Thm: (Prop 6.3) Let $0 \to M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to 0$ be a short exact seq. of left R-modules. Then M is Noeth. (resp Art.) iff M' & M'' are Noeth (resp. Art.)

Pf. ($\Rightarrow$) ($\uparrow$-increasing)

Let $N_1' \subseteq N_2' \subseteq \cdots$

be an $\uparrow$ chain of submodules in M'

$\ni$ $N_1'\alpha \subseteq N_2'\alpha \subseteq \cdots$

is an $\uparrow$ chain in M, which is Noeth.

$\therefore \exists\, p \in \mathbb{N}$: $N_p'\alpha = N_r'\alpha \quad \forall\, p \leq r$

$\Rightarrow N_p' = N_r'$ ($\because \alpha$ is 1-1) $\forall\, p \leq r$

$\therefore M'$ is Noeth.

M'' being the homo (homomorphic) image of M (as β is onto) is Noeth.

($\Leftarrow$) Let $N_1 \subseteq N_2 \subseteq \cdots$

be an $\uparrow$ chain in M.

Then $N_1\beta \subseteq N_2\beta \subseteq \cdots$

& $N_1\alpha^{-1} \subseteq N_2\alpha^{-1} \subseteq \cdots$

are $\uparrow$ chains in M'' & M'.

$\therefore \exists\, p \in \mathbb{N}$ s.t. $N_p\alpha^{-1} = N_r\alpha^{-1} \quad \forall\, p \leq r$

& $N_p\beta = N_r\beta \quad \forall\, p \leq r$

Let $r > p$. To show: $N_r \subseteq N_p$

Let $x \in N_r \Rightarrow x\beta \in N_r\beta = N_p\beta$

$\Rightarrow x\beta = y\beta$ when $y \in N_p$

$\Rightarrow x-y \in \mathrm{Ker}\beta = \mathrm{Im}\,\alpha$

$\Rightarrow x-y = z\alpha$ f.s. $z \in M'$

Now $z\alpha \in N_r$ ($\because x, y \in N_r$; $N_p \subseteq N_r$)

$\Rightarrow z \in N_r\alpha^{-1} = N_p\alpha^{-1}$

$\Rightarrow z\alpha \in N_p \Rightarrow x-y \in N_p \Rightarrow x \in N_p$ ($\because y \in N_p$)

$\therefore N_r \subseteq N_p$. Thus, $N_r = N_p \;\forall\; r \geq p$

$\therefore$ M is Noeth.

(The proof of Artinian is on the similar lines)

Corollary: If N is a submodule of M, then
M Noeth (resp. Art) $\Leftrightarrow$ N & M/N Noeth (resp Art)

Pf: We have short exact seq.

$$0 \to N \xrightarrow{i} M \xrightarrow{q} M/N \to 0$$

where i is the inclusion map & q is the canonical quotient map.

By last result, M is Noeth (resp. Art.) iff N & M/N are both Noeth. (resp. Art.)

Corollary: If M_i $(1 \le i \le n)$ are Noeth. (resp. Artinian) A-modules, then so is $\bigoplus_{i=1}^{n} M_i$.

Proof: The result is true for $n=1$.

Let $n=2$. Then we have a short exact sequence

$$0 \longrightarrow M_1 \xrightarrow{i} M_1 \oplus M_2 \xrightarrow{\pi} M_2 \longrightarrow 0$$

where i is the canonical injection and π is the canonical projection.

$\because$ M_1 & M_2 are both Noeth. (resp. Art.), by last theorem, $M_1 \oplus M_2$ is so.

For a general n $(n > 2)$

$$\bigoplus_{i=1}^{n} M_i = \bigoplus_{i=1}^{n-1} M_i \oplus M_n$$

Let the result be true for $n-1$ (Induction Hypothesis) then by case $n=2$, $\bigoplus_{i=1}^{n} M_i$ is Noeth (resp. Artinian) since $\bigoplus_{i=1}^{n-1} M_i$ is Noeth (resp. Art.)

