

(1)

Abstract Harmonic Analysis

ak7028581@gmail.com (Mar 17-20)

We know that  $L^1(G)$  has a bounded approximate identity. Given a net  $\{h_\alpha\}$ , we now find a necessary and sufficient condition for  $\{h_\alpha\}$  to be a bounded approximate identity.

Theorem Let  $G$  be a locally compact  $T_2$  group with left Haar measure  $\lambda$ .

Let  $\{h_\alpha\}$  be a net in  $L_+^1(G)$  such that  $\|h_\alpha\|_1 = 1$  for all  $\alpha$ . Then  $\{h_\alpha\}$  is a bounded left approximate identity in  $L^1(G)$  if and only if  $\lim_\alpha \|\int_{W'} h_\alpha(y) \chi_{W'}(y) d\lambda(y)\|_1 = 0$  for every neighbourhood  $W$  of  $e$ .

Proof Suppose that  $\lim_\alpha \|\int_{W'} h_\alpha(y) \chi_{W'}(y) d\lambda(y)\|_1 = 0$  for every ~~every~~ neighbourhood  $W$  of  $e$ .

For  $f \in L^1(G)$ , the map  $x \mapsto \int_x f$  is continuous so there is a neighbourhood  $W$  of  $e$  such that  $\|\int_x f - f\|_1 < \epsilon/2$ .

there is a neighbourhood  $W$  of  $e$  such that  $\|\int_x f - f\|_1 < \epsilon/2$ .  
for  $y \in W$ . Since  $\|h_\alpha\|_1 = 1$  and  $h_\alpha \geq 0$  so  $\int_G h_\alpha(y) d\lambda(y) = 1$ .

$$\begin{aligned}
 & \left| \int_G (h_\alpha * f)(x) - f(x) d\lambda(x) \right| \\
 &= \left| \int_G \left( \int_G f(y) h_\alpha(y) d\lambda(y) - \int_G f(x) h_\alpha(y) d\lambda(y) \right) d\lambda(x) \right| \\
 &\leq \left| \int_G \left( \int_G |f(y) - f(x)| h_\alpha(y) d\lambda(y) \right) d\lambda(x) \right| \\
 &= \int_G \left( \int_G |f(y) - f(x)| d\lambda(x) \right) h_\alpha(y) d\lambda(y) \\
 &= \int_G \|f_y - f\|_1 h_\alpha(y) d\lambda(y) \\
 &= \int_W \|f_y - f\|_1 h_\alpha(y) d\lambda(y) + \int_{W'} \|f_y - f\|_1 h_\alpha(y) d\lambda(y) \\
 &< \frac{\epsilon}{2} \int_W h_\alpha(y) d\lambda(y) + \int_{W'} 2\|f\|_1 h_\alpha(y) d\lambda(y) \\
 &\leq \frac{\epsilon}{2} + 2\|f\|_1 \int_{W'} h_\alpha(y) d\lambda(y)
 \end{aligned}$$

Since  $\lim_{\alpha} \|\rho_\alpha \bar{z}_{W'}\|_1 = 0$  so there exists  $\alpha_0$  such that (2)

$$\|\rho_\alpha \bar{z}_{W'}\|_1 < \frac{\epsilon}{2(1+\|f\|_1)}, \alpha \geq \alpha_0$$

Using (\*), for  $\alpha \geq \alpha_0$

$$\|\rho_\alpha * f - f\|_1 = \int_G |\rho_\alpha * f(y) - f(x)| d\lambda(x) < \epsilon$$

i.e.  $\{\rho_\alpha\}$  is a left approximate identity for  $L^1(G)$ .

Necessary Part. Let  $\{\rho_\alpha\}$  be a bounded left approximate identity for  $L^1(G)$ . Let  $W$  be a neighbourhood of  $e$ .

