Theory of Non-Commutative Rings

MATH 14 - 401(B)

MA/ MSC/Math/ Sem - IV

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# The Jacobson Radical

**Definition:-** Let $R$ be a ring. The Jacobson radical of a ring $R$ is defined to be the intersection of all the maximal left ideals of $R$ and denoted by $\operatorname{rad} R$. When $R=0$, Jacobson radical is defined to be $0$.

Note: that (1) when $R \neq 0$, by Zorn's lemma there always exists maximal left ideals. When $R=0$, there is no maximal left ideals.

(2) We will prove that $\operatorname{rad} R$ as $\bigcap$ max left ideals and $\bigcap$ max right ideals coincide, so the distinction between left radical (as it would have been called by above definition) and right radical is unnecessary.

**Lemma 4.1:-** For $y \in R$, the following statements are equivalent

(1) $y \in \operatorname{rad} R$

(2) $1-xy$ is left-invertible for any $x \in R$

(3) $yM=0$ for any simple left $R$-module $M$.

**Pf:-** (1) $\Rightarrow$ (2) Let $y \in \operatorname{rad} R$

Suppose $1-xy$ is not left invertible for some $x \in R$

then $r.(1-xy) \neq 1$ for any $r \in R$

$\Rightarrow R(1-xy) \neq R$

i.e. $R(1-xy) \subset R$

Then $R(1-xy)$ is contained in some maximal ideal $\mathfrak{m}$ (say)

But $1-xy \in \mathfrak{m}$ as $1.(1-xy) \in \mathfrak{m}$, $1 \in R$

Also $y \in \operatorname{rad} R = \bigcap_{\text{all}}$ max. ideals $\subseteq \mathfrak{m}$

$\Rightarrow 1 = (1-xy)+xy \in \mathfrak{m}$ which is a contradiction.

Thus $1-xy$ is left invertible for $x \in R$.

(2) $\Rightarrow$ (3) Let $M$ be any simple left $R$-module.

Assume that $yM \neq 0$ for some $m \in M$

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$R.yM \subseteq M$

As $M$ is simple and $\langle ym \rangle$ over $R$ is not zero

$R.ym = M$

In particular, $x.ym = m$ for some $x \in R$

$\Rightarrow (1-xy)m = 0$

Since $(1-xy)$ is left invertible by (2),

i.e. $\exists\, v \in R$ s.t. $v(1-xy)m = 0$

$\Rightarrow 1.m = 0$

$\Rightarrow m = 0$ implying $ym = 0$, contradiction.

<u>(3) $\Rightarrow$ (1)</u> We know that for any maximal ideal $\mathfrak{m}$, $R/\mathfrak{m}$ is simple left $R$-module.

(as any $R$-submodule (ideal) of $R/\mathfrak{m}$ is of the form $J/\mathfrak{m}$ where $J$ contains $\mathfrak{m}$, not possible as $\mathfrak{m}$ is maximal.)

By (3), $y.R/\mathfrak{m} = 0 \Rightarrow yR \subseteq \mathfrak{m}$

$\Rightarrow y \in \mathfrak{m}$ for any max ideal $\mathfrak{m}$

$\Rightarrow y \in \bigcap_{\text{all}}$ max. ideal of $R$

i.e. $y \in \operatorname{rad} R$. ∎

<u>Definition</u>:- Let $M$ be any left $R$-module, then

$\operatorname{ann} M = \{ r \in R \mid rM = 0 \}$

clearly $\operatorname{ann} M$ is an ideal of $R$.

