ADJOINT OPERATOR

Consider a bounded linear operator $T: X \to Y$ where X and Y are normed spaces.

Let g be a bounded linear functional on Y. Then, g is clearly defined for all $y \in Y$. Setting $y = Tx$ we obtain a functional on X call it f

s.t $$f(x) = g(Tx) \qquad x \in X \quad — (1)$$

* Then f is linear since g and T are linear
* f is bdd $\because$ $|f(x)| = |g(Tx)| \leq \|g\| \|Tx\|$

$$\leq \|g\| \|T\| \|x\|$$

Taking supremum over all $x \in X$ of norm 1 we get

$$\|f\| \leq \|g\| \|T\| \qquad — (2)$$

$\therefore$ $f \in X'$ (space of all bounded linear functionals on X).

$\therefore$ for $g \in Y'$ (1) defines an operator from Y' into X' called adjoint operator T and dentoted by T^{x}.

Thus $$\left.\begin{array}{l} T: X \to Y \\ T^{x} X' \leftarrow Y' \end{array}\right\} - (3)$$

T^{x} is defined on Y' whereas T is defined on X.

Def^n (Adjoint operator T^{x}). Let $T: X \to Y$ be bounded linear operator where X and Y are normed spaces. Then the adjoint operator $T^{x}: Y' \to X'$ of T is defined by

$$f(x) = (T^{x} g)(x) = g(Tx) \qquad (g \in Y') — (4)$$

where X' and Y' are dual spaces of X and Y resp.

Theorem: (Norm of the adjoint operator). The adjoint operator $T^\times$ is linear and bounded and

$$\|T^\times\| = \|T\|.$$

Proof: The operator $T^\times$ is linear since its domain Y' is a vector space and

$$T^\times(\alpha g_1 + \beta g_2)(x) = (\alpha g_1 + \beta g_2)(Tx) = \alpha g_1(Tx) + \beta g_2(Tx)$$

$$= \alpha(T^\times g_1)(x) + \beta(T^\times g_2)(x)$$

Also $\|T^\times g\| = \|f\| \leq \|g\|\|T\|$

Taking supremum over all $g \in Y'$ of norm 1 we get

$$\|T^\times\| \leq \|T\| \quad — (5)$$

Next To show $\|T^\times\| \geq \|T\|$

Using theorem 4.3-3 we have for every non zero $x_0 \in X$ (~~for~~ $Tx_0 \in Y$) $\exists$ $g_0 \in Y'$ such that

$$\|g_0\| = 1 \qquad g_0(Tx_0) = \|Tx_0\|.$$

Here $g_0(Tx_0) = (T^\times g_0)(x_0)$ by def^n of adjoint operator.

Writing $f_0 = T^\times g_0$ we obtain

$$\|Tx_0\| = g_0(Tx_0) = (T^\times g_0)(x_0) \leq \|T^\times g_0\|\|x_0\|$$

$$\leq \|T^\times\|\|g_0\|\|x_0\|$$

(This includes $x_0 = 0$ since $T0 = 0$)

But $\|Tx_0\| \leq \|T^\times\|\|x_0\|$ $[\because \|g_0\| = 1]$

(This includes $x_0 = 0$ since $T0 = 0$).

Also $\|Tx_0\| \leq \|T\|\|x_0\|$

and $c = \|T\|$ is the smallest const s.t

$$\|Tx_0\| \leq c\|x_0\| \quad \forall x_0 \in X$$

$$\therefore \|T^\times\| \geq \|T\| \quad — (6)$$

Combining (5) & (6) we have $\|T^\times\| = \|T\|$

Rk 1 If T is represented by a matrix T_G then the adjoint operator $T^\times$ is represented by the transpose of T_G

Rk 2 Let $S, T \in B(X, Y)$ Then

$$(S+T)^\times = S^\times + T^\times$$
$$(\alpha T)^\times = \alpha T^\times$$

Let X, Y, Z be normed spaces and $T \in B(X, Y)$, $S \in B(Y, Z)$ Then for the adjoint operators of the product ST we have

$$(ST)^\times = T^\times S^\times$$

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[rr, bend right, "ST"'] & Y \arrow[r, "S"] & Z \end{tikzcd} \qquad \begin{tikzcd} X' & Y' \arrow[l, "T^\times"'] & Z' \arrow[l, "S^\times"'] \arrow[ll, bend left, "T^\times S^\times"] \end{tikzcd}$$

Rk 3 For $T \in B(X, Y)$ if T^{-1} exist and belongs to $B(X, Y)$ then $(T^\times)^{-1}$ also exist and $(T^\times)^{-1} \in B(X', Y')$ s.t $(T^\times)^{-1} = (T^{-1})^\times$