Let  $V$  be a symmetric neighbourhood of  $e$  such that  $V^2 \subset W$  and  $\lambda(V) < \infty$ .

$$\text{Let } f(x) = \frac{1}{\lambda(V)} \bar{z}_V * \bar{z}_V(x) = \frac{1}{\lambda(V)} \int_G \bar{z}_V(y) \bar{z}_V(\bar{y}x) dy$$

$f$  is a continuous function as  $L^\infty(G) * L^1(G) \subset C(G)$

$$\begin{aligned} f(e) &= \frac{1}{\lambda(V)} \int_G \bar{z}_V(y) \bar{z}_V(\bar{y}) d\lambda(y) = \frac{1}{\lambda(V)} \int_V \bar{z}_V(\bar{y}) d\lambda(y) \\ &= \frac{1}{\lambda(V)} \lambda(V) = 1 \quad (V \text{ is symmetric}) \end{aligned}$$

For  $x \in W'$  so  $x \notin W \Rightarrow x \notin V^2 \Rightarrow x \notin V^2$

Now for  $y \in V$ ,  $y\bar{x}' \notin V$  (if  $y\bar{x}' \in V$  then  $\bar{x}' = y^{-1}(y\bar{x}') \in V^2$ )

$$f(\bar{x}') = \frac{1}{\lambda(V)} \int_V \bar{z}_V(y\bar{x}') d\lambda(y) = 0$$

$$\text{For } x \in G, f(x) \geq 0, f(x) = \frac{1}{\lambda(V)} \int_V \bar{z}_V(\bar{y}x) d\lambda(y) \leq \frac{1}{\lambda(V)} \lambda(V) = 1.$$

For each  $a$ ,

$$\begin{aligned} \|\rho_a \bar{z}_{W'}\|_1 &= \int_{W'} |\rho_a| d\lambda = 1 - \int_W |\rho_a| d\lambda \\ &\leq 1 - \int_W \rho_a(y) f(\bar{y}) dy \\ &= 1 - \int_W |\rho_a(y)| |f(\bar{y})| dy \end{aligned}$$

$$\begin{aligned}
&\leq \left| 1 - \int_W h_\alpha(y) f(\bar{y}) d\lambda(y) \right| \\
&= |f(e) - h_\alpha * f(e)| \\
&\leq \|f - h_\alpha * f\|_\infty \\
&= \frac{1}{\lambda(V)} \|\tilde{\beta}_V * \tilde{\beta}_V - h_\alpha * \tilde{\beta}_V * \tilde{\beta}_V\|_\infty \\
&\leq \frac{1}{\lambda(V)} \|\tilde{\beta}_V - h_\alpha * \tilde{\beta}_V\|_1, \|\tilde{\beta}_V\|_\infty \\
&= \frac{1}{\lambda(V)} \|h_\alpha * \tilde{\beta}_V - \tilde{\beta}_V\|_1, \rightarrow 0, \text{ since} \\
\end{aligned}$$

$\{h_\alpha\}$  is an approximate identity. Thus

$$\lim_\alpha \|h_\alpha \tilde{\beta}_W\|_1 = 0.$$

Next we study functional equations on compact groups.

Proposition Let  $G$  be a compact group and  $V$  be a  $d$ -dimensional representation of  $G$ . If  $X_V(x) = \text{tr}(U_x)$  then

$$X_V(x) X_V(y) = d \int_G X_V(tx t^{-1} y) dt$$

Proof Let  $\{\beta_1, \beta_2, \dots, \beta_d\}$  be an orthonormal basis for the Hilbert space of  $V$  (i.e. representation space of  $V$ ).

$$\begin{aligned}
\int_G X_V(tx t^{-1} y) dt &= \int_G \text{tr}(U_t U_x U_{t^{-1}} U_y) dt \\
&= \sum_{h=1}^d \int_G \langle U_t U_x U_{t^{-1}} U_y \beta_h, \beta_h \rangle dt \quad (*)
\end{aligned}$$