Two sided ideal as $r \in \operatorname{ann} M$, $arM = a0 = 0$

$\Rightarrow ar \in \operatorname{ann} M$

and $raM \subseteq rM = 0 \Rightarrow ra \in \operatorname{ann} M$

Recall that the left $R$-module $M$ is cyclic if there is an element $x \in M$ s.t. $M = Rx$ $(= \langle x \rangle)$

Define an $R$-module homomorphism

$\pi : R \longrightarrow M$ by

$\pi(r) = rx$ which will be surjective by assumption $M = \langle x \rangle$

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By First isomorphism theorem $\frac{R}{\ker \pi} \cong M$

$\ker \pi$ is an ideal (say) $Q$, then $M$ can be taken as $R/Q$

Then $\operatorname{ann} M = \{ r \in R \mid r . \frac{R}{Q} = 0 \}$

$= \{ r \in R \mid rR \subseteq Q \}$

$\Rightarrow \operatorname{ann} M \subseteq Q$ as $\left( \begin{array}{l} r \in \operatorname{ann} M \\ \Rightarrow rR \subseteq Q. \\ \Rightarrow r1 = r \in Q. \end{array} \right)$

which is in this case the largest ideal of $R$ contained in $Q$ as any ideal contained in $Q$ is contained in $\operatorname{ann} M$ as above.

This is sometimes called "core" of the left ideal $Q$

If $R$ is commutative, then

$Q \subseteq \operatorname{ann} M$ as for $r \in Q$, $rR \subseteq Q$ ($\because$ $R$ is commutative, $Q$ is right ideal also.)

Hence in this case

$\operatorname{ann} M = Q$

i.e. $\operatorname{ann}(R/Q) = Q$.

<u>Corollary 4.2</u>:- $\operatorname{rad} R = \bigcap \operatorname{ann} M$, where $M$ ranges over all the simple left $R$-modules.

In Particular, $\operatorname{rad} R$ is an ideal of $R$.

Pf:- clearly $y \in \operatorname{rad} R$ iff $yM = 0$ for any simple left $R$-module $M$

i.e. $y \in \operatorname{rad} R$ iff $y \in \operatorname{ann} M$ for any simple left $R$-mod $M$

i.e. $y \in \operatorname{rad} R$ iff $y \in \bigcap \operatorname{ann} M$, $M$ is simple left $R$-mod

i.e. $\operatorname{rad} R = \bigcap \operatorname{ann} M$

As $\operatorname{ann} M$ is an ideal and intersection of ideals is an ideal thus $\operatorname{rad} R$ is an ideal (two sided). ∎

<u>Lemma 4.3</u> (An refinement of the previous lemma)

For $y \in R$, the following are equivalent

(1) $y \in \operatorname{rad} R$

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(2') $1-xyz \in U(R)$, the group of units of $R$, for any $x, z \in R$
(this additional condition strengthens condition (2) in above lemma)

Pf:- By taking $z=1$,
(2') = (2)
It is sufficient to prove that equivalence ((1) ⇒ (2') ⇒ (2) ⇒ (3) ⇒ (1))
that (1) ⇒ (2')
Let $y \in \operatorname{rad} R$ and let $x, z \in R$
Since $\operatorname{rad} R$ is an ideal (two sided), $yz \in \operatorname{rad} R$
Then by (1) of Previous Lemma,
$1-xyz$ is left invertible
i.e. $\exists\, u \in R$ s.t. $u(1-xyz)=1$ i.e. $u$ is rt. invertible.
Again as $\operatorname{rad} R$ is an ideal, $xyz \in \operatorname{rad} R$
then for $(-u) \in R$, $1-(-u)xyz = 1+u(xyz)$ is left invertible
(by (2) of lemma 1)
From above $u - u(xyz) = 1$
i.e. $1+u(xyz) = u$
Hence $u$ is left invertible as well
$\Rightarrow u \in U(R)$
$\Rightarrow u^{-1} \in U(R)$
$\Rightarrow 1-xyz \in U(R)$ as $U(R)$ is a group under multiplication
and $u(1-xyz)=1$.

∎

Some consequences of the above results.

<u>Corollary</u>:- (A) $\operatorname{rad} R$ is the largest left ideal (hence the largest ideal) $Q \subseteq R$ s.t.
$1+Q \subseteq U(R)$

(B) The left radicals of $R$ agrees with the right radicals.