RELATION BETWEEN ADJOINT OPERATOR $T^\times$ AND THE HILBERT ADJOINT OPERATOR $T^\times$

we will show existence of such relation in the case of bounded linear operator $T: X \to Y$ if X and Y are Hilbert spaces say $X = H_1$ and $Y = H_2$

$$\begin{tikzcd} H_1 \arrow[r, shift left, "T"] & H_2 \arrow[l, shift left, "T^*"] \\ H_1' \arrow[u, "A_1"] & H_2' \arrow[l, "T^\times"] \arrow[u, "A_2"'] \end{tikzcd}$$

Then $T: H_1 \longrightarrow H_2$ and $T^\times: H_2' \longrightarrow H_1'$ where $T^\times$ is defined by

$$T^\times g = f \text{ and } g(Tx) = f(x)$$
$$f \in H_1', \; g \in H_2'$$

Since f and g are functionals on Hilbert spaces they have Riesz representation say

$$f(x) = \langle x, x_0 \rangle \text{ where } x_0 \in H_1 \text{ and } \|f\| = \|x_0\|$$
$$g(y) = \langle y, y_0 \rangle \text{ where } y_0 \in H_2 \text{ and } \|g\| = \|y_0\|$$

Where x_0, y_0 are uniquely determined by f and g respectively.

This defines operators

$A_1 : H_1' \to H_1$ by $A_1 f = x_0$

$A_2 : H_2' \to H_2$ $A_2 g = y_0$

where A_1 and A_2 are bijective and isometric

$\because \|A_1 f\| = \|x_0\| = \|f\|$

Also, A_1 and A_2 are conjugate linear because if we consider $f_1(x) = \langle x, x_1 \rangle$ and $f_2(x) = \langle x, x_2 \rangle$ then for all x and scalars α, β we have

$$(\alpha f_1 + \beta f_2)(x) = \alpha f_1(x) + \beta f_2(x) = \alpha \langle x, x_1 \rangle + \beta \langle x, x_2 \rangle$$
$$= \langle x, \bar{\alpha} x_1 + \bar{\beta} x_2 \rangle$$

then by defn of A_1

$$A_1(\alpha f_1 + \beta f_2) = \bar{\alpha} x_1 + \bar{\beta} x_2 = \bar{\alpha} A_1 f_1 + \bar{\beta} A_1 f_2$$

Illy we can prove for A_2 i.e.

$$A_2(\alpha f_1 + \beta f_2) = \bar{\alpha} A_2 f_1 + \bar{\beta} A_2 f_2$$

$\therefore$ we can define

$\boxed{T^* = A_1 T^\times A_2^{-1} : H_2 \to H_1}$ defined by $T^* y_0 = x_0$.

Relation between Hilbert adjoint operator and adjoint operator.

$\langle Tx, y_0 \rangle = g(Tx) = f(x) = \langle x, x_0 \rangle = \langle x\ T^* y_0 \rangle$

$\therefore T^*$ is Hilbert adjoint operator of T

and $\|T^*\| = \|T\|$ follows from isometry of A_1, A_2 and the fact that $\|T^\times\| = \|T\|$.

Difference between $T^{\times}$ and T^{*}

(1) $T^{\times}$ is defined on the dual of the space which contains the range of T whereas T^{*} is defined directly on the space containing range of T.

(2) for $T^{\times}$ $(\alpha T)^{\times} = \alpha T^{\times}$

but for T^{*} $(\alpha T)^{*} = \bar{\alpha} T^{*}$

(3) In finite dimensional case $T^{\times}$ is represented by the transpose of the matrix representing T whereas T^{*} is represented by the complex transpose of that matrix.

Reflexive Spaces

We consider a normed space X, its dual space X' and, moreover, the dual space $(X')'$ of X'. This space is denoted by X'' and is called the second dual space of X.

We define a functional g_x on X' by choosing a fixed $x \in X$ and setting

$$(1) \text{ --- } \qquad g_x(f) = f(x), \qquad f \in X' \text{ variable.}$$

Lemma (Norm of g_x): For every fixed x in a normed space X, the functional g_x defined by (1) is a bounded linear functional on X', so that $g_x \in X''$, and has the norm

$$(2) \text{ --- } \qquad \|g_x\| = \|x\|$$

<u>Proof</u>: To show g_x is linear, we have

$$g_x(\alpha f_1 + \beta f_2) = (\alpha f_1 + \beta f_2)(x), \quad f_1, f_2 \in X'$$

for some $x \in X$

$$= \alpha f_1(x) + \beta f_2(x) \qquad \because f_1, f_2 \text{ are linear}$$

$$= \alpha g_x(f_1) + \beta g_x(f_2)$$

Thus g_x is linear. Since f is bounded. Therefore, g_x is also bounded.

