Chain Conditions

Prop. On a partially ordered set $(X, \leq)$, the following conditions are equivalent:

1. Every increasing sequence $x_1 \leq x_2 \leq x_3 \leq \cdots$ in X is stationary, that is, $\exists\, n \in \mathbb{N}$ such that $x_n = x_{n+i}$ $\forall\, i \geq 1$ (or equivalently the sequence terminates at x_n).
2. Every non empty subset of X has a maximal element.

Proof. $1. \Rightarrow 2.$ Assume 1. holds.

Let S be any non empty subset of X.

Choose an element, say x_1 in S. If x_1 is maximal, then done. Otherwise $\exists\, x_2 \in S$ such that

$$x_1 \leq x_2.$$

If x_2 is maximal in S, then done.

Otherwise repeat the process.

Eventually we get an increasing sequence

$$x_1 \leq x_2 \leq x_3 \leq \cdots$$

By assumption 1., above sequence terminates at x_n, for some $n \in \mathbb{N}$.

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~~Assume the~~

By the construction process, the sequence can terminate only ~~at~~ when some maximal element of S picked, we conclude that x_n is a maximal element of S.

$2. \Rightarrow 1.$ Assume 2. holds.

Let $x_1 \leq x_2 \leq x_3 \leq$ — —

be an increasing sequence in X.

Let $S = \{x_n\}_{n \geq 1}$, that is, S is the set of all elements of X appearing in above sequence.

Then S is a non empty set and hence by assumption, S admits maximal element, say x_{n_0}

$\Rightarrow x_{n_0} = x_{n_0 + i} \quad \forall i,$

that is, the sequence terminates at x_{n_0}

By replace $\leq$ by $\geq$, one can prove the following:

Prop. On a partially ordered set $(X, \geq)$, the following conditions are equivalent:

1. Every decreasing sequence $x_1 \geq x_2 \geq x_3 \geq \cdots$ in X is stationary, that is, $\exists\ n \in \mathbb{N}$ such that $x_n = x_{n+i}$ $\forall\ i \geq 1$ (or equivalently, the sequence terminates at x_n).
2. Every non empty subset of X has a minimal element..

Proof: Do yourself. (Exercise).

Ex: Let M be a left R-module. Let X be the family of all submodules of M.

Then $(X, \subseteq)$ is a partially ordered set.

Thus by first proposition, every ascending chain of submodules of M is stationary $\Leftarrow\Rightarrow$ every non empty

Subset of X has a maximal element.

Definition: Let M be a left R-module. If every ascending chain of submodules of M is stationary, then M is said to satisfy ascending chain condition (a.c.c. for short) on its submodules.

M is said to be Noetherian if M satisfies a.c.c. on its submodules or equivalently if every non empty family of submodules of M has a maximal element.

Definition: Let M be a left R-module. If every descending chain of submodules of M is stationary, then M is said to satisfy descending chain condition (d.c.c. for short) on its submodules.

M is said to be an Artinian module if M satisfies d.c.c. on its submodules or equivalently if every nonempty family of submodules of M has a minimal element.

<u>Def</u>: A ring is said to be Noetherian (resp. Artinian) if it is Noetherian (resp. Artinian) as a module over itself.

<u>Examples</u>:

<u>1.</u> Let G be a finite abelian group. Then G is Noetherian and Artinian $\mathbb{Z}$-module as G has only finitely many submodules (subgroups).

<u>2.</u> The ring $\mathbb{Z}$ is Noetherian (Follows from next proposition).
However, $\mathbb{Z}$ is not Artinian as for any non zero $\alpha \in \mathbb{Z}$, the chain
$$\langle\alpha\rangle \supsetneq \langle\alpha^2\rangle \supsetneq \cdots$$
<u>is</u> non terminating.

<u>3.</u> Let k be a field. Then k has exactly two ideals and so k is Noetherian and Artinian.

4. For any field k, the ring $k[x]$ (polynomial ring in x over k, x - indeterminate) is Noetherian (Follows from Hilbert Basis Theorem that will be done after few lectures).
However, $k[x]$ is not Artinian as

$$\langle x \rangle \supsetneq \langle x^2 \rangle \supsetneq \text{———}$$

is a non-terminating sequence of ideals.

5. Let $R = k[x_1, x_2, \text{- - - -}]$ ~~be~~ be a polynomial ring in infinite indeterminates $x_1, x_2, \text{———}$ over k.

Then R is neither Noetherian nor Artinian as the sequences

$$\langle x_1 \rangle \subsetneq \langle x_1, x_2 \rangle \subsetneq \text{———}$$

$$\langle x_1 \rangle \supsetneq \langle x_1^2 \rangle \supsetneq \text{———}$$

are non-terminating.

Put $F = QF(R)$. Then F is Noetherian and Artinian for being a field.

But F contains a subring R that is ~~not~~ neither Noetherian nor Artinian.

Thus a subring of Noetherian (resp. Artinian) need not be Noetherian (resp. Artinian).

6. Fix a prime p.

Let $Q_p = \left\{ \frac{m}{p^n} \mid m, n \in \mathbb{Z}, n \geq 0 \right\}$.

Then Q_p is a subgroup of additive abelian group $(Q, +)$ containing $\mathbb{Z}$.

Then the quotient group $\frac{Q_p}{\mathbb{Z}}$ is also an additive abelian group.