Pf:- (A) Since $a \in \operatorname{rad} R\ (= Q)$
$\Rightarrow 1-a \in U(R)$ taking $x, z$ as unity of $R$
i.e. $1+Q \subseteq U(R)$

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It is sufficient to prove that equivalence ((1) ⇒ (2') ⇒ (2) ⇒ (3) ⇒ (1))

that (1) ⇒ (2')

Let $y \in \text{rad}\, R$ and let $x, z \in R$

Since rad$R$ is an ideal (two sided), $yz \in \text{rad}\, R$

Then by (1) of Previous Lemma,

$1-xyz$ is left invertible

i.e. $\exists\, u \in R$ s.t. $u(1-xyz)=1$ i.e. $u$ is rt. invertible.

Again as rad$R$ is an ideal, $xyz \in \text{rad}\, R$

then for $(-u) \in R$, $1-(-u)xyz = 1+u(xyz)$ is left invertible
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From above $u - u(xyz) = 1$

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Hence $u$ is left invertible as well

$\Rightarrow u \in U(R)$

$\Rightarrow u^{-1} \in U(R)$

$\Rightarrow 1-xyz \in U(R)$ as $U(R)$ is a group under multiplication and $u(1-xyz)=1$.

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Pf:- (A) Since $a \in \text{rad}\, R\ (= Q)$

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rad R is a left ideal satisfying above (Use (1), (2), (3) of above lemma)

rad R is largest in this sense as if $b \supset a$ then $1-b$ may not be in $U(R)$ thus the largest ideal satisfying the above.

Since the condition $y \in \text{rad } R$
$\Leftrightarrow 1 - xyz \in U(R)$ for any $x, z \in R$

is left-right symmetric, (A) gives a left-right symmetric characterization of rad R, hence B follows. ∎

Another property of Jacobson radical.

<u>Proposition 4.6</u> Let $a$ be any ideal of $R$ lying in rad R
Then $\text{rad}(R/a) = (\text{rad} R)/a$.

<u>Pf</u>:- Since rad R has left-right symmetric characterization both left and rt. radicals coincide.
rad R = $\cap$ all maximal ideals of R

Then $\text{rad}(R/a) = \cap$ all maximal ideals of $R/a$
$= (\cap$ all maximal ideals of $R)/a$
$= (\text{rad } R)/a$

for any ideal of $a$ of $R$ contained in rad R
(so that the quotient $(\text{rad} R)/a$ is meaningful) ∎

<u>Definition</u>:- A ring R is called Jacobson (or J)-semisimple if rad R = 0 (these J-semisimple rings generalize the semisimple rings)

<u>Note</u>:- These are also called semiprimitive rings. We use this term after we introduce primitive rings

<u>Example</u>:- $\frac{R}{\text{rad} R}$ is J-semisimple as $\text{rad}\left(\frac{R}{\text{rad} R}\right) = \frac{\text{rad } R}{\text{rad} R} = 0$

The rings R and $\frac{R}{\text{rad} R}$ share some common properties ——

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<u>Proposition 4.8</u>:- $R$ and $R/\text{rad}R$ have the same simple left modules. An element $x \in R$ is left invertible (resply. invertible) in $R$ iff $\bar{x} \in \bar{R}$ is left invertible (resply. invertible) in $\bar{R}$ $(= R/\text{rad}R)$.

<u>Pf</u>:- We know that if $M$ is a $R$-module, then $M$ is also left $(R/\text{ann}\,M)$-module by module action

$$(r + \text{ann}\,M)\,m = rm \qquad \text{since } am = 0 \ \forall a \in \text{ann}\,M \text{ and } \forall m \in M.$$

This action is well defined and one can easily check that it makes $M$ into an $(R/\text{ann}\,M)$-module (left)

Thus $R$ and $R/\text{rad}R$ has same left modules hence has same simple left modules.

