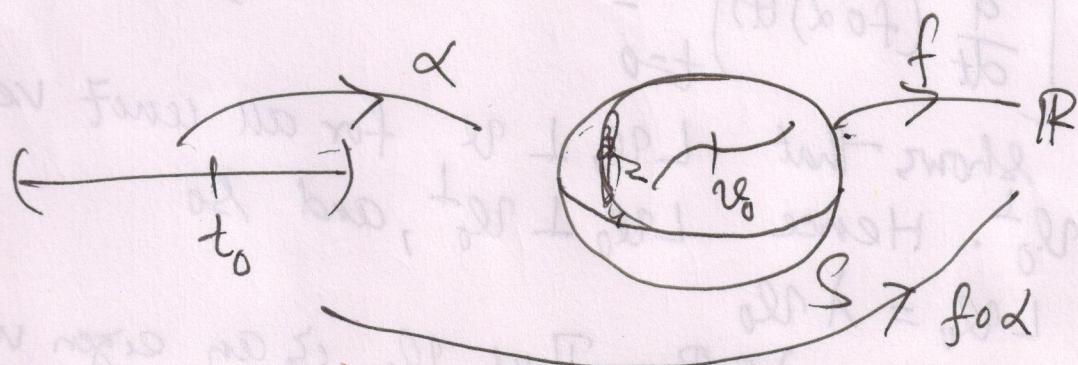


Lectures 1 and 2

(1)

Let S be an oriented n -surface in \mathbb{R}^n and $p \in S$. If $\vec{v} \in S_p$ and $\|\vec{v}\|=1$, we have ^{already defined} ~~seen~~ the normal curvature $k(\vec{v}) = L_p(\vec{v}) \cdot \vec{v}$, and also [^] have seen how the normal curvature $k(\vec{v})$ of S at p in the direction \vec{v} is actually the curvature of a (suitably oriented) plane curve. Since S_p is finite dimensional, the linear transformation $L_p: S_p \rightarrow S_p$ is continuous. Thus $k(\vec{v}) = L_p(\vec{v}) \cdot \vec{v}$ defines a continuous ~~functi~~ real valued function on the ~~compact~~ unit sphere of S_p which is compact. Hence k attains its maximum and minimum. The following lemma shows that these extrema are actually eigenvalues of the Weingarten map L_p .

Lemma 1: Let V be a finite dimensional vector space with dot product and $L: V \rightarrow V$ be a self-adjoint linear transformation on V . Let $S = \{v \in V : v \cdot v = 1\}$ and define $f: S \rightarrow \mathbb{R}$ by $f(v) = L(v) \cdot v$. Suppose f is stationary at $v_0 \in S$ (that is, for any parametrized curve $\alpha: I \rightarrow S$ with $\alpha(t_0) = v_0$, for some $t_0 \in I$,



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we have $(f \circ \alpha)'(t_0) = 0$. Then $\underline{f(v_0)}$.

$Lv_0 = f(v_0)v_0$ (i.e., v_0 is an eigen vector of L with eigenvalue $f(v_0)$).

Proof. If dimension of $V = 1$, then $L(v) = cv$,

and so $f(v_0) = L(v_0) \cdot v_0 = cv_0 \cdot v_0 = c$.

Thus $L(v_0) = f(v_0)v_0$.

So assume $\dim V \geq 2$. Let $v \in S, v \perp v_0$.