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~~Result (i) Subm~~ Homomorphic image

Result (1): Quotient of a Noetherian (or Artinian) is also Noeth. (or Artinian)

Pf: Let M be Noeth & N be a submodule of M.

~~Over~~ M/N Let

$$K_1 \subseteq K_2 \subseteq \cdots$$

be an ↑ chain of submodules of M/N

Then each $K_i = N_i/N$ where N_i is a submodule of M containing N. Also, (order preserving)

$$N_1 \subseteq N_2 \subseteq N_3 \subseteq \cdots$$

$\because$ M is Noeth, this chain is stationary

$\therefore$ $N_p = N_r \quad \forall\, r \geq p$ f.s. p;

$\Rightarrow$ $K_p = K_r$.

Result 2: Homomorphic image of a Noeth. (resp. Art) module is Noeth.

Pf. Let $f : M \to M'$ be an onto homo. & M be Noeth.

Then by 1st Iso theo.

$$\frac{M}{\mathrm{Ker} f} \cong M'$$

But $M/\mathrm{Ker} f$ is Noeth, by Result (1) above.

$\therefore$ M' is Noeth.

Result 3: Finite Sum of Noeth. (or Art.) submodule is Noetherian.

Pf: Let $M_1, \ldots, M_n$ be Noeth. submodules of M.

It is suff. (sufficient) to prove that $M_1 + M_2$ is Noeth. ~~(by ind.)~~) the proof then follows from Induction on n.

We know that

$$\frac{M_1 + M_2}{M_1} \cong \frac{M_2}{M_1 \cap M_2}$$

(2nd Iso thm) (II Isomorphism Theorem)

$\because$ M_2 is Noeth, so is $M_2/M_1 \cap M_2$ (by result 2)

$$\Rightarrow \frac{M_1 + M_2}{M_1} \text{ is Noeth.}$$

But M_1 & $\dfrac{M_1 + M_2}{M_1}$ both Noeth $\Rightarrow$ $M_1 + M_2$ is Noeth.

(by previous corollary)

$0 \longrightarrow M_1 \longrightarrow M_1 \oplus M_2 \longrightarrow$

$\boxed{\text{Thm: Let } R \text{ be a right Noetherian ring (or rt. Art.) If } M \text{ is finitely generated right } R\text{-module, then } M \text{ is Noeth (resp. Art.)}}$

Pf: Let M be generated by $x_1, \ldots, x_n$ (say)

Then $M = \sum_{i=1}^{n} x_i R$

It is suff. (sufficient) to prove ~~that~~ each $x_i R$ is Noeth.

($\because$ sum of Noeth is ~~so~~ Noeth - by Result 3)

Consider the mapping

$f : R \longrightarrow x_i R$ defined by

$(r)f = x_i r$

Then f is R-homo & onto

$\therefore$ R is Noeth $\Rightarrow$ $x_i R$ is Noeth. & hence the result. (since homomorphic image of Noeth is Noeth)

Def^n: An integral domain in which every right ideal ~~domain~~ is principal (i.e. generated as right ideal by a single elt.) is called a principal right ideal domain (P.R.I.D.)

~~Defn~~: Corollary ①: Every P.I.D. (P.R.I.D) is Noetherian; since every ideal (right ideal) is generated by a single element.

But P.I.D. need not be Artinian (e.g. $\mathbb{Z}$)

Cor. 2: A finitely generated module over a PID is Noetherian.

Theorem: Let R be Noetherian (resp. Artinian), I be an ideal of R. Then R/I is a Noetherian (resp. Art.) ring.
(i.e. R/I is Noeth. as R/I-module)

Proof: R is Noeth. as R-module

$\Rightarrow$ R/I is Noeth. as R-module
(quotient of Noeth. is Noeth.)

$\Rightarrow$ R/I is Noeth. as R/I-module.