For the second part, let $x \in R$ be left invertible

$\Rightarrow \exists\, y \in R$ s.t. $yx = 1$

$$\Rightarrow \bar{y}\bar{x} = (y + \text{rad}R)(x + \text{rad}R) = yx + \text{rad}R$$
$$= 1 + \text{rad}R = \bar{1} \in \bar{R} \quad (= R/\text{rad}R)$$

Thus $\bar{x} \in \bar{R}$ is left invertible.

Conversely, let $y \in R$ s.t. $\bar{y}\bar{x} = \bar{1} \in \bar{R}$

$\Rightarrow 1 - yx \in \text{rad}\,R$

$\Rightarrow -(1 - yx) \in \text{rad}R$

$\Rightarrow yx \in 1 + \text{rad}R \subseteq U(R)$ by 4.3 taking $x = 1, z = x$

i.e. $x$ has a left inverse in $R$

The proof is true for right inverse as well, thus it holds true for invertibility. ▨

<u>Definition</u>:- A one-sided (or two sided) ideal $Q \subseteq R$ is said to be nil if $Q$ consists of nilpotent elements. $Q$ is said to be nilpotent if $Q^n = 0$ for some natural number $n$ i.e. $a_1 \cdots a_n = 0$ for any set of elements $a_1, \ldots, a_n \in Q$.

This condition is much stronger than $Q$ being nil.

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<u>Example</u>:- Let $R$ be the commutative ring

$$R = \mathbb{Z}[x_1, x_2, \dots] / \langle x_1^2, x_2^3, x_3^4, \dots \rangle$$

Consider the ideal $\mathfrak{a} = \langle \bar{x}_1, \bar{x}_2, \bar{x}_3, \dots \rangle$ $\left( = \dfrac{\langle x_1, x_2, \dots \rangle}{\langle x_1^2, x_2^3, x_3^4, \dots \rangle} \right)$

Any element of $\mathfrak{a}$ being a finite sum of elements of the type $\sum_{\text{finite}} ra$, $r \in R$, $a \in \mathfrak{a} = \langle \bar{x}_1, \bar{x}_2, \dots \rangle$

we can get a maximal integer $n$ s.t.

$$\left(\sum ra\right)^n \in \langle x_1^2, x_2^3, \dots \rangle, \text{ the zero of } \mathfrak{a}$$

e.g. $\bar{x}_1^2 = (x_1 + \langle x_1^2, x_2^3, \dots \rangle)^2 = 0$

$x_2^3 = (x_2 + \langle x_1^2, x_2^3, \dots \rangle)^3 = 0$

Thus for any finite combination $\sum ra$, $\exists$ a maximal integer '$n$' s.t. $\left(\sum ra\right)^n = 0$

Thus every element is nilpotent, thus $\mathfrak{a}$ is nil.

But since $\mathfrak{a}$ has infinite generating set it is impossible to find one finite number '$n$' s.t. $\mathfrak{a}^n = 0$.

Thus $\mathfrak{a}$ is not nilpotent.

One advantage of nilpotent over nil is the following —

<u>Lemma 4.10</u>:- Let $\mathfrak{a}_i (1 \le i \le m)$ be a finite set of left ideals in $R$. If each $\mathfrak{a}_i$ is nilpotent, then $\mathfrak{a}_1 + \dots + \mathfrak{a}_m$ is also nilpotent.

<u>Pf</u>:- It is enough to prove the $m = 2$ case, so that the result will hold.

Let $\mathfrak{a}$ and $\mathfrak{b}$ be nilpotent left ideals.

i.e. $\mathfrak{a}^n = 0 = \mathfrak{b}^n$ for $n$ say.

Let $\mathfrak{c} = \mathfrak{a} + \mathfrak{b}$

We claim that $\mathfrak{c}^{2n} = 0$

Consider a product of any $2n$ elements in $\mathfrak{c}$

$$(a_1 + b_1) \cdots (a_{2n} + b_{2n})$$

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On expansion of this product, we get each term of the expansion is a product of $2n$ elements, at least $n$ from $\mathcal{A}$ or atleast $n$ from $\mathcal{B}$.