Now $\frac{Q_p}{\mathbb{Z}}$ can be considered as a $\mathbb{Z}$-module

Under the product:

$$q \cdot \left(\frac{m}{p^n} + \mathbb{Z} \right) = \frac{qm}{p^n} + \mathbb{Z}.$$

(every abelian group can be treated as a $\mathbb{Z}$-module)

Set $G_i = \langle \frac{1}{p^i} + \mathbb{Z} \rangle \quad \forall\, i = 0, 1, 2, \ldots$

Then

$$\langle \bar{0} \rangle = \mathbb{Z} = G_0 \subsetneq G_1 \subsetneq G_2 \subsetneq \cdots \quad \text{(1)}$$

as $\frac{1}{p^i} + \mathbb{Z} = p \cdot \left(\frac{1}{p^{i+1}} + \mathbb{Z} \right) \in G_{i+1}$

and hence $G_i \subseteq G_{i+1}$

Claim : $G_i \neq G_{i+1} \quad \forall\, i$.

If possible, suppose $G_n = G_{n+1}$ for some n. Then

$$\frac{1}{p^{n+1}} + \mathbb{Z} = r \cdot \left(\frac{1}{p^n} + \mathbb{Z} \right) \in G_n$$

$$\Rightarrow \frac{1}{p^{n+1}} = \frac{r}{p^n} + m \quad \text{for some } m \in \mathbb{Z}.$$

$$\Rightarrow 1 = rp + mp^{n+1} = p \cdot (r + mp^n) \quad \times$$

Thus (1) is strictly ascending chain of submodules of $\frac{\mathbb{Q}_p}{\mathbb{Z}}$ and is non termination.

$\Rightarrow \frac{\mathbb{Q}_p}{\mathbb{Z}}$ is not Noetherian $\mathbb{Z}$-module.

Clearly, $\frac{\mathbb{Q}_p}{\mathbb{Z}} = \bigcup_{n=0}^{\infty} G_n$

To show that $\mathbb{Q}_p/\mathbb{Z}$ is Artinian, enough to show that $G_0, G_1, G_2, \ldots, G_n, \ldots$ are the only proper submodules of $\mathbb{Q}_p/\mathbb{Z}$ as then each G_i will have only finitely many submodules.

Let H be a non zero proper submodule of $\mathbb{Q}_p/\mathbb{Z}$.

Let $\frac{m}{p^n} + \mathbb{Z}$ be a non zero element of H.

Then $n > 0$. WLOG, we may assume that $\gcd(m, p^n) = 1$.

$\Rightarrow \exists\ a, b \in \mathbb{Z}$ such that

$$am + bp^n = 1.$$

i.e. $\frac{am}{p^n} + b = \frac{1}{p^n}$.

$\Rightarrow a \cdot \left(\frac{m}{p^n} + \mathbb{Z}\right) = \frac{1}{p^n} + \mathbb{Z} \in H$.

$\Rightarrow G_n \subseteq H.$

Since H is proper submodule of $\mathbb{Q}_p/\mathbb{Z}$, we can find largest $m \in \mathbb{N}$ such that

$G_m \subseteq H.$

Claim: $G_m = H.$

Let $\frac{r}{p^t} + \mathbb{Z} \in H \setminus G_m.$

$\Rightarrow t > m$ as otherwise

$\frac{r}{p^t} + \mathbb{Z} \in G_t \subseteq G_m$ ⨳.

Also, as done earlier, we have

$\frac{1}{p^t} + \mathbb{Z} \in H.$

$\Rightarrow G_t \subseteq H$ ⨳ as this contradicts on the choice of m.

Thus $H = G_m.$

Thus $\mathbb{Q}_p/\mathbb{Z}$ is Artinian $\mathbb{Z}$-module.

Remark: Every proper submodule of $\mathbb{Q}_p/\mathbb{Z}$ is cyclic but $\mathbb{Q}_p/\mathbb{Z}$ is not even finitely generated $\mathbb{Z}$-module.

Prop: A left R-module M is Noetherian $\iff$ every submodule of M is finitely generated.

Proof: Suppose M is Noetherian.

Let N be a submodule of M.

Let $\mathcal{F}$ — family of all finitely generated submodules of N.

Then $\mathcal{F} \neq \phi$ as $(0) \in \mathcal{F}$.

By assumption, $\mathcal{F}$ has a maximal element, say N_0.

If $N_0 \neq N$, choose $x \in N \setminus N_0$.

Put $N_1 = N_0 + Rx$.

Then N_1 is a finitely generated submodule of N as N_0 is so, and hence $N_1 \in \mathcal{F}$.

But this contradicts the maximality of N_0 in $\mathcal{F}$.

Thus $N_0 = N$; and so N is finitely generated.

Conversely suppose that every submodule of M is finitely generated.

Let $M_1 \subseteq M_2 \subseteq$ — —

be an ascending chain of submodules of M.

Set $N = \bigcup_{i=1}^{\infty} M_i$

Then N is a submodule of M and hence N is finitely generated, by assumption.

Let $N = \langle x_1, \ldots, x_n \rangle$

Then $x_i \in M_{k_i} \quad \forall i$.

Put $m = \max.\{k_1, k_2, \ldots, k_n\}$.

Then $x_i \in M_m \quad \forall i = 1, \ldots, n$.

$\Rightarrow M_m = N$. i.e. $M_m = M_{m+i} \quad \forall i \geq 1$.

$\Rightarrow$ M is Noetherian.