Then $\alpha(t) = (\cos t)v_0 + (\sin t)v$, $t \in \mathbb{R}$

defines a parametrized curve in S through $\alpha(0) = v_0$. Then

$$f(\alpha(t)) = L(\alpha(t)) \cdot d\alpha(t)$$

$$= [(\cos t)Lv_0 + (\sin t)Lv] \cdot [(\cos t)v_0 + (\sin t)v]$$

$$= (\cos^2 t)Lv_0 \cdot v_0 + (\cos t \sin t)Lv \cdot v_0$$

$$+ (\cos t \sin t)Lv_0 \cdot v + (\sin^2 t)Lv \cdot v$$

$$= (\cos^2 t)Lv_0 \cdot v_0 + (2 \cos t \sin t)Lv \cdot v_0$$

$$+ (\sin^2 t)Lv \cdot v \quad (\because L \text{ is self-adjoint})$$

Therefore

$$0 = \left[\frac{d}{dt} (f \circ \alpha)(t) \right]_{t=0} = 2Lv \cdot v_0,$$

which shows that $Lv_0 \perp v$ for all unit vectors $v \in v_0^\perp$. Hence $Lv_0 \perp v_0^\perp$, and so

$$Lv_0 = \lambda v_0$$

for some $\lambda \in \mathbb{R}$. Thus v_0 is an eigen vector

of L . The eigen value λ is given by (3)

$$\lambda = \lambda v_0 \cdot v_0 = L(v_0) \cdot v_0 = f(v_0).$$

Therefore, $Lv_0 = f(v_0)v_0$. \square

Remark: converse of the above lemma is also true. That is, if $v_0 \in S$ is an eigen vector of L , then f is stationary at v_0 .

Proof: suppose $Lv_0 = \lambda v_0$, and $\alpha: I \rightarrow S$,

$$\alpha(t_0) = v_0. \text{ Then } \frac{d}{dt} (f \circ \alpha)(t) = \frac{d}{dt} [L(\alpha(t)) \cdot \alpha'(t)]$$

$$= \frac{d}{dt} L(\alpha(t)) \cdot \alpha'(t) + L(\alpha(t)) \cdot \frac{d}{dt} \alpha(t). \quad (1)$$

$$\text{Now } \frac{d}{dt} L(\alpha(t)) = \lim_{h \rightarrow 0} \frac{L(\alpha(t+h)) - L(\alpha(t))}{h}$$

$$= \lim_{h \rightarrow 0} L \left(\frac{\alpha(t+h) - \alpha(t)}{h} \right)$$

$$= L \left(\lim_{h \rightarrow 0} \frac{\alpha(t+h) - \alpha(t)}{h} \right) \quad (\because L \text{ is cont.})$$

$$= L \left(\frac{d\alpha}{dt} \right).$$

Substituting in (1),

$$(f \circ \alpha)'(t_0) = L \left(\frac{d\alpha}{dt} \right) \cdot v_0 + L(v_0) \cdot \frac{d\alpha}{dt}(t_0)$$

$$= 2 \frac{d\alpha}{dt}(t_0) \cdot Lv_0 \quad (\because L \text{ is self-adjoint})$$

$$= 2\lambda \frac{d\alpha}{dt}(t_0) \cdot \alpha(t_0)$$

$$= 0$$

$$(\because 0 = \frac{d}{dt} \| \alpha(t) \|^2 = 2\alpha(t) \cdot \frac{d\alpha(t)}{dt}, \forall t). \quad (4)$$

This shows that f is stationary at α_0 . \square

Applying Lemma 1 and the remark that follows it, to the case $V = S_p$, $L = L_p$ and $f = k$, the normal curvature function, we see that a unit vector $\vec{v} \in S_p$ is an eigenvector of L_p (with the corresponding eigenvalue $k(\vec{v})$) iff k is stationary at \vec{v} . It may also be noted that if $\vec{v} \in S_p$, $\|\vec{v}\|=1$ is an extreme point of k on the unit sphere of S_p , then by definition, k is stationary at \vec{v} .

The following result is a well known ^{result} in linear algebra.

Thm 2: Let $V \neq \{0\}$ be a finite dimensional vector space with dot product, and let $L: V \rightarrow V$ be a self-adjoint linear transformation on V . Then there exists an ON basis for V consisting of eigenvectors of L .

