n-sphere $\mathbb{S}^n$ is locally path-connected, because each point of $\mathbb{S}^n$ has a neighbourhood which is homeomorphic to an open subset of $\mathbb{R}^n$. It should be noted that a path-connected space need not be locally path-connected. This can be seen by the union of the closed topologist's sine curve with an arc connecting the points $(1, \sin 1)$ and $(0, 1)$. (see Figure 3.6.)

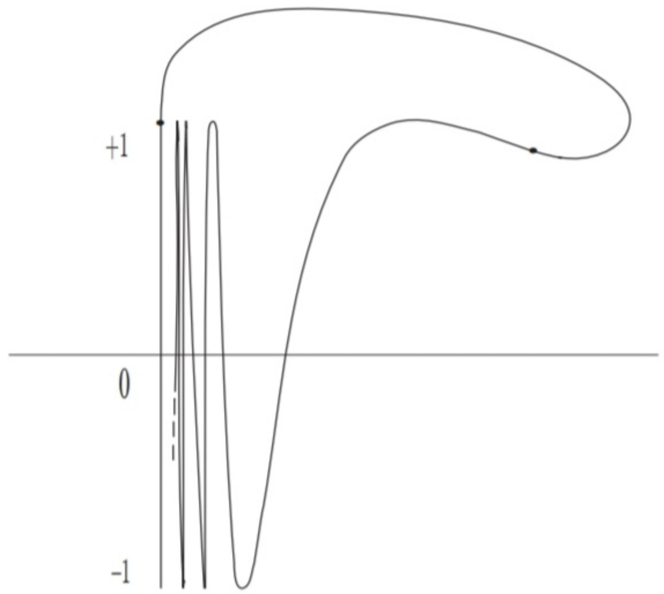

FIGURE 3.6: A path-connected space which is not locally path-connected.

Clearly, a space X is locally path-connected at $x \in X$ if and only if there exists a neighbourhood basis at x consisting of path-connected sets. A frequently used criterion for local path-connectedness is described by

Proposition 3.4.6 A space X is locally path-connected if and only if the path components of open subsets of X are open.

Proof. $\Rightarrow$: Suppose that X is locally path-connected, and let $U \subset X$ be open and P a path component of U. If $x \in P$, then there exists a path-connected neighbourhood V of x with $V \subset U$. As $x \in P \cap V$, $P \cup V$ is path-connected and contained in U. Since P is a maximal path-connected subset of U, we have $P \cup V = P$, which implies that $V \subset P$. Thus P is a neighbourhood of x, and P is open.

$\Leftarrow$: Obvious. $\diamondsuit$

From the preceding proposition, it is clear that a space X is locally path-connected if and only if it has a basis of path-connected open sets. Another equivalent formulation of local path-connectedness is given in Exercise 14.

Corollary 3.4.7 Let X be a locally path-connected space. Then each path component $P(x)$ of X is clopen, and therefore coincides with the component $C(x)$ of X.

Proof. By Proposition 3.4.6, $P(x)$ is open. The complement of $P(x)$ in X is the union of all path components of X which are different from $P(x)$, and is therefore open. Thus $P(x)$ is closed, too. Finally, $P(x)$ is connected, by Theorem 3.3.4, and hence a component of X. $\diamondsuit$

As a consequence of the preceding corollary, we obtain

Corollary 3.4.8 A connected, locally path-connected space is path-connected.

A discrete space with at least two points shows that the condition of connectedness in this corollary is essential.

It is clear from the definition that an open subspace of a locally path-connected space is locally path-connected. Hence every connected open subspace of $\mathbb{R}^n$ and of $\mathbb{S}^n$ is path-connected. It is also obvious that a locally path-connected space is locally connected, but the converse is not true (see Exercise 5).

The invariance properties of local path-connectedness are similar to those of local connectedness. The proof of the following theorems is similar to that of Theorem 3.4.3, and left to the reader.

Theorem 3.4.9 Let $f : X \to Y$ be a continuous closed or open surjection. If X is locally path-connected, then so is Y.

By the preceding theorem, the local path-connectedness is a topological invariant. However, the property is not continuous invariant.

Example 3.4.7 Consider the subspaces $X = S \cup \{(-1,0)\}$ and $Y = S \cup \{(0,0)\}$ of the closed topologist's sine curve (Ex. 3.1.7). Clearly, X is locally path-connected, but Y is not (see Ex. 3.4.5). Define $f : X \to Y$ by setting $f(x) = x$ for every $x \in S$, and $f(-1,0) = (0,0)$. Then f is a continuous surjection.

Theorem 3.4.10 Let X_α, $\alpha \in A$, be a collection of spaces. Then the product $\prod_\alpha X_\alpha$ is locally path-connected if and only if each X_α is locally path-connected, and all but finitely many X_α are also path-connected.