The products of the type (in 4-product)

$a_1 b_1 a_2 b_2 = 0$ as $\mathcal{B}$ is an left ideal so elements of the type $a_2 b_2$, $a_1 b_1 \in \mathcal{B}$ and their product is zero as $\mathcal{B}^2 = 0$

$a_1 a_2 b_1 b_2 \in \mathcal{B}$,

$a_1 b_1 b_2 a_2 = a_1 (b_1 b_2 a_2) \in \mathcal{A}\mathcal{A} = \mathcal{A}^2 = 0$

Thus since $\mathcal{A}$ and $\mathcal{B}$ are left $R$-ideals and the $n$-products $\mathcal{A}^n = \mathcal{B}^n = 0$

the above $2n$-product are zero.

$\Rightarrow \quad \mathcal{C}^{2n} = 0$

i.e. $\mathcal{C}$ is nilpotent. ∎

<u>Lemma 4.11</u> :- If a left (respectively right) ideal $\mathcal{A} \subseteq R$ is nil, then $\mathcal{A} \subseteq \operatorname{rad} R$

<u>Pf</u> :- Let $y \in \mathcal{A}$.

Since $\mathcal{A}$ is nil i.e. every element in $\mathcal{A}$ is nilpotent, the element $xy \in \mathcal{A}$ is also nilpotent for any $x \in R$.

We know the Geometric series $\sum_{i=0}^{\infty} (xy)^i = \frac{1}{1-xy}$

$xy$ nilpotent $\Rightarrow (xy)^n = 0$ for some $n$, thus the infinite sum $\sum_{i=0}^{\infty} (xy)^i$ has only finitely many terms and it belongs to $\mathcal{A}$.

Thus $(1-xy)$ is invertible $\left(\sum_{i=0}^{\infty} (xy)^i \text{ being the inverse}\right)$

$\Rightarrow \quad y \in \operatorname{rad} R$. ∎

An generalization of Wedderburn radical (the largest nilpotent ideal) being Jacobson radical i.e. rad.

In case $R$ is artinian, the two radicals coincide.

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<u>Theorem 4.12</u> :- Let $R$ be a left artinian ring. Then $\operatorname{rad} R$ is the largest nilpotent left ideal and it is also the largest nilpotent right ideal.

<u>Definition</u> :- A minimal element in a partially ordered set $(C, \leq)$ is defined as : $b \in C$ is minimal if every $c \in C$ which is comparable to $b$, then $b \leq c$.
Note that it is not necessarily true that $b \leq c \ \forall c \in C$.
Further, $C$ may contain many minimal elements or none at all.

<u>minimal Condition</u> :- A ring $R$ is said to satisfy the minimal condition on ideals if every non-empty set of ideals of $R$ contains a minimal element (w.r.t. set inclusion)

<u>Result</u> :- A ring $R$ satisfy the DCC on ideals iff $R$ satisfy the minimal condition on ideals.

<u>Pf</u> :- Suppose $R$ satisfies the minimal condition on ideals and $a_1 \supset a_2 \supset a_3 \supset \cdots$ is a chain of ideals.
Then the set $\{a_i \mid i \geq 1\}$ has a minimal element say $a_n$.
Consequently for $i \geq n$, $a_n \supset a_i$ by hypothesis
and $a_n \subset a_i$ by minimality
Thus $a_i = a_n$ for each $i \geq n$
Thus $R$ satisfies the DCC.
Conversely, suppose $R$ satisfies DCC
Let $S$ be a nonempty set of ideals of $R$
Then $\exists\ b_0 \in S$ as $S$ is non-empty
If $S$ has no minimal element, then for each ideal $b$ in $S$, $\exists$ at least one ideal $b'$ in $S$ s.t. $b \supsetneq b'$
For each $b$ in $S$, choose one such $b'$ (axiom of choice)
This choice is then defines a function

$$f : S \longrightarrow S \quad \text{by}$$
$$b \longmapsto b'$$

By Recursion Theorem: (If $S$ is a set, $a \in S$, and for each

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$n \in \mathbb{N}$, $f_n : S \longrightarrow S$ is a function then $\exists$ a unique function $\phi : \mathbb{N} \longrightarrow S$ s.t. $\phi(0) = a$ and $\phi(n+1) = f_n(\phi(n))$ $\forall n \in \mathbb{N}$.