Recall that  $\langle A_B \tilde{\beta}_1, \tilde{\beta}_2 \rangle = C_B(\beta_1, \beta_2)$  and  $A_B = d^{-1} \text{tr} B I$

so  $\langle A_B \tilde{\beta}_1, \tilde{\beta}_2 \rangle = d^{-1} \text{tr} B \langle \tilde{\beta}_1, \tilde{\beta}_2 \rangle$ . Thus applying with

$$\begin{aligned}
(*) \Rightarrow \int_G X_V(tx t^{-1} y) dt &= \sum_{h=1}^d C_B(U_y \beta_h, \beta_h) = \sum_{h=1}^d d^{-1} \text{tr} U_x \langle U_y \beta_h, \beta_h \rangle \\
&= d^{-1} \text{tr} U_x \sum_{h=1}^d \langle U_y \beta_h, \beta_h \rangle
\end{aligned}$$

$$\int_G x_U(tx t^{-1} y) dt = \text{tr } U_x \text{ tr } U_y = X_U(x) X_U(y)$$

Thus the trace function satisfies the functional equation in above proposition. We know prove a converse of the above Proposition. ( $G$  is a compact group)

Proposition Let  $0 \neq f \in L^1(G)$  and  $\tilde{f}$  is any function on  $G$  that satisfy

$$\tilde{f}(x) f(y) = \int_G f(tx t^{-1} y) dt \quad \forall x, y \in G \quad \dots (*)$$

then  $\tilde{f}_n = \frac{1}{n} X_U$ , for some continuous irreducible representation  $U$  of  $G$ .

Proof. Let  $g \in L_\infty(G)$  and  $\tilde{f}(x) = f(\bar{x}')$

$$\begin{aligned} \tilde{f}(x) \int_G \tilde{f}(ux) g(ux) d\lambda(ux) &= \tilde{f}(x) \int_G f(\bar{u}' u) g(\bar{u}' u) d\lambda(u) \quad (u \rightarrow \bar{u}') \\ &= \int_G \tilde{f}(x) f(u) g(\bar{u}') d\lambda(u) \\ &= \int_G \int_G f(tx t^{-1} u) g(\bar{u}') d\lambda(t) d\lambda(u) \quad (u \rightarrow \bar{u}') \\ &\quad u \rightarrow t u \\ &= \int_G \int_G f(tx u) g(\bar{u}' t^{-1}) d\lambda(t) d\lambda(u) \\ &\quad u \rightarrow \bar{u}' t^{-1} u \\ &= \int_G \int_G f(u) g(\bar{u}' t x t^{-1}) d\lambda(t) d\lambda(u) \quad (u \rightarrow \bar{u}') \\ &= \int_G \int_G f(\bar{u}') g(\bar{u}' t x t^{-1}) d\lambda(u) d\lambda(u) \quad \dots (***) \end{aligned}$$

Let  $U$  be a continuous irreducible representation of  $G$  on a Hilbert space  $H$ . For  $\bar{z}, \bar{n} \in M$ , let  $g(u) = \langle U_{\bar{u}} \bar{z}, \bar{n} \rangle$

Let  $P$  be the corresponding representation of  $L^1(G)$ .

~~$P(x) \leq C$~~

$$\begin{aligned}
h(x) \langle e(\tilde{f}) \rangle_{\beta, n} &= h(x) \int f(u') \langle u_n \rangle_{\beta, n} du \\
&= \int_G \int_G f(u') \langle u_n | U_{t(x+u')} \rangle_{\beta, n} du dt \quad (\text{use } (**)) \\
&= \int_G \langle e(\tilde{f}) | U_{t(x+u')} \rangle_{\beta, n} dt \\
&= \int_G \langle U_t U_x U_{-1} \rangle_{\beta, n} e(\tilde{f})^* n dt \\
&= C_B \langle \beta, e(\tilde{f})^* n \rangle = \langle A_B \beta, e(\tilde{f})^* n \rangle \quad (B=1) \\
&= \frac{1}{d} \operatorname{tr} U_x \langle \beta, e(\tilde{f})^* n \rangle \\
&= \frac{1}{d} X_U(x) \langle e(\tilde{f}) \rangle_{\beta, n}
\end{aligned}$$

