Tutorial for 23/03/20 (Monday)

Theory of Non-Commutative Rings

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2.1- Is any subring of a left semisimple ring left semisimple? Can any ring be embedded as a subring of a left semisimple ring?

Soln- Answer to first question is 'no', because $\mathbb{Q}$ is a semisimple ring being a field and $\mathbb{Z}$ is a subring of $\mathbb{Q}$ but $\mathbb{Z}$ is not semisimple as $\mathbb{Z}$ is not Artinian.

The answer to second question is also 'no': as $A = R_1 \times R_2 \times \cdots$ (where R_i are non-zero rings) cannot be embedded as a subring of a (left) semisimple ring. Because if A is a subring of a ring R, then R will have non-zero idempotents $e_1, e_2, \ldots$ with $e_i e_j = 0$ for $(i \neq j)$. But then

$$R \supseteq Re_1 \oplus Re_2 \oplus \cdots$$

and this implies R is not left noetherian, therefore not left semisimple.

2.2 Let $\{F_i : i \in I\}$ be a family of fields. Show that the direct product $R = \prod F_i$ is a semisimple ring iff I is finite.

Sol^n: First suppose R is semisimple. By previous exercise, if I is infinite, we can show that it is not noetherian and thus not semisimple. Therefore I must be finite.

Conversely, let I is finite and consider an ideal $A \subseteq R$. By exercise 1.8, $A = \bigoplus_{i \in J} F_i$ for a subset $J \subseteq I$. Clearly A is a direct summand of ${}_R R$, so ${}_R R$ is a semisimple module. ☒

2.3. A Determine which of the following are ~~semisimple~~ semisimple $\mathbb{Z}$-modules (where $\mathbb{Z}_n$ denotes $\mathbb{Z}/n\mathbb{Z}$)

$\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Q}/\mathbb{Z}$, $\mathbb{Z}_4$, $\mathbb{Z}_6$, $\mathbb{Z}_{12}$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots$ and

$\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 \oplus \cdots$, $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \cdots$

Sol^n: As simple $\mathbb{Z}$-modules are isomorphic to $\mathbb{Z}/p\mathbb{Z}$ for some prime p, therefore, as $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5$ are not torsion abelian groups, so they are not semisimple $\mathbb{Z}$-modules.

In $\mathbb{Z}_4$, $2\mathbb{Z}/4\mathbb{Z}$ is not a direct summand.

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Since $\mathbb{Z}/12\mathbb{Z}$ and $\mathbb{Q}/\mathbb{Z}$ both contain a copy of $\mathbb{Z}/4\mathbb{Z}$, they are also not semisimple.

$\mathbb{Z}_6 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3$ and $\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 \oplus \cdots$ are by definition, semisimple.

Finally as vector spaces are semisimple, so is $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots$ over $\mathbb{Z}_2$ and hence over $\mathbb{Z}$ too.

2.3.B : What are the semisimple $\mathbb{Z}$-modules?

Solⁿ: $\mathbb{Z}$-modules are just abelian groups and simple $\mathbb{Z}$-modules are abelian simple groups, namely, $\mathbb{Z}/p\mathbb{Z}$ where p is any prime.

$\therefore$ semisimple $\mathbb{Z}$-modules are just direct sums of copies of such $\mathbb{Z}/p\mathbb{Z}$'s and are, in particular, torsion abelian groups.

2.4 : Let R be the (commutative) ring of all real-valued continuous functions on $[0,1]$. Is R semisimple ring?

Solⁿ: Consider $I_n = \{ f \in R : f([1/n, 1]) = 0 \}$ where $n \geq 1$.

Then I_n are ideals. (Check)

Obviously, $I_1 \supsetneq I_2 \supsetneq I_3 \cdots\cdots$, so R is not artinian thus not semisimple.

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