with $f = f_n$ $\forall n$,

$\exists$ a function $\phi : \mathbb{N} \longrightarrow S$ s.t.

$$\phi(0) = b_0 \text{ and } \phi(n+1) = f(\phi(n)) = \phi(n')$$

Thus if $b_n \in S$ denotes $\phi(n)$ then there is a sequence

$$b_0, b_1, \ldots \text{ s.t. } b_0 \supsetneq b_1 \supsetneq b_2 \supsetneq \cdots$$

This contradicts DCC

Thus $S$ must have a minimal element when $R$ satisfies minimum condition

<u>Remark</u>:- This result is also true for ascending chain and maximal condition.

<u>Pf of the theorem</u> :-

clearly every nilpotent ($a^n = 0$) ideal is nil (all nilpotent) and by above lemma every nil hence nilpotent ideal is contained in rad $R$ ($J$ say)

Thus to show the result it is sufficient to prove that $J$ is nilpotent (then $J$ will be the largest nilpotent)

Consider the descending chain of ideals

$$J \supseteq J^2 \supseteq J^3 \supseteq \cdots$$

Since $R$ is artinian (left), $J$ is an ideal (left & right) applying the left DCC above, there exists an integer $k$ s.t. $J^k = J^{k+1} = \cdots = I$ (say)

We claim that $I = 0$ (i.e. $J^k = 0$)

Assume that $I \neq 0$.

Consider the set $\{$ left ideals $a \mid I.a \neq 0 \}$

By DCC (left), this set has a minimal element $a_0$ (say)

(note that a ring is DCC if it is both left and rt. DCC)

Fix an element $a \in a_0$ s.t. $I.a \neq 0$

Then $I.(Ia) = I^2 a = Ia \neq 0$

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Thus by minimality of $Q_0$ we have $\quad (Q_0 \supseteq I_a$ anyway

$$Q_0 \subseteq I_a$$

Thus $I \cdot a = Q_0$

$\Rightarrow a = ya$ for some $y \in I \subseteq \text{rad } R \ (= J)$

i.e. $(1-y)a = 0$

$\Rightarrow a = 0$ as $1-y \in U(R)$ $\quad \left(\begin{array}{l}\text{as } 1-xyz \in U(R) \\ \text{for } x = z = 1\end{array}\right)$

This is a contradiction.

Hence $I = J^k = 0$ satisfying DCC showing $J$ is nilpotent. Since $J (= \text{Rad } R)$ is both left and rt. radical, it is the largest nilpotent ideal. ∎

Combining the above theorem and the lemma preceding we have the following —

<u>Corollary 4.13</u> :- In a left artinian ring, any nil left ideal is nilpotent.

<u>Pf</u>:- Since any nil $Q$ is contained in rad $R$ and in an left artinian ring rad $R$ is the largest nilpotent making the nil $Q$ also a nilpotent.

<u>Remark</u>:- In a commutative ring $R$, as all nilpotent elements form a nil ideal which is contained in rad $R$ thus any nilpotent element is contained in rad $R$

But when $R$ is not commutative then this may not hold —

For instance, let $D$ be a division ring.

The only left ideals are $\{0\}$ and itself.

We know that the ring $R = M_n(D)$ has left ideals of the form $M_n(Q)$ where $Q$ is an left ideal of $D$.

Thus intersection of maximal ideals i.e rad $R = \{0\}$

Thus as $M_n(D)$ is left artinian, nil left ideals contained in rad $R$, show that $R$ has no nonzero left

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nil ideals. But nilpotent elements exist in large numbers. "nil $\Rightarrow$ nilpotent" fails in this case.