Since  $f \neq 0$  there exist  $U, \beta, n$  such that  $\langle e(\tilde{f}) \rangle_{\beta, n} \neq 0$

$$\text{Thus } h(x) = \frac{1}{d} X_U(x)$$

Exercise Write the interpretation of above proposition when  $G = T$  (the circle group)

Corollary If  $f \in L^1(G)$  satisfying the functional equation  $f(x) f(y) = \int_G f(t x t^{-1} y) dt$

$$\text{for all } x, y \in G \text{ then } f = 0 \text{ or } f = \frac{1}{d} X_U$$

Proof. Take  $h = f$  in the above Proposition.

Remark Let  $G$  be a compact group. For  $1 \leq p \leq \infty$ , the function space  $L^p(G)$  is a Banach algebra under convolution.

$$\begin{aligned}
\text{Proof} \quad \text{Since } L^p(G) \subseteq L^1(G) \\
\|f\|_p = \left( \int_G |f(x)|^p dx \right)^{1/p} \leq \left( \int_G 1 dx \right)^{1/p} \left( \int_G |f(x)|^p dx \right)^{1/p} = \|f\|_p
\end{aligned}$$

For  $f, g \in L^p(G)$ , we have  $f \in L^1(G)$  so  
 $f * g \in L^p(G)$  and  $\|f * g\|_p \leq \|f\|_1 \|g\|_p \leq \|f\|_p \|g\|_p$

Definition A function on a group  $G$  is called a central function if  $f$  is constant on conjugacy classes i.e.

$$f(yxy^{-1}) = f(x) \quad \forall x, y \in G. \text{ Equivalently } f(xy) = f(yx) \quad \forall x, y \in G.$$

E.g. characters of any finite dimensional representation is a central function since  $\text{tr}(\pi(x)\pi(y)) = \text{tr}(\pi(y)\pi(x))$

Denote  $Z(L^p(G)) = \text{the set of central functions in } L^p(G)$

$Z(C(G)) = \text{the set of central function in } C(G)$ .

Lemma 1.  $Z(L^p(G)) = \{f \in L^p(G) : f * g = g * f \quad \forall g \in L^p(G)\}$

$$Z(C(G)) = \{f \in C(G) : f * g = g * f \quad \forall g \in C(G)\}.$$

Proof If  $f \in L^p(G)$  then  $f * g = g * f$  if and only if

$$\begin{aligned} \int_G f(xy)g(y^{-1})dy &= \int_G g(y)f(y^{-1}x)dy \\ &= \int_G f(yx)g(y^{-1})dy \quad \forall g \in L^p(G) \end{aligned}$$

$$\Leftrightarrow f(xy) = f(yx).$$

Lemma 2. Let  $G$  be a compact group and  $f \in Z(L^1(G))$ ,

$\pi \in \hat{G}$  (dual space) then

$$d\pi \quad f * \chi_\pi = \left( \int f(x)\overline{\chi_\pi(\pi(x))}dx \right) \chi_\pi$$

Proof  $\hat{f}(\pi)\chi_\pi(x) = \int f(y)\pi(y^{-1})\pi(x)dy \quad (\hat{f}(\pi) = \pi(f))$

$$= \int f(y)\pi(\bar{y}x)dy, \quad y \rightarrow xy$$

$$= \int_G f(xy)\pi(y^{-1})dy$$

$$= \int_G f(yx) \pi(\bar{y}) dy \quad (y \rightarrow y\bar{x}')$$

$$= \int_G f(y) \pi(x\bar{y}') dy$$

$$= \pi(x) \int_G f(y) \pi(y)^* dy$$

$$= \pi(x) \hat{f}(\pi) \quad \forall x \in G$$

So by Schwarz's Lemma  $\hat{f}(\pi) = c_\pi I$ ,  $c_\pi$  is a scalar

Taking trace on both sides

$$c_\pi d\pi = \text{tr}(\hat{f}(\pi)) = \text{tr}\left(\int_G f(y) \pi(y)^* dy\right)$$