Proof (By induction on the dimension n of V)

For $n=1$, L is a constant multiple of identity, i.e. $Lv = \lambda v$, $\forall v \in V$.

choose $0 \neq v_0 \in V$. Then $\left\{ \frac{v_0}{\|v_0\|} \right\}$ is an ON basis

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of V consisting of eigen vector of L . vector spaces of

Suppose the result holds for all ~~\mathbb{F} with~~ dimension ~~$\leq k$~~ . Let V be of dimension $k+1$. Since $L_{\mathbb{F}, V}$ is continuous on the compact set S , $S = \{\mathbf{v} \in V : \|\mathbf{v}\|=1\}$, it attains its maximum say, at $\mathbf{v}_1 \in S$. Then by Lemma 1, \mathbf{v}_1 is an eigen vector of L , say $L\mathbf{v}_1 = \lambda_1 \mathbf{v}_1$.

Consider $W = \mathbf{v}_1^\perp$. Then W is a k dimensional vector space. Also, for any $w \in W$,

$$L(w) \cdot \mathbf{v}_1 = w \cdot L\mathbf{v}_1 = w \cdot \lambda_1 \mathbf{v}_1 = 0.$$

Thus $L|_W$ maps W into W , is linear, and self-adjoint. Therefore, by induction hypothesis, there exists an ON basis

$\{\mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_{k+1}\}$ of W consisting of eigen vectors of $L|_W$ and hence of L . Thus $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{k+1}\}$ is an ON basis for V consisting of eigen vectors of L .

This proves the theorem. \square

Note that if dimension $V=n$, then the self-adjoint linear transformation $L: V \rightarrow V$ has at most n eigen value, because each eigenvalue is a root of the characteristic polynomial $\det(L - \lambda I)$, which is a polynomial on λ of

degree n . So counting multiplicities, there are exactly n -eigenvalues of L (and all of them are real). (6)

Let S be an oriented n -surface in \mathbb{R}^{n+1} , $p \in S$. Then the eigen values $k_1(p), \dots, k_n(p)$ of $L_p : S_p \rightarrow S_p$ are called principal curvatures of S at p and the unit eigen vectors of L_p are called principal curvature directions. If we order the principal curvatures so that

$$k_1(p) \leq k_2(p) \leq \dots \leq k_n(p),$$

then the above theorem shows that

$$k_n(p) = \max \{ k(\vec{v}) : \vec{v} \in S_p, \|\vec{v}\|=1 \}$$

$$k_{n-1}(p) = \max \{ k(\vec{v}) : \vec{v} \in S_p, \|\vec{v}\|=1, \vec{v} \perp \vec{v}_n \},$$

where \vec{v}_n is a principal curvature direction corresponding to $k_n(p)$,

$$k_{n-2}(p) = \max \{ k(\vec{v}) : \vec{v} \in S_p, \|\vec{v}\|=1, \vec{v} \perp \{\vec{v}_n, \vec{v}_{n-1}\} \},$$

and so on. Furthermore, the Lemma 1 and the ~~Remark~~ Remark that follows it shows that all the principal curvatures $k_i(p)$ are stationary values of normal curvature, and

$$k_1(p) = \min \{ k(\vec{v}) : \vec{v} \in S_p, \|\vec{v}\|=1 \}.$$

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Example 1 consider the n -sphere S :

$$x_1^2 + x_2^2 + \dots + x_{n+1}^2 = r^2, r > 0, \text{ oriented by } \vec{N}(p)$$

$$= (p, \frac{p}{r}), p = (0, 0, \dots, 0, r) \text{ CS.}$$

Then we have seen that

$$L_p(\vec{v}) = -\frac{1}{r} \vec{v}$$

and the normal curvature $k(\vec{v}) = -\frac{1}{r}$, $\forall \vec{v} \in S_p, \|\vec{v}\|=1$.

Thus in this case, for any PCS, the principal curvatures are

$$k_1(p) = k_2(p) = -\frac{1}{r}$$

and for principal curvature directions we can choose any two ~~not~~ ON vectors \vec{v}_1 and \vec{v}_2

in S_p .