Connection between semisimple rings and J-semisimple rings:—

<u>Theorem 4.14</u>:— For any ring $R$, the following three statements are equivalent

(1) $R$ is semisimple
(2) $R$ is J-semisimple and left artinian
(3) $R$ is J-semisimple, and satisfies DCC on Principal left ideals.

<u>Pf</u>:- (1) $\Rightarrow$ (2) Assume that $R$ is semisimple and let $Q = \text{rad}\, R$

Since $Q$ is an ideal then semisimplicity implies that $\exists$ a left ideal $b$ s.t. $R = Q + b$.

Also $\exists$ idempotents $e, f$ s.t. $Q = Re$ and $b = Rf$ and $e + f = 1$ (by exercise 1.7)

Thus $e \in Q = \text{rad}\, R$

$\Rightarrow 1 - 1.e.1 = 1 - e \in U(R)$ i.e. is a unit for $x = 2 = 1$

But $f^2 = f$ (as is idempotent)

$\Rightarrow f = 1$

Then $f = 1 - e \Rightarrow e = 0$

In particular $Q = R.e = 0$

Thus $R$ is J-semisimple

Also a semisimple ring $R$ (which is clearly left semisimple) is left artinian (also left noetherian) by a result in semisimplicity

Thus the claim.

<u>(2) $\Rightarrow$ (3)</u> Trivial. Since $R$ is left artinian, then DCC holds for any left ideals hence for principal left ideals.

(3) $\Rightarrow$ (1) Assume that $R$ satisfies DCC on Principal left ideals. Such Ring $R$ satisfies the following two properties:—

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(a) Every left ideal ~~[illegible]~~ $Q \neq 0$ contains a minimal left ideal (13)

Since $R$ satisfies DCC on principal left ideals then it satisfies minimal property

i.e. any set of principal left ideals (nonzero) has a minimal.

i.e. If $I$ is a minimal number of the family of nonzero principal left ideals contained in $Q$.

i) $I$ is minimal as a left ideal in $Q$. (as if $I$ has one nonzero element in it the principal ideal generated by this has to be $I$.

(b) Every minimal left ideal $\mathfrak{b}$ is a direct summand of ${}_RR$ (left regular $R$-mod).

Since $\mathfrak{b} \neq 0 \subset \operatorname{rad} R$ i.e. $\mathfrak{b}$ is not intersection of all maximal ideals, $\exists$ a maximal left ideal $\mathfrak{M}$ not containing $\mathfrak{b}$. Then $\mathfrak{b} \cap \mathfrak{M} = 0$

$$\Rightarrow {}_RR = \mathfrak{b} \oplus \mathfrak{M}$$

Now on the contrary assume that $R$ is not semisimple

Let $\mathfrak{b}_1$ be the minimal left ideal

then (by (b)) let ${}_RR = \mathfrak{b}_1 \oplus Q_1$

$\Rightarrow Q_1 \neq 0$ and by (a) $\exists$ a minimal left ideal $\mathfrak{b}_2 \subseteq Q_1$ (say)

Again by (b) $\mathfrak{b}_2$ is a direct summand ~~of~~ in ${}_RR$

thus in $Q_1$ (if $\mathfrak{b}_2$ is direct summand in ${}_RR$ i.e.

$${}_RR = \mathfrak{b}_2 \oplus \mathfrak{b}_2'$$

i.e. $Q_1 = \mathfrak{b}_2 \oplus Q_2$

Continuing this way, we get a descending chain of left ideals $Q_1 \supsetneq Q_2 \supsetneq Q_3 \supsetneq \cdots$

which are direct summands of ${}_RR$.

i.e. ${}_RR = \mathfrak{b}_1 \oplus Q_1 \oplus Q_2 \oplus \cdots$

~~As~~ $Q_i$'s are direct summands of $R$ (by Problem 1.7, $Q_i = Re_i$, $e_i$, an idempotent)

so they are principal left ideals of $R$.

which contradicts (3).

Thus $R$ is semisimple. ∎

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