$$= \int_G f(y) \text{tr}(\pi(y)^*) dy$$

$$= \int_G f(y) \overline{x_\pi(y)} dy \quad \dots (A)$$

$$\left( \text{tr}(\pi(y)^*) = \sum_{h=1}^{d\pi} \langle \pi(y)^* \beta_h, \beta_h \rangle = \sum_{h=1}^{d\pi} \langle \beta_h, \pi(y) \beta_h \rangle \right)$$

$$= \sum_{h=1}^{d\pi} \overline{\langle \pi(y) \beta_h, \beta_h \rangle}$$

$$= \overline{\text{tr}(\pi(y))} = \overline{x_\pi(y)}$$

$$\text{tr}(\hat{f}(\pi) \pi(x)) = \int_G f(y) \text{tr}(\pi(\bar{y}x)) dy$$

$$= \int_G f(y) x_\pi(\bar{y}x) dy \quad \dots (B)$$

$$= f * x_\pi(x)$$

$$d\pi f * x_\pi = d\pi \text{tr}(\hat{f}(\pi) \pi)$$

$$= d\pi \text{tr}(c_\pi I \pi) = d\pi \text{tr}(c_\pi \pi)$$

$$= d\pi c_\pi \text{tr} \pi$$

$$= \left( \int_G f(y) \overline{x_\pi(y)} dy \right) x_\pi$$

□.

Proposition  $\{x_\pi : \pi \in \widehat{G}\}$  is an orthonormal basis for  $L^2(G)$ .

Proof.  $\chi_\pi$  are central and  $\lambda_\pi = \sum_{h=1}^{d_\pi} u_h v_h$

By orthogonality relations  $\chi_\pi$  are orthonormal

For  $f \in Z(L^2(G)) \subset L^2(G)$ , by Peter Weyl theorem

$$f = \sum_{\pi \in \widehat{G}} \sum_{i,j=1}^{d_\pi} c_{ij}^\pi \pi_{ij}, \text{ where}$$

$c_{ij}^\pi = d_\pi \int f(x) \overline{\pi_{ij}(x)} dx$ ,  $\pi_{ij}$  denotes the  $i,j$ th coordinate function of  $\pi$ .

$$\hat{f}(\pi) = \int f(x) \pi(x)^* dx$$

$$(\hat{f}(\pi))_{ij} = \int f(x) \overline{\pi_{ji}(x)} dx = \frac{1}{d_\pi} c_{ji}^\pi$$

$$\sum_{i,j} (c_{ij}^\pi) \pi_{ij}(x) = d_\pi \sum_{i,j} \hat{f}(\pi)_{ji} \pi_{ij}(x)$$

$$= d_\pi \operatorname{tr} (\hat{f}(\pi) \pi(x))$$

$$\text{So } f(x) = \sum_{\pi \in \widehat{G}} d_\pi \operatorname{tr} (\hat{f}(\pi) \pi(x))$$

$$= \sum_{\pi \in \widehat{G}} d_\pi f * \chi_\pi(x) \quad (\text{Lemma 2 (B)})$$

$$= \sum_{\pi \in \widehat{G}} \left( \int f \overline{\chi_\pi} \right) \chi_\pi(x) \quad (\text{Lemma 2})$$

$$\text{Thus } f = \sum_{\pi \in \widehat{G}} \langle f, \chi_\pi \rangle \chi_\pi.$$

Ex. White orthonormal basis for  $Z(L^2(SU(2)))$

Deduce ~~white~~ orthonormal basis of  $L^2(T)$ .