The norm of g_x is given by

$$\|g_x\| = \sup_{\substack{f \in X' \\ f \neq 0}} \frac{\|g_x(f)\|}{\|f\|} = \sup_{\substack{f \in X' \\ f \neq 0}} \frac{|f(x)|}{\|f\|} = \|x\|.$$

<u>Remark</u>: To every $x \in X$ there corresponds a unique bounded linear functional $g_x \in X''$ given by (1). This defines a mapping

$$C : X \longrightarrow X''$$

s.t. $C(x) = g_x$ or $C(x)(f) = g_x(f) = f(x)$ — (3)

or. $x \longmapsto g_x^f$.

C is called the canonical mapping of X into X''.

<u>Lemma</u>: The canonical mapping C given by (3) is an isomorphism of the normed space X onto the normed space $R(C)$, the range of C.

<u>Proof</u>: Linearity of C:

Let $x, y \in X$ and $\alpha, \beta \in \mathbb{K}$ then we have

$$\begin{aligned} C(\alpha x + \beta y)(f) &= g_{\alpha x + \beta y}(f) \\ &= f(\alpha x + \beta y) = \alpha f(x) + \beta f(y) \\ &= \alpha g_x(f) + \beta g_y(f) \\ &= (\alpha C(x) + \beta C(y))(f) \end{aligned}$$

$$\Rightarrow C(\alpha x + \beta y) = \alpha C(x) + \beta C(y)$$

Thus C is linear

In particular $g_x - g_y = g_{x-y}$

Hence by previous lemma, we get

$$\|g_x - g_y\| = \|g_{x-y}\| = \|x-y\|$$

This shows that C is isometric and it preserves the norm. Isometry implies the injectivity. Indeed if $x \neq y$ then $g_x \neq g_y$. Hence C is bijective regarded as a mapping onto its range. Therefore C is an isomorphism.

Reflexivity: A normed space X is said to be reflexive if

$$R(C) = X''$$

where $C : X \to X''$ is the canonical mapping

Theorem: If a normed space X is reflexive, it is complete. (hence a Banach space).

Proof. We know that dual space of a normed space is a Banach space. Since X'' is the dual space of X', it is complete. Reflexivity of X means that

$$R(C) = X''$$

where $C : X \to X''$ is the canonical mapping. Hence C is an isomorphism. Thus completeness of X'' implies that X is also complete.

Remark: The converse of above theorem is not true in general i.e a complete may not be reflexe.

For example; ℓ^1 is complete but not reflexive $\because (\ell^1)' = \ell^\infty$.

<u>Theorem</u>: Every finite dimensional normed space X is reflexive

<u>Proof</u>: If $\dim(X) < \infty$, then every ~~bounded~~ linear functional on X is bounded so that $X' = X^*$. Thus algebraic reflexivity of X implies that every finite dim. normed space is reflexive.

<u>Theorem</u>: Every Hilbert space H is reflexive.

<u>Proof</u>: We shall prove surjectivity of the canonical mapping $C: H \longrightarrow H''$ by showing that for every $g \in H''$ there is an $x \in H$ such that $g = Cx$.

We define $A: H' \longrightarrow H$ by

$$Af = z$$

where z is given by Riesz-representation

$$f(x) = \langle x, z \rangle.$$

Since for every $f \in H'$, $f(x) = \langle x, z \rangle$ where z depends on f, is uniquely determined.

Thus A is bijective.

To see A is isometric, we have

$$\|Af\| = \|z\| = \|f\| \quad \forall f \in H'$$

For conjugate linearity of A, we have

$$f_1(x) = \langle x, z_1 \rangle \quad \forall x \in H, \ f_1 \in H'$$

$$f_2(x) = \langle x, z_2 \rangle \quad \forall x \in H, \ f_2 \in H'$$

Where z_1 and z_2 are unique. Then for all $x \in H$ and $\alpha, \beta \in \mathbb{K}$, we have

$$(\alpha f_1 + \beta f_2)(x) = \alpha f_1(x) + \beta f_2(x)$$
$$= \alpha \langle x, z_1 \rangle + \beta \langle x, z_2 \rangle$$
$$= \langle x, \bar{\alpha} z_1 \rangle + \langle x, \bar{\beta} z_2 \rangle$$
$$= \langle x, \bar{\alpha} z_1 + \bar{\beta} z_2 \rangle$$

From the definition of A, we obtain ~~a~~ the conjugate linearity of A i.e

$$A(\alpha f_1 + \beta f_2) = \bar{\alpha} A f_1 + \bar{\beta} A f_2$$

We know that dual space of a normed space is complete. Hence H' is complete. Space H' is a Hilbert space with inner product defined by

$$\langle f_1, f_2 \rangle_1 = \langle A f_2, A f_1 \rangle$$

clearly $\langle \cdot, \cdot \rangle_1$ satisfies all the postulates of an inner product.

