Assignment- Free module

Q.1 Let R be a commutative ring. If every submodule of a free R-module is free then show that R is PID.

Q.2 (i) Give an example of a free module ${}_RM$ having a maximal linearly independent subset which is not a basis.

(ii) Let M be a module over a division ring R. Show that a maximal linearly independent subset of M is a basis of M.

Q.3 (i) Give an example of a free module ${}_RM$ having a linearly independent set which is not contained in a basis.

(ii) Let M be a module over a skew field R. Show that a linearly independent set is contained in a basis.

Q.4 (i) Give an example of a free module ${}_RM$ having a subset X which spans M but X does not contain a basis of M.

(ii) Let M be a module over a division ring R. If $X \subset M$ and X spans M then show that X contains a basis of M.

Q.5 Show that $\mathbb{Q}$ cannot be finitely generated $\mathbb{Z}$-module. Show that $\mathbb{Q}$ do not have any minimal generating set as a $\mathbb{Z}$-module.