Example 2: consider the hyperboloid S :

$$-x_1^2 + x_2^2 + x_3^2 = -1$$

in \mathbb{R}^3 , oriented by $\vec{N}(p) = (p, \frac{-x_1}{\|p\|}, \frac{x_2}{\|p\|}, \frac{x_3}{\|p\|})$,

$p = (x_1, x_2, x_3) \text{ CS. Then we have seen earlier}$

in this chapter, for $\Rightarrow p = (0, 0, 1) \text{ CS}$,

$$S_p = \{(p, x_1, x_2, 0) : x_1, x_2 \in \mathbb{R}\}, \text{ and for } \|\vec{v}\|=1,$$

$$k(\vec{v}) = x_1^2 - x_2^2.$$

So what are the max. & min. of k on the

unit sphere of S_p . To know this we

$$\text{minimize } g(x_1, x_2) = x_1^2 - x_2^2$$

$$\text{subject to } f(x_1, x_2) = x_1^2 + x_2^2 = 1.$$

So the Lagrange multiplier method of calculus tells us if (u_1, u_2) is an extreme points then

$$\nabla g(u_1, u_2) = \lambda \nabla f(u_1, u_2)$$

for some Lagrange multiplier λ . Thus

$$(2u_1, -2u_2) = \lambda (2u_1, 2u_2)$$

$$\Rightarrow (1-\lambda)u_1 = 0$$

$$(1+\lambda)u_2 = 0$$

Since $u_1^2 + u_2^2 = 1$, $(u_1, u_2) \neq (0, 0)$. Thus either

$$\lambda = 1 \text{ or } \lambda = -1. \text{ Hence}$$

$$(\pm 1, 0) \text{ and } (0, \pm 1)$$

are the extreme points.

Therefore, $k(\vec{v})$ attains its maximum value $k_g(p) = 1$ on $\vec{v}_2 = (p, \pm 1, 0, 0)$ and minimum value $k_l(p) = -1$ on $\vec{v}_1 = (p, 0, \pm 1, 0)$.

Theorem 3: let S be an oriented n -surface in \mathbb{R}^{n+1} ,

let $\{k_1(p), \dots, k_n(p)\}$ be the principal curvatures of S at p with corresponding principal curvature directions $\{\vec{v}_1, \dots, \vec{v}_n\}$.

Then the normal curvature $k(\vec{v})$ in the direction $\vec{v} \in S_p$ ($\|\vec{v}\| = 1$) is given by

$$k(\vec{v}) = \sum_{i=1}^n k_i(p) (\vec{v} \cdot \vec{v}_i)^2 = \sum_{i=1}^n k_i(p) \cos^2 \theta_i,$$

where $\theta_i = \cos^{-1}(\vec{v} \cdot \vec{v}_i)$ is the angle between \vec{v} and \vec{v}_i . (9)

Proof: v_i^i 's are ON

$$\Rightarrow \vec{v} = \sum_{i=1}^n a_i \vec{v}_i = \sum_{i=1}^n (\vec{v} \cdot \vec{v}_i) \vec{v}_i = \sum_{i=1}^n (\cos \theta_i) \vec{v}_i.$$

$$\Rightarrow k(\vec{v}) = L_p(\vec{v}) \cdot \vec{v} = \sum_{i=1}^n (\cos \theta_i) L_p(\vec{v}_i) \cdot \vec{v}$$

($\because L_p$ is linear)

$$= \sum_{i=1}^n (\cos \theta_i) k_i(p) \vec{v}_i \cdot \vec{v} = \sum_{i=1}^n k_i(p) (\cos^2 \theta_i) \quad \square$$

The numbers $\cos \theta_i = \vec{v} \cdot \vec{v}_i$ such that
 $\vec{v} = \sum_{i=1}^n (\cos \theta_i) \vec{v}_i$ are called the direction cosines
of \vec{v} w.r.t the ON basis $\{\vec{v}_1, \dots, \vec{v}_n\}$.