Let $g \in H''$ be arbitrary. Let its Riesz representation be

$$g(f) = \langle f, f_0 \rangle_1 = \langle A f_0, A f \rangle$$

Writting $A f_0 = x$, we thus have

$$g(f) = \langle x, Af \rangle = \langle x \cdot z \rangle \quad \because Af = z$$
$$= f(x)$$

i.e $g = Cx$ by definition of C.

Since $g \in H''$ was arbitrary, C is surjective, so that H is reflexive.

Lemma: Let Y be a proper closed subspace of a normed space X. Let $x_0 \in X - Y$ be arbitrary and

$$\delta = \inf \|\tilde{y} - x_0\|$$

the distance from x_0 to Y. Then there exists an $\tilde{f} \in X'$ such that

$$\|\tilde{f}\| = 1, \quad \tilde{f}(y) = 0 \text{ for all } y \in Y, \quad \tilde{f}(x_0) = \delta.$$

Proof. Let Z be a subspace of X spanned by Y and x_0. i.e. $Z = \text{span}(Y, \{x_0\})$

Let $z \in Z$ then z has a unique representation

$$z = y + \alpha x_0 \quad , \quad y \in Y$$

Define a linear bounded functional f on Z by

$$f(z) = f(y + \alpha x_0) = \alpha \delta, \quad y \in Y$$

Since Y is closed, $\delta > 0$, so that $f \neq 0$.

Now $\alpha = 0$ gives $f(y) = 0$ for all $y \in Y$.

For $\alpha = 1$ and $y = 0$ we have

$$f(x_0) = \delta.$$

Now we show that f is bounded. $\alpha = 0$ gives $f(z) = 0$. Let $\alpha \neq 0$ then $-\left(\frac{1}{\alpha} y\right) \in Y$.

Thus we obtain

$$|f(z)| = |\alpha| \delta = |\alpha| \inf_{\tilde{y} \in Y} \|\tilde{y} - x_0\|$$

$$\leq |\alpha| \left\| -\frac{1}{\alpha} y - x_0 \right\|$$

$$= \|y + \alpha x_0\| = \|z\|$$

Hence f is bounded and $\|f\| \leq 1$.

Now we show that $\|f\| \geq 1$

By definition of an infimum, Y contains a sequence (y_n) such that

$$\|y_n - x_0\| \longrightarrow \delta$$

Let $z_n = y_n - x_0$. Then we have

$$f(z_n) = f(y_n - x_0) = -1 . \delta = -\delta$$

Also we have

$$\|f\| = \sup_{\substack{z \in Z \\ z \neq 0}} \frac{|f(z)|}{\|z\|} \geq \frac{|f(z_n)|}{\|z_n\|} = \frac{\delta}{\|z_n\|} \rightarrow \frac{\delta}{\delta}$$

$$= 1 \text{ as } n \to \infty$$

Hence $\|f\| \geq 1$. So that $\|f\| = 1$. By the Hahn-Banach theorem for normed space, we can extend f to X without increasing the norm.

<u>Theorem</u>: If the dual space X' of a normed space X is separable, then X itself is separable space

Proof. We assume that X' is separable. We know that every subspace of a normed space is separable. Then the unit sphere $U' = \{f : \|f\| = 1\} \subset X'$ is also separable. Thus U' contains a countable dense subset, say, (f_n). Since $f_n \in U'$, we have

$$\|f_n\| = \sup_{\|x\|=1} |f_n(x)| = 1$$

By definition of a supremum, we can find points $x_n \in X$ of norm 1 such that

$$|f_n(x_n)| \geq \frac{1}{2}.$$

Let Y be the closure of span (x_n). Then Y is separable because Y has a countable dense subset, namely, the set of all linear combinations of the x_n's with coefficients whose real and imaginary parts are rational.

We show that $Y = X$. Suppose $Y \neq X$. Then since Y is closed, by previous lemma, there exists an $\hat{f} \in X'$ with $\|\hat{f}\| = 1$ and $\hat{f}(y) = 0$ for all $y \in Y$. Since ~~x_n~~ $x_n \in Y$, we have $\hat{f}(x_n) = 0$ and for all n,

$$\begin{aligned}\frac{1}{2} \leq |f_n(x_n)| &= |f_n(x_n) - \hat{f}(x_n)| \\ &= |(f_n - \hat{f})(x_n)| \\ &\leq \|f_n - \hat{f}\| \, \|x_n\|,\end{aligned}$$

where $\|x_n\| = 1$. Hence $\|f_n - \hat{f}\| \geq \frac{1}{2}$ but this contradicts the assumption that (f_n) is dense in U' because $\hat{f}$ is itself in U'; in fact $\|\hat{f}\| = 1$.

Hence $Y = X$, therefore X is separable.